Physics 5714 – Problem set 8

1. (AWH 1.1.5a-c) Test the following for convergence:

   (a)
   $$\sum_{n=2}^{\infty} (\log n)^{-1}$$

   (b)
   $$\sum_{n=1}^{\infty} \frac{n!}{10^n}$$

   (c)
   $$\sum_{n=1}^{\infty} \frac{1}{(2n)(2n+1)}$$

2. (AWH 1.1.9) (Olbers' paradox) Assume a static universe in which the stars are uniformly distributed. Divide all space into shells of constant thickness; the stars in any one shell by themselves subtend a solid angle of ω_0. Allowing for the blocking out of distant stars by nearer stars, show that the total net solid angle subtended by all stars, shells extending to infinity, is exactly 4π. (Therefore the night sky should be ablaze with light.)

3. (AWH 1.1.15a) Show that
   $$\sum_{n=2}^{\infty} (\zeta(n) - 1) = 1$$
   where $\zeta(n)$ is the Riemann zeta function.

4. (AWH 1.3.10) The displacement x of a particle of rest mass m_0, resulting from a constant force $m_0 g$ along the x axis, is
   $$x = \frac{c^2}{g}\left\{\left[1 + \left(g\frac{t}{c}\right)^2\right]^{1/2} - 1\right\}$$
   including relativistic effects. Find the displacement x as a power series in time t. Compare with the classical result, $x = (1/2)gt^2$.

5. The functions $u(x, y)$, $v(x, y)$ are the real, imaginary parts of a holomorphic (equivalently, analytic) function $w(z)$.

(a) Assuming the required derivatives exist, show that

$$\nabla^2 u \;=\; \nabla^2 v \;=\; 0$$

Solutions of Laplace's equation such as $u(x, y)$, $v(x, y)$ are called *harmonic functions.*

(b) Show that

$$u_x u_y \;+\; v_x v_y \;=\; 0$$

6. (AWH 11.2.2) Having shown that the real part $u(x, y)$ and the imaginary part $v(x, y)$ of an analytic function $w(z)$ each satisfy Laplace's equation, show that $u(x, y)$, $v(x, y)$ cannot both have either a maximum or a minimum in the interior of any region in which $w(z)$ is analytic.

7. Show that

$$\exp(iz) \;=\; \cos z \;+\; i \sin z$$

for every complex number z.

8. For $z = x + iy$, show that

$$|\sin z| \;\geq\; |\sin x|$$

9. Find all roots of the equation $\cos z = 2$.

10. For a complex number z, define

$$\sinh z \;=\; \frac{1}{2}\left(\exp(z) \;-\; \exp(-z)\right), \quad \cosh z \;=\; \frac{1}{2}\left(\exp(z) \;+\; \exp(-z)\right)$$

Show that

$$\sinh(2z) \;=\; 2 \sinh z \cosh z$$

11. Show that

$$-i \sinh(iz) \;=\; \sin z, \quad \cosh(iz) \;=\; \cos z$$

12. For complex numbers z_1, z_2, show that

$$\begin{aligned} \sinh(z_1 + z_2) &= \sinh z_1 \cosh z_2 \;+\; \cosh z_1 \sinh z_2 \\ \cosh(z_1 + z_2) &= \cosh z_1 \cosh z_2 \;+\; \sinh z_1 \sinh z_2 \end{aligned}$$

13. For $z = x + iy$, show that

$$\begin{aligned} \sinh z &= \sinh x \cos y \;+\; i \cosh x \sin y \\ \cosh z &= \cosh x \cos y \;+\; i \sinh x \sin y \end{aligned}$$

14. For $z = x + iy$, show that

$$\begin{aligned} |\sinh z|^2 &= \sinh^2 x + \sin^2 y \\ |\cosh z|^2 &= \sinh^2 x + \cos^2 y \end{aligned}$$

15. Show that the holomorphic function

$$f_2(z) = \frac{1}{z^2 + 1} \quad (z \neq \pm i)$$

is the analytic continuation of the function

$$f_1(z) = \sum_{n=0}^{\infty} (-)^n z^{2n} \quad (|z| < 1)$$

into the domain consisting of all points in the z plane except $z = \pm i$.

16. Show that the function $f_2(z) = z^{-2}$ $(z \neq 0)$ is the analytic continuation of the function

$$f_1(z) = \sum_{n=0}^{\infty} (n+1)(z+1)^n \quad (|z+1| < 1)$$

into the domain consisting of all points in the z plane except $z = 0$.

17. Find the analytic continuation of the function

$$f(z) = \int_0^{\infty} t \exp(-zt) dt \quad (\text{Re } z > 0)$$

into the domain consisting of all points in the z plane except the origin.

18. Show that the function $(z^2+1)^{-1}$ is the analytic continuation of the function

$$f(z) = \int_0^{\infty} \exp(-zt)(\sin t)\, dt \quad (\text{Re } z > 0)$$

into the domain consisting of all points in the z plane except $z = \pm i$.

19. Rodrigues' formula for the Legendre polynomials $P_n(z)$ says that

$$P_n(z) = \frac{1}{2^n n!} \left(\frac{d}{dz}\right)^n \left(z^2 - 1\right)^n, \quad n = 0, 1, 2, \cdots$$

(a) Show that the Legendre polynomials can also be expressed as

$$P_n(z) = \frac{1}{2^{n+1}\pi i} \int_C \frac{(s^2 - 1)^n}{(s - z)^{n+1}} ds, \quad n = 0, 1, 2, \cdots$$

for some contour C enclosing z. This is known as the Schlaefli integral.

(b) Show that $P_n(1) = 1$ and $P_n(-1) = (-)^n$ for all n, using the Schlaefli integral representation of $P_n(z)$.

20. Describe a Riemann surface for the triple-valued function

$$w = (z - 1)^{1/3}$$

and point out which third of the w plane represents the image of each sheet of the surface.

21. Describe the curve, on a Riemann surface for $z^{1/2}$, whose image is the entire circle $|w| = 1$ under the transformation $w = z^{1/2}$.