Physics 5714 – Problem set 10

1. Find the value of the integral

$$\int_C \frac{3z^3 + 2}{(z-1)(z^2+9)} dz$$

taken counterclockwise around the circles below:

   (a) $|z - 2| = 2$

   (b) $|z| = 4$

2. Let C be the circle $|z| = 2$, described in the positive sense, and evaluate the integral

   (a)

$$\int_C \tan z \, dz$$

   (b)

$$\int_C \frac{dz}{\sinh 2z}$$

   (c)

$$\int_C \frac{\cosh \pi z}{z(z^2+1)} dz$$

3. Compute the following improper integrals with residues:

   (a)

$$\int_0^\infty \frac{dx}{x^2+1}$$

   (b)

$$\int_0^\infty \frac{dx}{(x^2+1)^2}$$

4. Compute

$$\int_{-\infty}^\infty \frac{\cos x dx}{(x^2+a^2)(x^2+b^2)}, \quad (a > b > 0)$$

5. Compute

$$\int_{-\infty}^\infty \frac{x \sin ax}{x^4+4} dx, \quad (a > 0)$$

6. Compute

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2}$$

7. Compute

$$\int_{-\infty}^{\infty} \frac{\sin x dx}{x^2+4x+5}$$

8. Compute

$$\int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2\theta}$$

9. Compute

$$\int_{0}^{\pi} \frac{d\theta}{(a + \cos\theta)^2}, \quad (a > 1)$$

10. Show that

$$\int_{0}^{\infty} \frac{dx}{\sqrt{x}\,(x^2 + 1)} = \frac{\pi}{\sqrt{2}}$$

11. (a) By considering the integral of $\exp(iz^2)$ around the positively-oriented boundary of the sector $0 \le r \le R$, $0 \le \theta \le \pi/4$, show that

$$\int_0^R \exp(ix^2)dx = \exp(i\pi/4)\int_0^R \exp(-r^2)dr - \int_{C_R} \exp(iz^2)dz$$

where C_R is the arc $z = R\exp(i\theta)$, $(0 \le \theta \le \pi/4)$.

(b) Show that the integral above along C_R tends to zero as $R \to \infty$.

(c) Use the results in parts (a) and (b), together with the known integration formula

$$\int_0^{\infty} \exp(-x^2)dx = \frac{\sqrt{\pi}}{2}$$

to evaluate the Fresnel integrals

$$\int_0^{\infty} \cos(x^2)dx = \int_0^{\infty} \sin(x^2)dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

which are important in diffraction theory.