Another approach to meaning of holomorphicity:

It's a condition for $\frac{\partial}{\partial z}$ to exist.

$$\frac{\partial f}{\partial z} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{z + \Delta z - z}$$

In order for this limit to exist, it must be path-independent.

An ex where a limit doesn't exist: $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$. Along the line $y = mx$, $\lim = \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}$ not path ind!

Write $\Delta z = \Delta x + i \Delta y$

S'pose $\Delta y = 0$.

Then, $$\frac{\partial f}{\partial z} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$
$$= \frac{\partial f}{\partial x} = u_x + i v_x \quad \text{where } f = u + iv$$

S'pose $\Delta x = 0$

Then, $$\frac{\partial f}{\partial z} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{i \Delta y}$$
$$= -i \frac{\partial f}{\partial y} = -i u_y + v_y$$

Require $u_x + i v_x = v_y - i u_y$

$$\Rightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \rightarrow \text{the Cauchy-Riemann equ's.}$$

~~[illegible]~~

Exs Trig functions

x real $\quad e^x = 1 + x + \frac{x^2}{2!} \cdots \quad \sin x = x - \frac{x^3}{3!} \cdots \quad \cos x = 1 - \frac{x^2}{2!} \cdots$

z complex $\quad e^z = 1 + z + \frac{z^2}{2!} \cdots \quad \sin z = z - \frac{z^3}{3!} \cdots \quad \cos z = 1 - \frac{z^2}{2!} \cdots$

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}), \qquad \cos x = \frac{1}{2}(e^{ix} + e^{-ix})$$

Sim'ly: $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}), \quad \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$

(check)

Properties: $\sin(-z) = -\sin z, \qquad \cos(-z) = +\cos z$

y real $\quad \sin(iy) = \frac{1}{2i}(e^{-y} - e^{+y}) = \frac{i}{2}(e^{y} - e^{-y})$

$$= i \sinh y$$

$$\cos(iy) = \frac{1}{2}(e^{-y} + e^{+y}) = \cosh y$$

$$\frac{\partial}{\partial z} \sin z = \cos z, \qquad \frac{\partial}{\partial z} \cos z = -\sin z$$

e^{iz}, e^{-iz} are entire, so, since $\sin z$ & $\cos z$ are linear comb's, $\sin z$ & $\cos z$ are entire too.

Trig f'ns, cont'd

Expected identities hold

Ex $2\sin z_1 \cos z_2 = \sin(z_1+z_2) + \sin(z_1-z_2)$

Check

$$2\sin z_1 \cos z_2 = 2\left(\frac{e^{iz_1}-e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2}+e^{-iz_2}}{2}\right)$$

$$= \frac{1}{2i}\left[e^{i(z_1+z_2)} - e^{-i(z_1+z_2)} - e^{-i(z_1-z_2)} + e^{i(z_1-z_2)}\right]$$

$$= \sin(z_1+z_2) + \sin(z_1-z_2) \checkmark$$

Sim'ly:

$$\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

$$\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

$$\sin^2 z + \cos^2 z = 1$$

$$\sin 2z = 2\sin z\cos z, \quad \cos 2z = \cos^2 z - \sin^2 z$$

$$\sin\left(z+\frac{\pi}{2}\right) = \cos z$$

If take $z_1 = x$, $z_2 = iy$ above, x, y real, $z = x+iy$, then can read off

$$\sin z = \sin x \cosh y + i\cos x \sinh y$$

$$\cos z = \cos x \cosh y - i\sin x \sinh y$$

Sim'ly,
$\sin(z+2\pi) = \sin z \qquad \sin(z+\pi) = -\sin z$
$\cos(z+2\pi) = \cos z \qquad \cos(z+\pi) = -\cos z$

$$|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$$

$$= \sin^2 x\,(1+\sinh^2 y) + \cos^2 x \sinh^2 y = \sin^2 x + \sinh^2 y$$

& sim'ly $|\cos z|^2 = \cos^2 x + \sinh^2 y$

Trig f'ns, cont'd

Zeroes:

$\sin z = 0 \Leftrightarrow z = n\pi, \quad n \in \mathbb{Z}$

$\cos z = 0 \Leftrightarrow z = (n+\frac{1}{2})\pi, \quad n \in \mathbb{Z}$

Verify 1st statement:

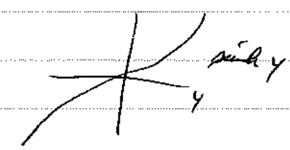
$\Leftarrow$ clear

$\Rightarrow$: $|\sin z|^2 = \sin^2 x + \sinh^2 y$

For this to be 0 $\Rightarrow \sin x = 0, \sinh y = 0$

But $\sinh y = 0 \Rightarrow y = 0$

$\sin x = 0 \Rightarrow x = n\pi, \quad n \in \mathbb{Z}$.



$\tan z = \dfrac{\sin z}{\cos z}$, $\cot z = \dfrac{\cos z}{\sin z}$

$\sec z = \dfrac{1}{\cos z}$, $\csc z = \dfrac{1}{\sin z}$

analytic everywhere
except where $\cos z = 0$

$\frac{d}{dz} \tan z = \sec^2 z$ $\qquad$ $\frac{d}{dz} \cot z = -\csc^2 z$

$\frac{d}{dz} \sec z = \sec z \tan z$ $\qquad$ $\frac{d}{dz} \csc z = -\csc z \cot z$

Analytic Continuation

See Arfken & Weber pp 432-434 (in section 6.5 on Laurent expansions) for some of the material below. Our description here will closely follow [1].

1 Definition

The *intersection* of two domains (regions in the complex plane) D_1, D_2, denoted $D_1 \cap D_2$, is the set of all points common to both D_1 and D_2. The *union* of two domains D_1, D_2, denoted $D_1 \cup D_2$, is the set of all points in either D_1 or D_2.

Now, suppose you have two domains D_1 and D_2, such that the intersection is nonempty and connected, and a function f_1 that is analytic over the domain D_1. If there exists a function f_2 that is analytic over the domain D_2 and such that $f_1 = f_2$ on the intersection $D_1 \cap D_2$, then we say f_2 is an *analytic continuation* of f_1 into domain D_2.

Now, whenever an analytic continuation exists, it is unique. The reason for this is a basic mathematical result from the theory of complex variables:

A function that is analytic in a domain D is uniquely determined over D by its values over a domain, or along an arc, interior to D.

Define the function $F(z)$, analytic over the union $D_1 \cup D_2$, as

$$F(z) = \begin{cases} f_1(z) & \text{when } z \text{ is in } D_1 \\ f_2(z) & \text{when } z \text{ is in } D_2 \end{cases}$$

In other words, F is given by f_1 over D_1 and by f_2 over D_2, and since $f_1 = f_2$ over the intersection of D_1 and D_2, this is a well-defined, holomorphic function. By the mathematical result quoted above, since F is analytic in $D_1 \cup D_2$, it is uniquely determined by f_1 on D_1. (For that matter, it is also uniquely determined by f_2 on D_2.) In other words, there is only one possible holomorphic function on $D_1 \cup D_2$ that matches f_1 on D_1.

In this case, the function $F(z)$ is said to be the analytic continuation over $D_1 \cup D_2$ of either f_1 or f_2.

Example: Consider first the function

$$f_1(z) = \sum_{n=0}^{\infty} z^n$$

This power series converges when $|z| < 1$ to $1/(1-z)$, and is not defined when $|z| \geq 1$. (In particular, this is just a geometric series, so we can sum it as a geometric series, so long as we're in the region of convergence.)

Note series only diverges for $z \to 1$, merely ill-defined elsewhere on unit circle.

On the other hand, the function

$$f_2(z) = \frac{1}{1-z}$$

is defined and analytic everywhere except $z = 1$.

Since $f_1 = f_2$ on the disk $|z| < 1$, we can view f_2 as the analytic continuation of f_1 to the rest of the complex plane (minus the point $z = 1$).

Example: Consider the function

$$f_1(z) = \int_0^\infty \exp(-zt)dt$$

This integral exists only when Re $z > 0$, and for such z, this integral has value $1/z$.

Since the function $1/z$ matches f_1 on the domain Re $z > 0$, the function $1/z$ is the analytic continuation of f_1 to nonzero complex numbers.

While we're at it, define

$$f_2(z) = i\sum_{n=0}^{\infty}\left(\frac{z+i}{i}\right)^n$$

This series converges for $|z+i| < 1$, and so f_2 is defined only within that disk centered on $-i$. Within that unit disk, one can show that $f_2(z) = 1/z$, using the fact that the series is a geometric series.

Since f_2 matches $1/z$ on a disk, we can view $1/z$ as the analytic continuation of f_2 to nonzero complex numbers.

Also, we can view f_2 as the analytic continuation of f_1 to the disk $|z+i| < 1$.

Example: The Gamma function.

Recall the second definition of the Gamma function,

$$\Gamma(z) = \int_0^\infty \exp(-t)t^{z-1}dt$$

is valid for Re $z > 0$. Other definitions, such as the Weierstrass form

$$\frac{1}{\Gamma(z)} = z\exp(\gamma z)\prod_{n=1}^{\infty}\left(1 + \frac{z}{n}\right)\exp(-z/n)$$

are valid more generally. Thus, we can view the Weierstrass form as an analytic continuation of the Euler integral form.