Physics 5714 – Problem set 3

Note: my conventions (and those of the lecture notes 5714vc.pdf) differ slightly from AWH. I'll follow my own conventions.

1. Use Green's theorem to evaluate the line integral

$$\oint_C (2xy dx \; + \; y^2 dy)$$

where C is the closed curve formed by $y = x/2$ and $y = \sqrt{x}$ between $(0, 0)$ and $(4, 2)$.

2. Use Green's theorem to evaluate the line integral

$$\oint_C (xy dx \; + \; (x + y) dy)$$

where C is the triangle with vertices $(0, 0)$, $(2, 0)$, $(0, 1)$.

3. Evaluate the surface integral

$$\int \int_G g(x, y, z) dS$$

for $g(x, y, z) = 2y^2 + z$, and where G is the surface defined by $z = x^2 - y^2$, $0 \leq x^2 + y^2 \leq 1$.

4. Let G be the sphere $x^2 + y^2 + z^2 = a^2$. Evaluate each of the following: (Hint: use symmetries to make each trivial.)

   (a)

$$\int \int_G z dS$$

   (b)

$$\int \int_G \frac{x \; + \; y^3 \; + \; \sin z}{1 + z^4} dS$$

   (c)

$$\int \int_G (x^2 \; + \; y^2 \; + \; z^2) dS$$

   (d)

$$\int \int_G x^2 dS$$

(e)

$$\int\int_G (x^2 + y^2) dS$$

5. Use Gauss's divergence theorem to compute

$$\int\int_{\partial S} \vec{F} \cdot \hat{n} dS$$

for $\vec{F} = \langle z, x, y \rangle$, and S the hemisphere $0 \le z \le \sqrt{9 - x^2 - y^2}$.

6. Let $\vec{F} = \langle x, y, z \rangle$ and let S be a solid for which Gauss's divergence theorem applies. Show the volume of S is given by

$$\text{Vol}(S) = \frac{1}{3} \int\int_{\partial S} \vec{F} \cdot \hat{n} dS$$

7. Use Stokes's theorem to compute

$$\int\int_S (\nabla \times \vec{F}) \cdot \vec{n} dS$$

where $\vec{F} = \langle xz^2, x^3, \cos(xz) \rangle$ and S is the part of the ellipsoid $x^2 + y^2 + 3z^2 = 1$ *below* the xy plane, $\vec{n}$ the lower normal.

8. Use Stokes's theorem to compute

$$\oint_C \vec{F} \cdot \vec{T} ds$$

for $\vec{F} = \langle 2z, x, 3y \rangle$, C the ellipse that is the intersection of the plane $z = x$ and the cylinder $x^2 + y^2 = 4$, oriented clockwise as seen from above.

9. Compute

$$\int_{-\infty}^{\infty} \left(\frac{d^2}{dx^2} \delta(x) \right) f(x) dx$$

10. If $\mathbf{C}$ is the field of complex numbers, which vectors in $\mathbf{C}^3$ are linear combinations of $(1, 0, -1)$, $(0, 1, 1)$, $(1, 1, 1)$?

11. Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Define

$$\begin{aligned} (x, y) + (x_1, y_1) &= (x + x_1, y + y_1) \\ c(x, y) &= (cx, y) \end{aligned}$$

Is V, with these operations, a vector space over the field of real numbers?

12. On $\mathbf{R}^n$, define the operations

$$\begin{aligned} \alpha \oplus \beta &= \alpha - \beta \\ c \cdot \alpha &= -c\alpha \end{aligned}$$

for $\alpha, \beta \in \mathbf{R}^n$ and c a scalar. Which of the axioms for a vector space are satisfied by $(\mathbf{R}^n, \oplus, \cdot)$?

13. Show that the only subspaces of $\mathbf{R}$ are $\mathbf{R}$ itself and the zero subspace.

14. Let V be the vector space of all functions from $\mathbf{R}$ to $\mathbf{R}$. Let V_e be the subset of even functions, $f(-x) = f(x)$. Let V_o be the subset of odd functions, $f(-x) = -f(x)$. Show that V_e, V_o are subspaces of V.