Linear algebra

A *field* = set of numbers,
on which $+$, $-$, $\cdot$, & $/$ (except by 0) are defined.

Ex $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, ...

Not ex: $\mathbb{Z}$, $\mathbb{N}$

Vector space over a field F

= set V with operations $+: V \times V \to V$ & scalar multiplication (by an element of the field), such that

- $x + y = y + x \quad \forall x, y \in V$
- $x + (y + z) = (x + y) + z \quad \forall x, y, z \in V$
- $\exists!\ 0 \in V$ s.t. $x + 0 = x \quad \forall x \in V$
- $\forall x \in V\ \exists!$ "$-x$" s.t. $x + (-x) = 0$
- $1 x = x \quad \forall x \in V \quad (1 \in F)$
- $(ab)x = a(bx) \quad \forall x \in V, \forall a, b \in F$
- $a(x + y) = (ax) + (ay) \quad \forall a \in F, x, y \in V$
- $(a + b)x = (ax) + (bx) \quad \forall a, b \in F, x \in V$

Call elements of V, vectors.

Exs of things that form vector spaces:

vectors in the ordinary sense
matrices
tensors
functions (mention QM application)
polynomials

$\rightsquigarrow$ outline $+$ operation for each of these.

Linear combination

abstract sense

A vector β in a vector space V is said to be a linear combination of the vectors $\alpha_1, \ldots, \alpha_n$ in V if $\exists$ scalars $c_1, \ldots, c_n \in F$ s.t.

$$\beta = c_1\alpha_1 + \cdots + c_n\alpha_n$$

(give some exs)

~~Cont'd~~

Subspaces

Def'n: Let V be a vector space over the field F.
A subspace of V is a subset W of V which is itself a vector space with the operations of vector add'n & scalar multiplication inherited from V.

(give exs to build intuition)

Thm A nonempty subset $W \subseteq V$ is a subspace of V
iff for each $\alpha, \beta \in W$, each $c \in F$, $c\alpha + \beta \in W$

– explain to give intuition

Pf S'pose W is a nonempty subset of V st $c\alpha+\beta \in W$.
Since W is nonempty, $\exists \rho \in W$, so $(-1)\rho + \rho = 0 \in W$.
Then, $\forall \alpha \in W, c \in F$ ~~$c\alpha = c\alpha + 0$~~ $c\alpha + 0 = c\alpha \in W$
In particular, $(-1)\alpha = -\alpha \in W$
& the rest follows sim'ly.
[Explain]

Check def'n of vector space:

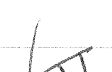
Conversely, s'pose ~~[illegible]~~ W is a subspace.
Then triv'ly, $c\alpha+\beta \in W$.

Ex $\{0\} \subseteq V$ is the zero subspace

Ex (symmetric matrices) $\subseteq$ matrices
↳ explain

Ex Hermitian matrices $\subseteq$ matrices
↳ explain

Ex polynomials $\subseteq$ functions

Ex not a subspace: vectors in  in 2D (~~[illegible]~~ $-v \notin$ region but $v \in$ region)

Ex Sol'n space of a system of homogeneous linear equ'ns.

Let A be $m \times n$ matrix,
then sol'n space $= \{x \mid Ax = 0\}$

If $Ax=0$, then $A(cx+y) = cAx + Ay$
$= 0+0=0$

Thm Let V be a vector space over a field F.
The intersection of any collection of subspaces, is a subspace.

Pf Let $\{W_\alpha\}$ be a collection of subspaces of V,
set $W = \bigcap_\alpha W_\alpha$.

$0 \in W_\alpha \ \forall \alpha \Rightarrow 0 \in W$
S'pose $x, y \in W$, $c \in F$
then for all α, $x, y \in W_\alpha$ & $cx+y \in W_\alpha$
$\Rightarrow cx+y \in W$.

Start here Mon

Def'n Let S be a set of vectors in a vector space V.
The subspace spanned by S is the intersection of all subspaces that contain S.
When $S =$ finite set $\{\alpha_1, \ldots \alpha_n\}$,
say $W =$ subspace spanned by $\{\alpha_1, \ldots \alpha_n\}$.

Thm The subspace spanned by a nonempty subset $S \subseteq V$ is the set of all linear combinations of vectors in S.

Pf

Let W = subspace spanned by S.
Note all linear combinations $\subseteq W$.
Furthermore, set of all lin' comb's is a subspace of V
$(x, y \in \{\text{lin comb's}\} \Rightarrow cx + y \in \{\text{lin comb's}\})$
$\Rightarrow W \subseteq \{\text{lin' comb's}\}$

$\Rightarrow W = \{\text{lin' comb's}\}$.

Def'n If $S_1, \dots S_k$ are subsets of V,
the set of all sums $\alpha_1 + \dots + \alpha_k$ $(\alpha_i \in S_i)$
is called sum of the subsets & is denoted $S_1 + \dots + S_k$

Def'n Let V be a vector space over F.
A subset S of V is said to be linearly dependent
if $\exists$ distinct vectors $\alpha_1, \cdots \alpha_n$ & scalars $c_1 \neq 0, \cdots c_n \neq 0$
s.t.

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0$$

A set which is not linearly dependent, is linearly independent.

° give some exs

Facts

- any set which contains a linearly dependent set, is itself linearly dependent
- any subset of a linearly independent set is linearly independent
- any set which contains the 0 vector is linearly dependent
- a set of vectors is linearly independent iff every finite subset is linearly independent

Def'n Let V be a vector space.
A basis for V is a linearly independent set of vectors in V which spans V.
V is said to be finite-dimensional if it has a finite basis.

- give some exs

Standard basis for $F^n = \{ (1,0,\cdots 0), (0,1,\cdots 0), \cdots, (0,0,\cdots 1)\}$

Ex let P be an invertible $n\times n$ matrix.
The columns of P are a basis for F^n.

→ If X is a column matrix,
$PX = x_1 P_1 + \cdots + x_n P_n$, $\quad$ x_i entries of X
$\quad$ P_i the columns of P

Since $PX = 0 \Rightarrow X = 0$,
the P_i are LI.
why does it span?
let Y be any column vector.
If $X = P^{-1}Y$, then $Y = PX = x_1 P_1 + \cdots + x_n P_n$
& so spans.

Ex Consider $\mathbb{C}[x]$ = poly's in x

A basis is $\{1, x, x^2, \cdots\}$
Check:
- spans: clear, all poly's are lin' comb' of ~~monomial~~
- LI: show each finite subset LI.
Suffices to show, for each n, $\{1, x, \cdots x^n\}$ LI.
S'pose $c_0 + c_1 x + \cdots + c_n x^n = 0$
→ would have to hold $\forall x \in F$,
ie, every $x\in F$ a root of the poly' above,
but can have no more than n distinct roots
$\Rightarrow c_0 = \cdots = c_n = 0 \quad \Rightarrow$ LI.

Thm Let V be a v.s. spanned by a finite set of vectors $\beta_1, \cdots \beta_n$. Then any LI set of vectors in V is finite & contains no more than n elements.

Pf Suffices to show any subset containing $> n$ vectors is lin' dep'.
Let S be such a subset, $\alpha_1, \cdots \alpha_m$ distinct vectors, $m > n$.
Since $\beta_1, \cdots \beta_n$ span V, $\exists$ scalars A_{ij} st

$$\alpha_i = \sum_j A_{ij} \beta_j$$

Then for any m scalars x_i,

$$\sum_i x_i \alpha_i = \sum_{ij} x_i A_{ij} \beta_j = \sum_j \left(\sum_i x_i A_{ij} \right) \beta_j$$

From (solving algebraic eqn's), $\exists$ scalars x_i st $\sum_i x_i A_{ij} = 0 \ \forall j$

$$\Rightarrow \sum x_i \alpha_i = 0, \quad x_i \text{ not all } 0$$

$$\Rightarrow \alpha_i \ \cancel{LI}.$$

Cor If V is a finite-dim'l v.s., then any two bases of V have the same (finite) number of elements.

Pf Since V is finite-dim'l, it has a finite basis $\{\beta_1, \cdots \beta_n\}$.
By thm above, every basis is finite & has no more than n elements.
So if $\{\alpha_1, \cdots \alpha_m\}$ is a basis, $m \leq n$.
By same argument, $n \leq m \Rightarrow \underline{m = n}$.

Def'n dimension of a vector space $=$ # of elements in a basis.

~~[illegible]~~

Cor Let V be a finite-dim'l v.s. & let $n = \dim V$. Then

a) any subset of V which contains more than n vectors is linearly dependent

b) no subset of V which contains fewer than n vectors can span V.

Lemma Let S be a LI subset of a vs V.
S'pose $\beta \in V$, $\beta \notin$ span of S.
Then the set obtained by adding β to S is LI.

Pf S'pose $\alpha_1, \cdots \alpha_n$ are distinct vectors in S,
and $c_1 \alpha_1 + \cdots + c_n \alpha_n + b\beta = 0$

Then $b = 0$, else $\beta = \left(-\frac{c_1}{b}\right)\alpha_1 + \cdots + \left(-\frac{c_n}{b}\right)\alpha_n \Rightarrow \beta \in$ span of S.

Thus $c_1 \alpha_1 + \cdots + c_n \alpha_n = 0$, & since S is LI, all $c_i = 0$.

$\Rightarrow$ LI.

Thm If W is a subspace of a finite-dim'l v.s. V,
then every LI subset of W is finite & is part of a (finite) basis for W.

Pf see text

Cor If W is a proper subspace of a finite-dim'l vector space V,
then W is finite-dim'l and $\dim W < \dim V$.

Pf S'pose $W \ni \alpha \neq 0$. By thm above, there is a basis of W containing α which has $\leq (\dim V)$ elements.
$\Rightarrow$ W is finite-dim'l, & $\dim W \leq \dim V$.
Since W is a proper subspace,
there is a vector $\beta \in V$ which is not in W.
Adjoining β to any basis of W,
we get a LI subset of V. $\Rightarrow$ $\dim W < \dim V$.

Cor In a finite-dim'l v.s. V, every nonempty LI ~~subset~~ set of vectors is part of a basis.

Cor Let A be an $n \times n$ matrix over a field F,
& s'pose the row vectors of A form a LI set of vectors in F^n.
Then A is invertible.

Pf Let $\alpha_1, \cdots \alpha_n$ be the row vectors of A,
& s'pose W is the subspace of F^n spanned by $\alpha_1, \cdots \alpha_n$.
Since they are LI, $\dim W = n$, so $W = F^n$.
If $\epsilon_1, \cdots \epsilon_n$ is std basis for F^n,
we then have $\epsilon_i = \sum_j B_{ij} \alpha_j$ for some scalars B_{ij}.

Thus, for the matrix B w/entries B_{ij}, we have

$$BA = I.$$

Thm If W_1, W_2 are finite-dim'l subspaces of a vector space V, then $W_1 + W_2$ is finite-dim'l and

$$\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2).$$

– explain intuition, refer to text for pf.

(§2.4 - coordinates)

Def'n If V is a finite-dim'l v.s., an ordered basis for V is a finite sequence of vectors which is LI & spans V.

Let $\mathcal{B} = \{\alpha_1, \cdots \alpha_n\}$ be an ordered basis for V.

For any $\alpha \in V$, $\exists!$ n-tuple $(x_1, \cdots x_n) \in \mathbb{F}^n$ s.t.

$$\alpha = \sum_i x_i \alpha_i$$

Uniqueness: if $\alpha = \sum_i z_i \alpha_i$,

then $\sum_i (x_i - z_i)\alpha_i = 0$ but α_i LI $\Rightarrow$ $x_i - z_i = 0$ $\forall i$

Call x_i the i^{th} coordinate of α relative to the ordered basis $\mathcal{B}$.

There is a 1-1 correspondence

vectors α $\longleftrightarrow$ coordinates $(x_1, \cdots, x_n)$

Change of basis:

Suppose $\mathcal{B} = \{\alpha_1, \cdots, \alpha_n\}$, $\mathcal{B}' = \{\alpha_1', \cdots, \alpha_n'\}$
are two ordered bases for V.
There are unique scalars P_{ij} s.t. $\alpha_j' = \sum_i P_{ij} \alpha_i$

If $\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n = x_1'\alpha_1' + \cdots + x_n'\alpha_n'$

$$= \sum_i x_i' \left(\sum_j P_{ij} \alpha_j\right) = \sum_j \left(\sum_i P_{ij} x_i'\right) \alpha_j$$

So $X = PX'$ where $X = (x_i)$, $X' = (x_i')$

& P is invertible.

(§ 2.5) (need to describe row reduction)

Let A be an $m \times n$ matrix;
the row vectors are

$$\alpha_i = (A_{i1}, \ldots, A_{in})$$

row space of A = subspace of F^n spanned by these vectors

row rank = dim of row space

Let P be a $k \times m$ matrix,
so $(PA) = k \times n$ matrix

Row vectors of PA are linear comb's

$$\beta_i = P_{i1}\alpha_1 + \cdots + P_{im}\alpha_m$$

$\rightsquigarrow$ so row space of (PA) is a subspace of row space of A

If P is invertible,
then row space of PA = row space of A
(as by above, row space of $A = P^{-1}PA$ is subspace of row space of PA)

Turn to row operations in solving systems of linear eqn's....

Row operations in solving systems of linear algebraic eqn's:

$$A_{11}x_1 + \cdots + A_{1n}x_n = y_1$$
$$A_{21}x_1 + \cdots + A_{2n}x_n = y_2$$
$$\vdots$$
$$A_{m1}x_1 + \cdots + A_{mn}x_n = y_m$$

or, $Ax = y$

How to solve?
- replace with linear comb's.
  (do an ex)

Specifically, we can:

- multiply one row of A by a nonzero scalar
- replace r^{th} row by (row r) + c(row s), $c \in F$, $r \neq s$
- interchange 2 rows

$\rightsquigarrow$ "elementary row operations"
- realized by multiplication by "elementary matrices," which are elem'-row-op'd versions of I (by def'n)

  eg $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$

  Fact: elementary matrices are invertible

Two matrices A, B are said to be row-equivalent if B can be obtained from A by a finite sequence of elementary row ops.

An $m \times n$ matrix A is called row-reduced if
a) the first nonzero entry in each nonzero row of $A = 1$
b) each column of A which contains the leading non-zero entry of some row has all its other entries $= 0$.

Ex not row-reduced $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$ is row-reduced: $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 9 \end{bmatrix}$

An $m \times n$ matrix A is called <u>row-reduced echelon</u> if

a) A is row-reduced

b) every row of A which has all its entries 0 occurs below every row which has a nonzero entry

c) if rows $1, \ldots, r$ are the nonzero rows of A, and if the leading nonzero entry of row i occurs in column k_i, then $k_1 < k_2 < \cdots < k_r$

<u>Ex</u> $\begin{pmatrix} 0 & 1 & -3 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ is row-reduced echelon

Back to row spaces:

~~Defn Two matrices A, B are row-equivalent iff~~

<u>Thm</u> Row-equivalent matrices have the same row space.

<u>Pf</u> If A, B are row-equivalent,
then $B = PA$ for P = product of elementary matrices.
Since elem' matrices are invertible, P is invertible.
$\Rightarrow A, B$ have same row space

<u>Thm</u> Let R be a nonzero row-reduced echelon matrix.
Then the nonzero row vectors of R form a basis for the row space of R.

<u>Pf</u> Let $\rho_1, \dots, \rho_r$ be the nonzero row vectors.
Clearly <u>span</u> the row space; need merely show LI.
Since R is row-reduced echelon,
there are positive integers $k_1, \dots, k_r$ st for $i \le r$,

- $R(i,j) = 0$ if $j < k$
- $R(i, k_j) = \delta_{ij}$
- $k_1 < \cdots < k_r$

Suppose $\rho = (b_1, \dots, b_n)$ is a vector in the row space:
$$\rho = c_1\rho_1 + \cdots + c_r\rho_r$$
Claim $c_j = b_{k_j}$. After all, $b_{k_i} = \sum_{j=1}^{r} c_j R(j, k_i)$
$$= \sum_j c_j \delta_{ij} = c_i$$

so $\rho = 0 \Rightarrow$ all $c_i = 0 \Rightarrow \rho_i$ LI.

(§2.6)

Given a set of vectors, how to determine if LI?

Fast way: write a ~~matrix~~ matrix whose rows are those vectors,
then put in row-reduced echelon form.
If there are any zero rows, then, <u>not</u> LI.

<u>Ex</u> $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\{(1,0), (0,1), (1,1)\}$ ~~LI~~

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\{(1,0,0), (0,1,0), (0,1,1)\}$ LI

→ same method gives dim of space spanned by those vectors.

→ in this fashion, can also get a nearly-standard basis for the subspace spanned by the rows.

(§3.1 Linear transformations)

Def'n Let V, W be vector spaces over the field F.
A linear transformation from V into W
is a function $T: V \to W$ such that
$$T(c\alpha + \beta) = c(T\alpha) + T\beta \qquad \forall \alpha, \beta \in V, \ \forall c \in F$$

Ex For any vector space V,
the identity transformation $I: \alpha \mapsto \alpha$
the zero transformation $0: \alpha \mapsto 0$

— check both are linear

Ex Let V be the space of polynomials over field F,
commonly denoted $F[x]$.

The differentiation transformation D takes derivatives:
for $f(x) = c_0 + c_1 x + \cdots + c_n x^n$,
$(Df)(x) = c_1 + 2c_2 x + \cdots + n c_n x^{n-1}$

— check linear

Ex Let A be an $m \times n$ matrix,
$T: F^{n\times 1} \to F^{m \times 1}$ by $T\alpha = A\alpha$

— check linear.

Ex Let V = vector space of cont' functions $\mathbb{R} \to \mathbb{R}$.
Define $(Tf)(x) = \int_0^x f(t)\,dt$

— check linear

Note $T(0)=0$:

$$T(0) = T(0+0) = T(0) + T(0)$$

Also $T(a-a) = T(0)$
$= T(a) - T(a)$
$= 0$

So $T: x \mapsto ax+b$ for $b \neq 0$

is not linear!

$\rightsquigarrow$ so watch out, this notion of linear maybe slightly counterintuitive

Also note linear trans' preserve linear combinations:

$$T(c_1\alpha_1 + \cdots + c_n\alpha_n) = c_1(T\alpha_1) + \cdots + c_n(T\alpha_n)$$

Thm Let V be a finite-dim'l vector space,
let $\{\alpha_1, \cdots \alpha_n\}$ be an ordered basis for V.
Let W be a vector space over the same field,
& let $\beta_1, \cdots \beta_n$ be <u>any</u> vectors in W.

Then there is precisely one linear transformation $T : V \to W$
st $T\alpha_i = \beta_i \quad \forall i$

Since $\{\alpha_1, \cdots \alpha_n\}$ form a basis,
any vector $\alpha \in V$ can be written $\alpha = c_1\alpha_1 + \cdots + c_n\alpha_n$.

<u>Define</u> $T\alpha = c_1\beta_1 + \cdots + c_n\beta_n$

$\rightsquigarrow$ check linear.

Furthermore, if $U : V \to W$ is any other lin' trans' st $U\alpha_i = \beta_i$,
then, for $\alpha = c_1\alpha_1 + \cdots + c_n\alpha_n$,

$$U\alpha = c_1\beta_1 + \cdots + c_n\beta_n \quad \text{by linearity}$$
$$= T\alpha$$

so $U = T$,
& the lin' trans' is unique.

Ex The vectors $\alpha_1 = (1,2)$, $\alpha_2 = (8,7)$ form a basis for $\mathbb{R}^2$.

(Check: why LI? $\rightsquigarrow$ b/c not proportional)

$\exists!$ lin' trans $\mathbb{R}^2 \to \mathbb{R}^3$ s.t. $T\alpha_1 = (1,2,3)$
$T\alpha_2 = (0,0,1)$

What is $T(1,0)$?

1^{st}, write $(1,0) = c_1(1,2) + c_2(8,7)$

$$\Rightarrow c_1 + 8c_2 = 1, \quad 2c_1 + 7c_2 = 0$$
$$\Rightarrow c_2 = -\frac{2}{7}c_1$$
$$\Rightarrow c_1 - \frac{16}{7}c_1 = 1 \Rightarrow c_1 = -\frac{7}{9}, \quad c_2 = +\frac{2}{9}$$

so $T(1,0) = -\frac{2}{9}(1,2,3) + \frac{2}{9}(0,0,1)$

Ex Recall T !'ly determined by images of a basis
— so take standard basis.

Define $\beta_i = T\varepsilon_i$, $\quad \varepsilon_i = (0, \dots 1, \dots 0)$
↳ i^{th} position

Then T can be represented by a matrix:

$$T(c_1\varepsilon_1 + \dots + c_n\varepsilon_n) = c_1\beta_1 + \dots + c_n\beta_n$$

becomes

$$T\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \underbrace{\begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{bmatrix}}_{\sim T}\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$\rightsquigarrow$ will come back to this later.

If $T: V \to W$ is a linear transformation,
then, $\operatorname{im} T$ is a <u>subspace</u> of W: (called <u>range</u>)

s'pose $\alpha, \beta \in \operatorname{im} T$, so $\exists$ $a, b \in V$ st $\alpha = Ta$, $\beta = Tb$.
then $c\alpha + \beta = cTa + Tb = T(ca + b) \in \operatorname{im} T$
& nonempty b/c $T(0) = 0$.

The <u>null space</u> of a linear trans' $T: V \to W$
is the set of vectors $\alpha \in V$ s.t. $T\alpha = 0$.

Claim the null space is a subspace of V:

- nonempty since $T(0) = 0$
- if $\alpha, \beta \in$ null space,
  then $c\alpha + \beta \in$ null space since $T(c\alpha + \beta) = 0$.

<u>Define</u> <u>rank</u> of a linear tran' T = dim range

<u>nullity</u> " " " " " = dim null space.

<u>Thm</u> Let V, W be vector spaces over field F and let $T : V \to W$ be a linear transformation. S'pose V is finite-dim'l.
<u>Then</u>

$$\text{rank}(T) + \text{nullity}(T) = \dim V$$

<u>Pf</u> Let $\{\alpha_1, \cdots \alpha_k\}$ be a basis for the null space of T.
There are vectors $\alpha_{k+1}, \cdots \alpha_n$ s.t. $\{\alpha_1, \cdots \alpha_n\}$ is a basis for V.
<u>Claim</u> $\{T\alpha_{k+1}, \cdots T\alpha_n\}$ is a basis for the range of T.

Certainly $T\alpha_1, \cdots T\alpha_n$ span the range of T,
& since $T\alpha_j = 0$ for $j \leq k$, $T\alpha_{k+1}, \cdots T\alpha_n$ span the range of T.
Now, show LI.

S'pose $c_{k+1} T\alpha_{k+1} + \cdots + c_n T\alpha_n = 0$

$$\Rightarrow T(c_{k+1}\alpha_{k+1} + \cdots + c_n \alpha_n) = 0$$

$$\Rightarrow c_{k+1}\alpha_{k+1} + \cdots + c_n\alpha_n \in \text{null space of } T.$$

Since $\alpha_1, \cdots \alpha_k$ form a basis for the null space,
$\exists$ b_i s.t.

$$c_{k+1}\alpha_{k+1} + \cdots + c_n\alpha_n = b_1\alpha_1 + \cdots + b_k\alpha_k$$

$$\Rightarrow b_1\alpha_1 + \cdots + b_k\alpha_k - c_{k+1}\alpha_{k+1} - \cdots - c_n\alpha_n = 0$$

But the α_i are LI $\Rightarrow$ <u>$b_i = c_j = 0$</u>

$\Rightarrow T\alpha_{k+1}, \cdots T\alpha_n$ are LI, hence form a basis.

---

start here Mon

Thm Let A be an $m \times n$ matrix.
Then: row rank(A) = column rank(A).

Pf

Let T be the linear transformation $F^{n\times 1} \to F^{m\times 1}$ defined by $T(X) = AX$.

Null space of $T = \{X \mid AX = 0\}$.

If $A_1, \dots A_n$ are the columns of A, then $AX = x_1A_1 + \dots + x_nA_n$.

$\Rightarrow$ range T = column space of A
$\Rightarrow$ rank T = column rank A

(see p. 42 then → 72)

~~Now, consider putting A in row-echelon form.~~
~~row rank (A) + (# zero rows) =~~

Let S be the solution space of the system $\{AX=0\}$. (= null space)

~~row rank = #~~

Put A in row reduced echelon form, call it R.
(Same sol'n space S.)
row rank = # nontrivial equ'ns (others identically zero)
n = # unknowns

so, $\dim S = n -$ row rank(A)

Also, since S = null space of T,

$\underbrace{\dim S}_{=\text{nullity } T} + \text{rank } T = n$

so $\dim S = n - \text{rank } T = n -$ row rank A

Since rank T = column rank A,
we see

column rank A = row rank A

(§3.2 Algebra of linear transformations)

<u>Thm</u> Let V, W be vector spaces over a field.
Let T, U be linear transformations $V \to W$.

- The function $(T+U)$ defined by
$$(T+U)(\alpha) = \cancel{T\alpha + U\alpha}$$
$$= T\alpha + U\alpha$$
is a linear transformation $V \to W$.
- If c is a scalar, the function (cT) defined by $(cT)(\alpha) = c(T\alpha)$ is a linear transformation.
- The set of all <u>linear transformations $V \to W$ is a vector space</u>.

<u>Pf</u>

- Check $(T+U)(c\alpha+\beta) = c(T+U)(\alpha) + (T+U)(\beta)$
- Ditto for cT

· Vector space: most clear (outline some on board).
Zero vector = <u>zero transformation $0: \alpha \mapsto 0$</u>

<u>Notation</u> $L(V, W)$ = vector space of all linear trans' $V \to W$.

<u>Thm</u> Let V be an n-dim'l vector space over F,
& W an m-dim'l " " " "
Then the space $L(V,W)$ is finite-dim'l & of dimension mn.

<u>Pf</u>
Let $\mathcal{B} = \{\alpha_1, \cdots \alpha_n\}$, $\mathcal{B}' = \{\beta_1, \cdots \beta_m\}$
be ordered bases for V, W, resp.

For each pair of integers (p, q), $1 \le p \le m$, $1 \le q \le n$,
define a linear transformation $E^{p,q}: V \to W$ by

$$E^{p,q}(\alpha_i) = \delta_{iq} \beta_p$$

(There is ! lin' trans' w/ this property.)

Claim the $E^{p,q}$ form a basis for $L(V,W)$.

Check span:
Let $T: V \to W$ be a linear transformation.
For each j, $1 \le j \le n$, let $A_{1j}, \cdots A_{mj}$ be the coord's of $T\alpha_j$ in basis $\mathcal{B}'$,
ie, $T\alpha_j = \sum_p A_{pj} \beta_p$

<u>Claim</u> $T = \sum_p \sum_q A_{pq} E^{p,q}$

Check: $\sum_p \sum_q A_{pq} E^{p,q}(\alpha_j) = \sum_{p,q} A_{pq} \delta_{jq} \beta_p = \sum_p A_{pj} \beta_p = T\alpha_j$

& so $T = \sum_{p,q} A_{pq} E^{p,q}$

$\Rightarrow E^{p,q}$ span $L(V,W$

Need to show LI:
Suppose $\sum_{p,q} A_{pq} E^{p,q} = 0$ transformation

$\Rightarrow \sum_{pq} A_{pq} E^{p,q}(\alpha_j) = 0 \ \forall j \Rightarrow \sum_p A_{pj} \beta_p = 0$

Since β's are LI, $\Rightarrow A_{pj} = 0 \ \forall p, j$.

Thm Let V, W, Z be vector spaces over a field F.
Let T be a linear transformation $V \to W$ $U: W \to Z$ a lin' trans'.
Define the composition $(UT)(\alpha) = U(T(\alpha))$.
Then UT is a linear trans' $V \to Z$.

– check on board.

Def'n A lin' trans' $V \to V$ is called a linear operator on V.

Lemma Let V be a vector space over a field F,
let U, T_1, T_2 be linear operators on V, let $c \in F$.

- $IU = UI = U$
- $U(T_1 + T_2) = UT_1 + UT_2$
  $(T_1 + T_2)U = T_1 U + T_2 U$
- $c(UT_1) = (cU)T_1 = U(cT_1)$

Check part of this:

$$\begin{aligned} U(T_1 + T_2)(\alpha) &= U(T_1\alpha + T_2\alpha) && \text{by def'n of } + \\ &= U(T_1\alpha) + U(T_2\alpha) && \text{by linearity of } U \\ &= (UT_1)(\alpha) + (UT_2)(\alpha) && \text{by def'n of product.} \end{aligned}$$

& so forth.

Claim composition of linear transformations $\leftrightarrow$ matrix multiplication

Pf

Let V be a vector space, $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ a basis,
consider the linear operators

$$E^{p,q}(\alpha_i) = \delta_{iq}\alpha_p$$

$\rightsquigarrow$ form a basis for $L(V,V)$

$$(E^{p,q}E^{r,s})(\alpha_i) = E^{p,q}(\delta_{is}\alpha_r) = \delta_{is}\delta_{rq}\alpha_p$$

$$\Rightarrow E^{p,q}E^{r,s} = \begin{cases} 0 & r \neq q \\ E^{p,s} & r = q \end{cases}$$

a linear operator $T = \sum_{p,q} A_{pq} E^{p,q}$ for some A_{pq}

$U = \sum_{rs} B_{rs} E^{r,s}$ for some $B_{r,s}$,

then

$$TU = \sum_{pqrs} A_{pq} B_{rs} E^{p,q}E^{r,s}$$

$$= \sum_{prs} A_{pr} B_{rs} E^{p,s}$$

$$= \sum_{ps} \underbrace{\left(\sum_r A_{pr} B_{rs}\right)}_{\text{product of matrices}} E^{p,s}$$

A linear function $T: V \to W$ is invertible
if there exists a ~~~~ function $U: W \to V$ s.t. $UT = id_V$
$TU = id_W$

If T is invertible, its inverse is unique & labelled T^{-1}!

T is invertible iff
- T is 1-1, ie, $T\alpha = T\beta \Rightarrow \alpha = \beta$
- T is onto, ie, range of T = all of W

Thm If T is invertible, then T^{-1} is a linear transformation $W \to V$.

Pf Let $\beta_1, \beta_2 \in W$, $c \in F$
Claim $T^{-1}(c\beta_1 + \beta_2) = cT^{-1}\beta_1 + T^{-1}\beta_2$.

$$T(cT^{-1}\beta_1 + T^{-1}\beta_2) = cTT^{-1}\beta_1 + TT^{-1}\beta_2 \quad \text{since } T \text{ linear}$$
$$= c\beta_1 + \beta_2$$

since $T(cT^{-1}\beta_1 + T^{-1}\beta_2) = c\beta_1 + \beta_2$,
it follows that
$$T^{-1}(c\beta_1 + \beta_2) = cT^{-1}\beta_1 + T^{-1}\beta_2.$$

Note $(UT)^{-1} = T^{-1}U^{-1}$

(check)

Call a linear transformation T nonsingular if $T\gamma = 0 \Rightarrow \gamma = 0$
($\Leftrightarrow$ null space $= \{0\}$)
($\Leftrightarrow$ 1-1)

Thm Let T be a linear transformation $V \to W$.
Then T is nonsingular iff T maps each LI subset of V onto a LI subset of W.

Pf
$\Rightarrow$: S'pose T is nonsingular.
Let S be a LI subset of V.
Let $\alpha_1, \ldots, \alpha_k$ be vectors in S.
Note $c_1 T\alpha_1 + \cdots + c_k T\alpha_n = 0$

$= T(c_1\alpha_1 + \cdots + c_n\alpha_n)$
$\Rightarrow c_1\alpha_1 + \cdots + c_n\alpha_n = 0$ since T is nonsingular
$\Rightarrow c_1 = \cdots = c_n = 0$ since α_i are LI
$\Rightarrow T\alpha_1, \ldots, T\alpha_n$ are LI

$\Leftarrow$: S'pose T maps LI subsets to LI subsets.
If $\alpha \neq 0$, then $\{\alpha\}$ is a LI subset,
so $T\alpha \neq 0$ (since $\{0\}$ is ~~LI~~)
Hence null space $= \{0\}$
so T is nonsingular.

Thm Let V, W be finite-dim'l vector spaces s.t. $\dim V = \dim W$.
If $T: V \to W$ is a linear transformation,
then FAE:

i) T is invertible
ii) T is nonsingular (1-1)
iii) T is onto

p 81 Pf Let $n = \dim V = \dim W$.
We know $\operatorname{rank} T + \operatorname{nullity} T = n$.

T is nonsingular $\Leftrightarrow$ $\operatorname{nullity} T = 0$ $\Leftrightarrow$ $\operatorname{rank} T = n$ $\Leftrightarrow$ T is onto

T is invertible $\Leftrightarrow$ nonsingular & onto.

Start here Wed

(§3.3 Isomorphism)

If V, W are vector spaces over a fixed field F, then any one-to-one linear transformation of V onto W is called an isomorphism of V onto W.

If $\exists$ isomorphism $V \to W$, call them isomorphic.

Ex V isomorphic to itself (identity op)
If V iso to W, via T,
then T^{-1} exists & defines iso W to V.

Thm Every n-dim'l vector space over a field F is isomorphic to F^n.

Pf Let V be an n-dim'l v.s., $\mathcal{B} = \{\alpha_1, \cdots \alpha_n\}$ an ordered basis.
Define T as follows:
$$T(\alpha) = (x_1, \cdots, x_n) \text{ where } \quad \alpha = x_1\alpha_1 + \cdots + x_n\alpha_n.$$

$\rightsquigarrow$ check linear, onto, 1-1.

(§3.4 Representation of transformations by matrices)

Let V be an n-dim'l v.s.,
W " m-dim'l "

Let $\mathcal{B} = \{\alpha_1, \cdots \alpha_n\}$ be an ordered basis for V
$\mathcal{B}' = \{\beta_1, \cdots, \beta_m\}$ " " " " " W

If $T: V \to W$ is any linear transformation,
then T is determined by its action on the vectors α_i.

Write $T\alpha_j = \sum_{i=1}^{n} A_{ij} \beta_i$

If $\alpha = x_1 \alpha_1 + \cdots + x_n \alpha_n$, then

$$T\alpha = T\left(\sum_j x_j \alpha_j\right) = \sum_j x_j (T\alpha_j) = \sum_{j,i} x_j A_{ij} \beta_i$$
$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij} x_j\right) \beta_i$$

Thm For each linear transformation $T: V \to W$,
there is an $m \times n$ matrix A such that

$$[T\alpha]_{\mathcal{B}'} = A[\alpha]_{\mathcal{B}}$$

for every vector α in V.
Also, $T \to A$ is a 1-1 correspondence between the set of all linear transformations $V \to W$ & the set of all $m \times n$ matrices over the field F.

The matrix A called the matrix of T relative to the ordered bases $\mathcal{B}, \mathcal{B}'$.

In fact, there is an ~~...~~ isomorphism between $L(V, W)$ & space of all $m \times n$ matrices.

Notation: $[T]_{\mathcal{B}, \mathcal{B}'}$ is the matrix
(to emphasize dependence on bases)

Ex Let V be the space of poly's $\mathbb{R}\to\mathbb{R}$ of deg ≤ 3:

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

Consider the differentiation operator D, as a linear op $V \to V$.
w.r.t. the basis $\{1, x, x^2, x^3\}$,
compute the matrix representing D.

$$D(1) = 0$$
$$D(x) = 1$$
$$D(x^2) = 2x$$
$$D(x^3) = 3x^2$$

$$\Rightarrow \quad [D] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Check: Represent $f = c_0 + c_1 x + c_2 x^2 + c_3 x^3$
by the vector $\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$$[D]\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ 2c_2 \\ 3c_3 \\ 0 \end{bmatrix}$$

which corresponds to $c_1 + 2c_2 x + 3c_3 x^2$

$$= D(c_0 + c_1 x + c_2 x^2 + c_3 x^3) \checkmark$$

Composition of linear transformations
$\leftrightarrow$ product of matrices

As previously discussed,

if $A = [T]_{\mathcal{B}\mathcal{B}'}$, $B = [U]_{\mathcal{B}'\mathcal{B}''}$

then $[UT]_{\mathcal{B}\mathcal{B}''} = BA$

Change of basis:

For simplicity, let $T: V \to V$ (rather than W)

Let $\mathcal{B} = \{\alpha_1, \dots \alpha_n\}$, $\mathcal{B}' = \{\alpha_1', \dots \alpha_n'\}$
be two ordered bases for V.

How are $[T]_{\mathcal{B}}$, $[T]_{\mathcal{B}'}$ related?

As we saw (in ch 2),
$\exists!$ invertible $n \times n$ matrix P s.t. $[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'}$

(Specifically, it's the matrix $P = [P_1, \dots P_n]$
where $P_j = [\alpha_j']_{\mathcal{B}}$.)

so: $[T\alpha]_{\mathcal{B}} = [T]_{\mathcal{B}} [\alpha]_{\mathcal{B}}$

also $= P[T\alpha]_{\mathcal{B}'}$

$$\Rightarrow [T]_{\mathcal{B}} P[\alpha]_{\mathcal{B}'} = P[T\alpha]_{\mathcal{B}'} = P[T]_{\mathcal{B}'} [\alpha]_{\mathcal{B}'}$$

$$\Rightarrow [T]_{\mathcal{B}} P = P[T]_{\mathcal{B}'}$$

$$\Rightarrow [T]_{\mathcal{B}'} = P^{-1} [T]_{\mathcal{B}} P$$

This is how change-of-basis acts on matrices representing linear transformations.

Ex S'pose T expressed wrt std basis $\mathcal{B} = \{e_1, e_2\}$ for $\mathbb{R}^2$, has the form

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Express T in the basis $\mathcal{B}' = \{e_1 + e_2,\ e_1 - e_2\}$.

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \text{(columns are the basis elements).}$$

$$P^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

so from previous analysis,

$$[T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}}P = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}$$

When to go back to A&W?

<u>Def'n</u> Let A, B be $n \times n$ (square) matrices.
We say B is <u>similar</u> to A
if there is an invertible $n \times n$ matrix P s.t. $B = P^{-1}AP$.