1.5.2 Verify the expansion of the triple vector product

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$$

$$B \times C = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{bmatrix}$$

$$= \hat{x}(B_y C_z - B_z C_y) - \hat{y}(B_x C_z - B_z C_x) + \hat{z}(B_x C_y - B_y C_x)$$

$$A \times (B \times C) = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_y C_z - B_z C_y & B_z C_x - B_x C_z & B_x C_y - B_y C_x \end{bmatrix}$$

$$= \hat{x}\left[A_y(B_x C_y - B_y C_x) - A_z(B_z C_x - B_x C_z)\right]$$
$$- \hat{y}\left[A_x(B_x C_y - B_y C_x) - A_z(B_y C_z - B_z C_y)\right]$$
$$+ \hat{z}\left[A_x(B_z C_x - B_x C_z) - A_y(B_y C_z - B_z C_y)\right]$$

$$= \hat{x}\left[B_x(A \cdot C - \cancel{A_x B_x}) - C_x(A \cdot B - \cancel{A_x B_x})\right]$$
$$- \hat{y}\left[-B_y(A \cdot C - \cancel{A_y B_y}) + C_y(A \cdot B - \cancel{A_y B_y})\right]$$
$$+ \hat{z}\left[B_z(A \cdot C - \cancel{A_z B_z}) - C_z(A \cdot B - \cancel{A_z B_z})\right]$$

$$= B(A \cdot C) - C(A \cdot B)$$

1.6 e) If $S(x, y, z) = (x^2+y^2+z^2)^{-3/2}$, find

a) ∇S at the point $(1, 2, 3)$

$$\nabla S = -\tfrac{3}{2}(x^2+y^2+z^2)^{-5/2} \langle 2x, 2y, 2z \rangle$$

$$= -3(x^2+y^2+z^2)^{-5/2} \langle x, y, z \rangle$$

$$\nabla S(1,2,3) = -3(1+4+9)^{-5/2} \langle 1, 2, 3 \rangle = \underline{-3(14)^{-5/2} \langle 1, 2, 3 \rangle}$$

b) The magnitude of the gradient of S, $|\nabla S|$, at $(1, 2, 3)$.

$$|\nabla S(1,2,3)| = +3(14)^{-5/2}\sqrt{14}$$

$$= \underline{+3(14)^{-2}} = +\frac{3}{196}$$

c) the direction cosines of ∇S at $(1,2,3)$

$$\cos\alpha = \frac{\nabla S \cdot \hat{x}}{|\nabla S|} = \frac{-3(14)^{-5/2}}{+3(14)^{-2}} = -(14)^{-1/2}$$

$$\cos\beta = \frac{\nabla S \cdot \hat{y}}{|\nabla S|} = \frac{-3(14)^{-5/2}(2)}{+3(14)^{-2}} = -2(14)^{-1/2}$$

$$\cos\gamma = \frac{\nabla S \cdot \hat{z}}{|\nabla S|} = \frac{-3(14)^{-5/2}(3)}{+3(14)^{-2}} = -3(14)^{-1/2}$$

1.6.2 a) Find a unit vector perpendicular to the surface

$$x^2+y^2+z^2=3$$

at the point $(1,1,1)$.

A ~~perpe~~ perpendicular vector is defined by the gradient:

$$\bar{v} = \nabla g \quad \text{for} \quad g = x^2+y^2+z^2-3$$
$$= \langle 2x, 2y, 2z \rangle$$

@ $(1,1,1)$, $\bar{v} = \langle 2,2,2 \rangle$

Unit vector: $\dfrac{\bar{v}}{|\bar{v}|} = \dfrac{\langle 2,2,2\rangle}{2\sqrt{3}} = \dfrac{\langle 1,1,1\rangle}{\sqrt{3}}$

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b) Derive the equation of the plane tangent to the surface at $(1,1,1)$.

In general, the equation of a plane is $(\bar{r}-\bar{r}_0)\cdot\bar{n} = 0$

for $\bar{r}_0$ a vector to a point in the plane, & $\bar{n}$ is normal to the plane.

Here: $\bar{r} = \langle x, y, z\rangle$
$\bar{r}_0 = \langle 1,1,1\rangle$
$\bar{n} = \frac{1}{\sqrt{3}}\langle 1,1,1\rangle$ (from (a))

$$\Rightarrow \frac{1}{\sqrt{3}}(1)(x-1) + \frac{1}{\sqrt{3}}(1)(y-1) + \frac{1}{\sqrt{3}}(1)(z-1) = 0$$

$$\Rightarrow x-1+y-1+z-1 = 0$$

$$\Rightarrow x+y+z = 3$$

1.7.1 For a particle moving in a circular orbit $\bar{r} = \hat{x} r \cos\omega t + \hat{y} r \sin\omega t$

(r, ω constant)

a) evaluate $\bar{r} \times \dot{\bar{r}}$

$$\dot{\bar{r}} = \hat{x} r (-\omega) \sin\omega t + \hat{y} r (\omega) \cos\omega t$$

$$\bar{r} \times \dot{\bar{r}} = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ r\cos\omega t & r\sin\omega t & 0 \\ -\omega r \sin\omega t & \omega r \cos\omega t & 0 \end{bmatrix}$$

$$= \hat{z} (\omega r^2)(\cos^2\omega t + \sin^2\omega t)$$

$$= \hat{z}\, \omega r^2$$

---

b) Show that $\ddot{\bar{r}} + \omega^2 \bar{r} = 0$

$$\ddot{\bar{r}} = \hat{x} r (-\omega)(\omega) \cos\omega t \quad + \hat{y} r (\omega)(-\omega) \sin\omega t$$

$$= -\omega^2 (+\hat{x} r \cos\omega t + \hat{y} r \sin\omega t)$$

$$= -\omega^2 \bar{r} \quad \checkmark$$

1.7.5 Show $\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$

$$A \times B = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix}$$

$$= \hat{x}(A_y B_z - B_y A_z) - \hat{y}(A_x B_z - A_z B_x) + \hat{z}(A_x B_y - A_y B_x)$$

$$\nabla \cdot (A \times B) = \frac{\partial}{\partial x}(A_y B_z - B_y A_z) - \frac{\partial}{\partial y}(A_x B_z - A_z B_x) + \frac{\partial}{\partial z}(A_x B_y - A_y B_x)$$

$$= B_z \frac{\partial}{\partial x} A_y - B_y \frac{\partial}{\partial x} A_z - B_z \frac{\partial}{\partial y} A_x + B_x \frac{\partial}{\partial y} A_z + B_y \frac{\partial}{\partial z} A_x - B_x \frac{\partial}{\partial z} A_y$$
$$+ A_y \frac{\partial}{\partial x} B_z - A_z \frac{\partial}{\partial x} B_y - A_x \frac{\partial}{\partial y} B_z + A_z \frac{\partial}{\partial y} B_x + A_x \frac{\partial}{\partial z} B_y - A_y \frac{\partial}{\partial z} B_x$$

$$= B_x\left(\frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y\right) + B_y\left(\frac{\partial}{\partial z} A_x - \frac{\partial}{\partial x} A_z\right) + B_z\left(\frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x\right)$$
$$+ A_x\left(\frac{\partial}{\partial z} B_y - \frac{\partial}{\partial y} B_z\right) + A_y\left(\frac{\partial}{\partial x} B_z - \frac{\partial}{\partial z} B_x\right) + A_z\left(\frac{\partial}{\partial y} B_x - \frac{\partial}{\partial x} B_y\right)$$

$$= B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$$

1.8.11 Verify the vector identity

$$\nabla \times (A \times B) = (B \cdot \nabla) A - (A \cdot \nabla) B - B(\nabla \cdot A) + A(\nabla \cdot B)$$

$$A \times B = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix}$$

$$= \hat{x}(A_y B_z - A_z B_y) - \hat{y}(A_x B_z - A_z B_x) + \hat{z}(A_x B_y - A_y B_x)$$

$$\nabla \times (A \times B) = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \partial/\partial z \\ A_y B_z - A_z B_y & A_z B_x - A_x B_z & A_x B_y - A_y B_x \end{bmatrix}$$

$$= \hat{x}\left(\frac{\partial}{\partial y}(A_x B_y - A_y B_x) - \frac{\partial}{\partial z}(A_z B_x - A_x B_z)\right)$$
$$- \hat{y}\left(\frac{\partial}{\partial x}(A_x B_y - A_y B_x) - \frac{\partial}{\partial z}(A_y B_z - A_z B_y)\right)$$
$$+ \hat{z}\left(\frac{\partial}{\partial x}(A_z B_x - A_x B_z) - \frac{\partial}{\partial y}(A_y B_z - A_z B_y)\right)$$

$$= \hat{x}\Big(B_y \frac{\partial}{\partial y} A_x + B_z \frac{\partial}{\partial z} A_x - A_y \frac{\partial}{\partial y} B_x - A_z \frac{\partial}{\partial z} B_x$$
$$+ A_x \frac{\partial}{\partial y} B_y - B_x \frac{\partial}{\partial y} A_y - B_x \frac{\partial}{\partial z} A_z + A_x \frac{\partial}{\partial z} B_z \Big)$$
$$+ \hat{y}\Big(B_x \frac{\partial}{\partial x} A_y + B_z \frac{\partial}{\partial z} A_y - A_x \frac{\partial}{\partial x} B_y - A_z \frac{\partial}{\partial z} B_y$$
$$+ A_y \frac{\partial}{\partial x} B_x + A_y \frac{\partial}{\partial z} B_z - B_y \frac{\partial}{\partial x} A_x - B_y \frac{\partial}{\partial z} A_z \Big)$$
$$+ \hat{z}\Big(B_x \frac{\partial}{\partial x} A_z + B_y \frac{\partial}{\partial y} A_z - A_x \frac{\partial}{\partial x} B_z - A_y \frac{\partial}{\partial y} B_z$$
$$+ A_z \frac{\partial}{\partial x} B_x + A_z \frac{\partial}{\partial y} B_y - B_z \frac{\partial}{\partial x} A_x - B_z \frac{\partial}{\partial y} A_y \Big)$$

(cont'd)

(cont'd)

$$\hat{x}\text{ component} = (B\cdot\nabla)A_x - B_x\frac{\partial}{\partial x}A_x - (A\cdot\nabla)B_x + A_x\frac{\partial}{\partial x}B_x$$
$$+ A_x\frac{\partial}{\partial y}B_y - B_x\frac{\partial}{\partial y}A_y - B_x\frac{\partial}{\partial z}A_z + A_x\frac{\partial}{\partial z}B_z$$

$$= (B\cdot\nabla)A_x - (A\cdot\nabla)B_x - (\nabla\cdot A)B_x + (\nabla\cdot B)A_x$$

$$\hat{y}\text{ component} = (B\cdot\nabla)A_y - B_y\frac{\partial}{\partial y}A_y - (A\cdot\nabla)B_y + A_y\frac{\partial}{\partial y}B_y$$
$$+ A_y\frac{\partial}{\partial x}B_x + A_y\frac{\partial}{\partial z}B_z - B_y\frac{\partial}{\partial x}A_x - B_y\frac{\partial}{\partial z}A_z$$
$$= (B\cdot\nabla)A_y - (A\cdot\nabla)B_y - (\nabla\cdot A)B_y + (\nabla\cdot B)A_y$$

$$\hat{z}\text{ component} = (B\cdot\nabla)A_z - B_z\frac{\partial}{\partial z}A_z - (A\cdot\nabla)B_z + A_z\frac{\partial}{\partial z}B_z$$
$$+ A_z\frac{\partial}{\partial x}B_x + A_z\frac{\partial}{\partial y}B_y - B_z\frac{\partial}{\partial x}A_x - B_z\frac{\partial}{\partial y}A_y$$
$$= (B\cdot\nabla)A_z - (A\cdot\nabla)B_z - (\nabla\cdot A)B_z + (\nabla\cdot B)A_z$$

$$\Rightarrow \nabla\times(A\times B) = (B\cdot\nabla)A - (A\cdot\nabla)B - (\nabla\cdot A)B + (\nabla\cdot B)A$$