Physics 5714 – Problem set 2

(AWH = Arfken-Weber-Harris, 7th edition)

Note: my conventions (and those of the lecture notes 5714vc.pdf) differ slightly from AWH. I'll be following my own conventions.

1. The velocity of a two-dimensional flow of liquid is given by

$$\vec{V} = \hat{x}u(x,y) - \hat{y}v(x,y)$$

   If the liquid is incompressible and the flow is irrotational, show that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

2. (AWH 3.6.10) Show that $\nabla \times (\varphi \nabla \varphi) = 0$.
3. (AWH 3.6.15) Show that any solution of the equation

$$\nabla \times (\nabla \times \vec{A}) - k^2 \vec{A} = 0$$

   automatically satisfies the vector Helmholtz equation

$$\nabla^2 \vec{A} + k^2 \vec{A} = 0$$

   and the solenoidal condition $\nabla \cdot \vec{A} = 0$. (Hint: let $\nabla\cdot$ operate on the first equation.)
4. Compute the line integral

$$\int_C (x^3 + y) ds$$

   for C the curve described by $x = 3t$, $y = t^3$, $t \in [0, 1]$.
5. Compute the line integral

$$\int_C (\sin x + \cos y) ds$$

   where C is the line segment from $(0, 0)$ to $(\pi, 2\pi)$.
6. Compute the line integral

$$\int_C (y dx + x dy)$$

   for C the curve $y = x^2$, $x \in [0, 1]$.

7. Compute the line integral

$$\int_C (xzdx \; + \; (y+z)dy \; + \; xdz)$$

for C the curve $x = \exp t$, $y = \exp(-t)$, $z = \exp(2t)$, $0 \le t \le 1$.

8. For the vector field

$$\vec{F} \; = \; -(\exp(-x))(\ln y)\hat{x} \; + \; (\exp(-x))(y^{-1})\hat{y}$$

show that it is conservative (meaning, $\nabla \times \vec{F} = 0$) and then find a function $f(x,y)$ such that $\vec{F} = \nabla f$.

9. For the vector field

$$\vec{F} \; = \; 3x^2\hat{x} \; + \; 6y^2\hat{y} \; + \; 9z^2\hat{z}$$

show that it is conservative (meaning, $\nabla \times \vec{F} = 0$) and then find a function $f(x,y)$ such that $\vec{F} = \nabla f$.

10. Show that the line integral

$$\int_{(-1,2)}^{(3,1)} \left[(y^2 \; + \; 2xy)dx \; + \; (x^2 \; + \; 2xy)dy \right]$$

is independent of path, and then use that fact to evaluate the line integral. (Hint: the last couple of problems are what I have in mind here.)

11. Show that if

$$\vec{F}(x,y,z) \; = \; g(x^2+y^2+z^2)(x\hat{x} \; + \; y\hat{y} \; + \; z\hat{z})$$

for some function $g(t)$, then $\vec{F}$ is conservative, meaning $\nabla \times \vec{F} = 0$. Hint: show that $\vec{F} = \nabla f$, $f(x,y,z) = \frac{1}{2}h(x^2+y^2+z^2)$, $h(u) = \int g(u)du$.