Physics 5714 – Problem set 9

1. Describe a Riemann surface for the multiple-valued function

$$f(z) = \left(\frac{z-1}{z}\right)^{1/2}$$

2. Let C describe the positively-oriented circle $|z-2| = 1$ on the Riemann surface described for $z^{1/2}$, where the upper half of the circle lies on one sheet and the lower half on another. Note that, for each point z on C, one can write

$$z^{1/2} = \sqrt{r}\exp\left(\pm\frac{i\theta}{2}\right), \quad \text{where } 4\pi - \frac{\pi}{2} < \theta < 4\pi + \frac{\pi}{2}$$

State why it follows that

$$\int_C z^{1/2} dz = 0$$

Generalize this result to fit the case of the other simple closed curves that cross from one sheet to another without enclosing the branch point.

3. Let a function f be continuous in a closed bounded region R, and let it be analytic and not constant in the interior of R. Assuming that $f(z) \neq 0$ anywhere in R, show that $|f(z)|$ has a *minimum* value in R which occurs on the boundary and never in the interior. Hint: think about $g(z) = 1/f(z)$.

4. Suppose that $f(z)$ is entire and the harmonic function

$$u(x, y) = \text{Re } f(z)$$

has an upper bound, *i.e.*, $u(x, y) \leq u_0$ for all $z = x + iy$. Show that $u(x, y)$ must be constant. Hint: apply Liouville's theorem to $g(z) = \exp(f(z))$.

5. Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(z-1)}$$

and specify the regions in which they are valid.

6. Write two Laurent series expansions in powers of z that represent the function

$$f(z) = \frac{1}{z(1+z^2)}$$

in certain domains, and specify the domains.

7. Bessel functions.

(a) Let z be any complex number, and let C denote the unit circle $w = \exp(i\phi)$, $-\pi < \phi < \pi$, in the w plane. Then, using a Laurent series of the form

$$\sum_{n=-\infty}^{\infty} c_n w^n = f(w)$$

with

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{w^{n+1}} dw$$

evaluated along C above, show that

$$\exp\left(\frac{z}{2}\left(w - \frac{1}{w}\right)\right) = \sum_{n=-\infty}^{\infty} J_n(z) w^n, \quad 0 < |w| < \infty$$

where

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(-i\left[n\phi - z\sin\phi\right]\right) d\phi, \quad n \text{ an integer}$$

$J_n(z)$ is called the *nth Bessel function of the first kind*, and $\exp((z/2)(w - 1/w))$ is known as the generating functional for the Bessel functions of the first kind.

(b) Show that

$$J_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos\left(n\phi - z\sin\phi\right) d\phi$$

Hint: this has nothing to do with Laurent series, but rather is merely a problem in symmetry properties of integrals.

(c) From the expression

$$\exp\left(\frac{z}{2}\left(w - \frac{1}{w}\right)\right) = \sum_{n=-\infty}^{\infty} J_n(z) w^n, \quad 0 < |w| < \infty$$

show that for $n \geq 0$,

$$J_n(z) = \sum_{s=0}^{\infty} \frac{(-)^s}{s!(n+s)!} \left(\frac{z}{2}\right)^{n+2s}$$

Hint: this also has nothing to do with Laurent series per se, though it is an exercise in infinite series.

8. In each case, write the principal part of the function at its isolated singular point and determine whether that point is a pole, an essential singular point, or a removable singular point.

(a) $z\exp(1/z)$

(b) $z^2/(1+z)$

(c) $(\sin z)/z$

(d) $(\cos z)/z$

(e) $(2-z)^{-3}$

9. Show that the singular point of each of the following functions is a pole. Determine the order of that pole and the corresponding residue.

   (a) $z^{-3}(1-\cosh z)$

   (b) $z^{-4}(1-\exp(2z))$

   (c) $(z-1)^{-2}\exp(2z)$

10. Find the residue at $z=0$ of the following functions:

    (a) $(z+z^2)^{-1}$

    (b) $z\cos(1/z)$

    (c) $z^{-1}(z-\sin z)$

    (d) $z^{-4}\cot z$

    (e) $z^{-4}(1-z^2)^{-1}\sinh z$

11. Use residues to evaluate the integrals of the following functions about the circle $\{|z|=3\}$ oriented positively:

    (a) $z^{-2}\exp(-z)$

    (b) $z^{-2}\exp(1/z)$

    (c) $(z^2-2z)^{-1}(z+1)$

12. Let $f(z)$ be a function which is analytic at z_0.

    (a) Show that if $f(z_0)=0$, then z_0 is a removable singular point of the function

    $$g(z) = \frac{f(z)}{z-z_0}$$

    (b) Show that if $f(z_0)\neq 0$, then z_0 is a simple pole of the function $g(z)$ above, with residue $f(z_0)$.

13. In each case, show that the singular points of the function are poles. Determine the order of each pole, and find the corresponding residue.

    (a) $(z^2+z)/(z-1)$

    (b) $(z/(2z+1))^3$

(c) $\coth z = \cosh z / \sinh z$

(d) $(\exp z)/(z^2 + \pi^2)$

(e) $z/\cos z$

(f) $z^{1/4}/(z+1)$