Generalized Complete Intersection Calabi-Yau (gCICY)

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Work with: Lara B. Anderson, Fabio Apruzzi, James Gray, Seung-Joo Lee,

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Outline

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Why Calabi-Yau

From string to the real wold: 10D $\rightarrow$ 4D

What we want: $\mathcal{N}=1$ Supersymmetry with chiral spectrum

Best under control: $\mathcal{N}=1$ Flux Compactification

- Het string on CY_3
- Type IIA/B on CY_3 with orientifold (include Type I $\cong$ Type IIB orientifold with $O9$-plane)
- (Aux 12D) F-theory on CY_4
- (11D) M-theory on $CY_3 \times S^1/\mathbb{Z}_2$ or on $\mathcal{M}^7$ with G_2 holonomy
- ...

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$\Rightarrow$ Calabi-Yau threefold CY_3 or fourfold CY_4.

What is Calabi-Yau

Calabi-Yau n-folds is a complex n-dimentional compacted Kähler Manifold satisfied:

- $c_1(M) = 0 \in H^2(M, \mathbb{Z})$.
- $K_M = \wedge^n T^*(1,0)(M)$ is trivial since $c_1(K_M) = -c_1(M)$.
- Unique nowhere vanishing holomophic n-form, $\Omega_n \in \Omega^{n,0}(M), d\Omega_n = 0$
- The Ricci tensor vanish, i.e. $R_{mn} = 0$
- The holonomy group of M is $SU(n)$

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Calabi-Yau 3,4-folds

How to construct Calabi-Yau calculable

- Toric Calabi-Yau Borisov, Batyrev, Cox, Kreuzer, Skarke ......
  - Hypersurface $\hookrightarrow$ 473,800,776 reflexive polyhedra in 4D
    Kreuzer,Skarke,Altman,Gray,He,Jejjala,Nelson,...
  - Hypersurface $\hookrightarrow$ weighted project space Kreuzer,Skarke,...
- Complete Intersection Calabi-Yau (CICY)
  - Complete intersection hypersurfaces $\hookrightarrow$ Product of projective spaces
    7,890 configuration matrices for CY3
    Hubsch,Candelas,Dale,Lutaken,Schimmrigk,Green,...
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Generalized Complete Intersection Calabi-Yau Manifolds (gCICY)

CICY 3-folds

$$X = \left[\begin{array}{c||ccc} \mathbb{P}^2 & 1 & 1 & 1 \\ \mathbb{P}^4 & 3 & 1 & 1 \end{array} \right]$$

- $X \equiv X^1 \cap X^2 \cap X^3 \hookrightarrow \mathcal{A} \cong \mathbb{P}^2 \times \mathbb{P}^4$
- $X^a : p^a(\mathbf{x_1}, \mathbf{x_2}) = 0, \ a = 1, 2, 3. \ ||p^1|| = (1, 3), ||p^{2,3}|| = (1, 1).$
  $\mathbf{x_1} = (x_1^0 : x_1^2 : x_1^3), \quad \mathbf{x_2} = (x_2^0 : \cdots : x_2^5).$
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Generalized To

Drop: Positive semi-definite entries.

gCICY

$$X = \left[\begin{array}{c||cc|cc} \mathbb{P}^1 & 1 & 1 & -1 & 1 \\ \mathbb{P}^1 & 1 & 1 & 1 & -1 \\ \mathbb{P}^5 & 3 & 1 & 1 & 1 \end{array}\right]$$

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- Still complete intersection?
  $h^0(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^5, \mathcal{O}(1,-1,1)) = 0.$
  $X \hookrightarrow \mathcal{A}$ is not algebraic complete intersection.
- Calabi-Yau?
- Smooth?

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  ②: $h^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(1,-1,1)) = h^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(-1,1,1)) = 1$
  $\Rightarrow$ Polynomial description in $\mathcal{M}$ " $\equiv$ " Rational description by $\mathbf{x} \in \mathcal{A}$
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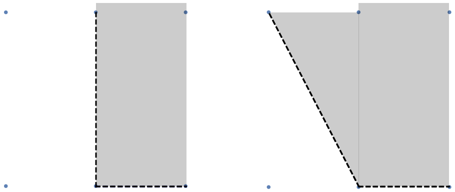
- The effective cone of $\mathcal{M}$ is larger than the one in $\mathcal{A}$



Outline

Definition

$$\mathcal{M} = [\,\mathbf{n}\,||\,\{\mathbf{a}_\alpha\}\,] = \begin{bmatrix} \mathbb{P}^{n_1} \\ \mathbb{P}^{n_2} \\ \vdots \\ \mathbb{P}^{n_m} \end{bmatrix}\left\| \begin{matrix} a^1_1 & \cdots & a^1_K \\ a^2_1 & \cdots & a^2_K \\ \vdots & \ddots & \vdots \\ a^m_1 & \cdots & a^m_K \end{matrix} \right]$$

$$\dim_{\mathbb{C}} \mathcal{M} = \sum_{r=1}^{m} n_r - K \ ,$$

- Standard Complete Intersection $\mathcal{M}$:
  $\{p_\alpha(\mathbf{x_r}) = 0\}, \quad \alpha = 1, 2, \ldots, K; \quad r = 1, \ldots, m.$

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- Standard Complete Intersection $\mathcal{M}$: $\{p_\alpha(\mathbf{x_r}) = 0\}, \quad \alpha = 1, 2, \ldots, K; \quad r = 1, \ldots, m.$
- Adding $\mathcal{L}_1 = \mathcal{O}(b_1^1, \ldots, b_1^m)$ by $h^0(\mathcal{M}, \mathcal{L}_1) \neq 0$, $\mathcal{M}_1 \hookrightarrow \mathcal{M}$.
- Adding $\mathcal{L}_2 = \mathcal{O}(b_2^1, \ldots, b_2^m)$ by $h^0(\mathcal{M}_1, \mathcal{L}_2) \neq 0$, $\mathcal{M}_2 \hookrightarrow \mathcal{M}_1$. ...
- $X \hookrightarrow \mathcal{M}_{L-1} \hookrightarrow \ldots \hookrightarrow \mathcal{M}_1 \hookrightarrow \mathcal{M} \hookrightarrow \mathcal{A}$

$$X = [\,\mathbf{n}\,||\,\{\mathbf{a}_\alpha\}\,|\,\{\mathbf{b}_\mu\}\,] = \left[\begin{array}{c||ccc|ccc} \mathbb{P}^{n_1} & a_1^1 & \cdots & a_K^1 & b_1^1 & \cdots & b_L^1 \\ \mathbb{P}^{n_2} & a_1^2 & \cdots & a_K^2 & b_1^2 & \cdots & b_L^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{P}^{n_m} & a_1^m & \cdots & a_K^m & b_1^m & \cdots & b_L^m \end{array}\right]$$

$$\dim_{\mathbb{C}} X = \sum_{r=1}^{m} n_r - K - L \ , \quad \text{Codim } (K, L)$$

Topology I: Hodge Number $X \hookrightarrow \mathcal{M} \hookrightarrow \mathcal{A}$

Bundle-valued cohomology on the smooth $\mathcal{M}$, and then on X.

e.g: codim=$(K,1)$: $\dim_{\mathbb{C}} X = \sum_{r=1}^{m} n_r - K - 1$. $h^0(\mathcal{M}, \mathcal{L}_1) \neq 0$

- Adjunction Formula: ${\mathcal{K}^\vee}_{\mathcal{M}} = \mathcal{O}_{\mathcal{M}}(\mathcal{L}_1)$

$$0 \to TX \to T\mathcal{M}|_X \to \mathcal{O}_{\mathcal{M}}(\mathcal{L}_1)|_X \to 0$$

- Koszul short exact sequence:

$$0 \to \mathcal{O}_{\mathcal{M}}(-\mathcal{L}_1) \to \mathcal{O}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}|_X \to 0,$$

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$$0 \to \mathcal{O}_{\mathcal{M}}(-\mathcal{L}_1) \otimes \mathcal{V} \to \mathcal{V} \to \mathcal{V}|_X \to 0$$

- $H^*(X,TX)$

$$0 \to H^0(X,TX) \to H^0(X,T\mathcal{M}|_X) \to H^0(X,O(\mathcal{L}_1)|_X) \to H^1(X,TX) \to \cdots$$

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Remark: $\mathcal{L}_1$ is effective divisor but not neccesarily be ample.

$\implies \mathcal{M}$ may not be Fano, X may not be smooth.

Topology I: Hodge Number - Example

$$X = \left[\begin{array}{c||c|c} \mathbb{P}^1 & 3 & -1 \\ \mathbb{P}^4 & 2 & 3 \end{array} \right] , \quad \mathcal{M} = \left[\begin{array}{c||c} \mathbb{P}^1 & 3 \\ \mathbb{P}^4 & 2 \end{array} \right] .$$

- $h^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(-1,3)) = 15$
- $0 \to TX \to T\mathcal{M}|_X \to \mathcal{O}_{\mathcal{M}}(-1,3)|_X \to 0$

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    $0 \to \mathcal{O}_{\mathcal{A}}^{\oplus 2} \to \mathcal{O}_{\mathcal{A}}(1,0)^{\oplus 2} \oplus \mathcal{O}_{\mathcal{A}}(0,1)^{\oplus 5} \to T\mathcal{A} \to 0$
    $0 \to \mathcal{O} \to \mathcal{O}_{\mathcal{A}}(3,2) \to \mathcal{O}_{\mathcal{A}}(3,2)|_{\mathcal{M}} \to 0$,
    $0 \to T\mathcal{M} \to T\mathcal{A}|_{\mathcal{M}} \to \mathcal{O}_{\mathcal{A}}(3,2)|_{\mathcal{M}} \to 0$
  - $h^*(T\mathcal{M} \otimes \mathcal{O}_{\mathcal{M}}(1,-3)) = (0,0,0,2,0)$
  - $h^*(X, T\mathcal{M}|_X) = (0,32,2,0)$
- $h^*(X, TX) = \{0,46,2,0\}, \quad h^{2,1}(X) = 46, h^{1,1}(X) = 2.$

Topology II: Chern class and Triple intersection numbers

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- $c(\mathcal{M}) = c_1^r J_r + c_2^{rs} J_r J_s + c_3^{rst} J_r J_s J_t + \ldots \qquad J_r \in H^{1,1}(\mathcal{M})$
  - $c_1^r = (n_r + 1) - \sum_{j=1}^K a_j^r, \quad c_2^{rs} = \ldots, \quad c_3^{rst} = \ldots$

$\Longrightarrow \quad c_1^r(X) = (n_r + 1) - \sum_{j=1}^K a_j^r - \sum_{k=1}^L b_k^r$ CY condition remains

$$
\begin{aligned}
d_{rst} &= \int_X J_r \wedge J_s \wedge J_t = \int_{\mathcal{M}} J_r \wedge J_s \wedge J_t \wedge \mu_X \\
&= \int_{\mathcal{A}} J_r \wedge J_s \wedge J_t \wedge \mu_X \wedge \mu_{\mathcal{M}}
\end{aligned}
$$

Topology II: Chern class and Triple intersection numbers - Example

$$X = \left[\begin{array}{c||c|c} \mathbb{P}^1 & 3 & -1 \\ \mathbb{P}^4 & 2 & 3 \end{array} \right], \quad \mathcal{M} = \left[\begin{array}{c||c} \mathbb{P}^1 & 3 \\ \mathbb{P}^4 & 2 \end{array} \right].$$

- $c_1^r = 0, \quad r = 1, 2.$
- $(h^{1,1}, h^{2,1}) = (2, 46)$, twice in CICY list and not in Kreuzer-Skarke list.
- $c_2(TX)^{rs} d_{rst} = (24, 46)$,
- $d_{122} = 6, \quad d_{222} = 7.$

$\Longrightarrow$ Inequivalent to other known CYs by linear basis chages.

$\Longrightarrow$ New Calabi-Yau threefold (Wall's thm)

Topology III: Kähler and Mori cone

$$X = \left[\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^4 \end{array} \middle\| \begin{array}{c|c} 3 & -1 \\ 2 & 3 \end{array} \right] , \quad \mathcal{M} = \left[\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^4 \end{array} \middle\| \begin{array}{c} 3 \\ 2 \end{array} \right] .$$

- "Favorable": $h^{1,1}(X) = h^{1,1}(\mathcal{A}) = 2$, forms/divisors are descend from $\mathcal{A}$
- $\int_{\mathcal{M}} J \wedge J \wedge J \wedge J > 0 \;, \quad \int_{\mathcal{L}} J \wedge J \wedge J > 0 \;, \quad \int_{S} J \wedge J > 0 \;, \quad \int_{C} J > 0$

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  - The effective cone on $\mathcal{A}$: $aH_1 + bH_2, \quad \{a, b \geq 0\}$
  - The effective cone on $\mathcal{M}$ enlarged: $H^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(aH_1 + bH_2)) > 0$

$$\{a, b \geq 0\} \ \ || \ \ \{a = -1, \ b \geq 2\} \ \ || \ \ \{a \leq -2, b \geq \frac{1}{6}(-5 + 8|a|) + \frac{1}{6}\sqrt{-119 + 64|a| + 64a^2}\}$$

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  - The cone of ample divisors also changes in the same way?
    $J = \mathcal{K}_{\mathcal{M}}^{-1} = \mathcal{O}_{\mathcal{M}}(-1, 3), \quad \mathcal{L} = -H_1 + 3H_2.$

$$S = \{p_1 = 0\} \cap \{p_2 = 0\}, \qquad [p_i = 0] \in [-H_1 + 2H_2]$$

$$\int_S J_{K^{-1}} \wedge J_{K^{-1}} = \int_{\mathcal{A}} J_{K^{-1}} \wedge J_{K^{-1}} \wedge \mu_S \wedge \mu_{\mathcal{M}} < 0$$

$\Longrightarrow \quad \mathcal{K}_{\mathcal{M}}^{-1}$ is effective but not ample.

$\Longrightarrow$ $\mathcal{M}$ is not Fano. No garauntee X is smooth.

Outline

Construct Global Sections in $\mathcal{M}$

- $h^0(\mathcal{M},\mathcal{L}) \neq 0, \quad \mathcal{L} = \mathcal{O}_{\mathcal{M}}(\mathbf{b}), \quad \mathbf{b} = (b^1, \ldots, b^m)$.
- $q \in H^0(\mathcal{M},\ \mathcal{O}_{\mathcal{M}}(\mathbf{b}))$, $\mathbf{x_r} = (x_r^0 : \ldots, x_r^{n_r}) \in \mathbb{P}^{n_r} \in \mathcal{A}$.

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Regular Condition: N vanish at the same order when D goes to zero on $\mathcal{M}$

$$N\in\langle D\rangle\cap\mathbb{C}\left[\mathbf{x}_1,\cdots,\mathbf{x}_m\right]\Longrightarrow q\text{ is polynomial on }\mathcal{M},\text{ no poles}.$$

$\langle D\rangle$ is generated by $D(\mathbf{x_r})$ in $R(\mathcal{M}):=\mathbb{C}\left[\mathbf{x}_1,\cdots,\mathbf{x}_m\right]/\langle p_1,\cdots,p_K\rangle$

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- If $h^0(\mathcal{M},\mathcal{L}_\mu) \neq 0$, generate all $q_\mu \in H^0(\mathcal{M},\ \mathcal{O}_{\mathcal{M}}(\mathbf{b}_\mu))$, $\mu = 1,\ldots,L$.

$$\begin{aligned} X &= \{x \in \mathcal{M} \mid q_1(x) = \cdots = q_L(x) = 0\} \\ &= \{x \in \mathcal{A} \mid p_1(x) = \cdots = p_K(x) = q_1(x) = \cdots = q_L(x) = 0\} \end{aligned}$$

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- $h^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(-1,3)) = 15$, $p_1 \in H^0(\mathcal{A}, \mathcal{O}_{\mathcal{A}}(3,2))$, $q_1 \in H^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(-1,3))$.

  $p_1(\mathbf{x}_1, \mathbf{x}_2) = (x_1^0)^3 P_{11}(\mathbf{x}_2) + (x_1^0)^2 x_1^1 P_{12}(\mathbf{x}_2) + x_1^0 (x_1^1)^2 P_{13}(\mathbf{x}_2) + (x_1^1)^3 P_{14}(\mathbf{x}_2)$

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- Linearly independent $\Longrightarrow q_1 = \sum_{i=0}^{4} \alpha_i\, s_i + \sum_{i=0}^{4} \beta_i\, t_i + \sum_{i=0}^{4} \gamma_i\, u_i$

Construct Global Sections in $\mathcal{M}$ - Numerical

- $q_1 = \frac{N(\mathbf{x}_1, \cdots, \mathbf{x}_m)}{D(\mathbf{x}_1, \cdots, \mathbf{x}_m)} \in H^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(\mathbf{b}_1)), \qquad N \in \langle D \rangle \cap \mathbb{C}[\mathbf{x}_1, \cdots, \mathbf{x}_m]$

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- Generic $D(\mathbf{x}_1, \cdots, \mathbf{x}_m)$ with deg $[\mathbf{b_1}]_-$,
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- Pick out the zero locus of D on $\mathcal{M}$:
  If X is 3-folds, $\dim_{\mathbb{C}} \mathcal{M} = L + 3$. Intersect the divisor D with $\mathcal{M}$
  $\Rightarrow$ Intersect with $L + 2$ generic multilinear hypersurfaces
  $\mathbf{h} = \{h_i(\mathbf{x}_1, \cdots, \mathbf{x}_m) = 0, i = 1, \cdots, L + 2.\}$

$\mathcal{I}_{\mathbf{h}} = \{ x \in \mathcal{A} \mid D(x) = 0 \ , \ p_\alpha(x) = 0$ for $1 \le \alpha \le K \ , \ h_i(x) = 0$ for $1 \le i \le L+2\}$

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  If X is 3-folds, $\dim_{\mathbb{C}} \mathcal{M} = L + 3$. Intersect the divisor D with $\mathcal{M}$
  $\Rightarrow$ Intersect with $L + 2$ generic multilinear hypersurfaces
  $\mathbf{h} = \{h_i(\mathbf{x}_1, \cdots, \mathbf{x}_m) = 0, i = 1, \cdots, L + 2.\}$

$\mathcal{I}_{\mathbf{h}} = \{ x \in \mathcal{A} \mid D(x) = 0 \ , \ p_\alpha(x) = 0$ for $1 \leq \alpha \leq K \ , \ h_i(x) = 0$ for $1 \leq i \leq L+2\}$

- Regular $\Rightarrow N(\mathbf{x}_1, \cdots, \mathbf{x}_m)|_{x \in \mathcal{I}_{\mathbf{h}}} = \sum_{\deg \mathbf{m} = [\mathbf{b}_1]_+} c_{\mathbf{m}} \ \mathbf{m}|_{x \in \mathcal{I}_{\mathbf{h}}} = 0$

Construct Global Sections in $\mathcal{M}$ - Numerical

- $q_1 = \frac{N(\mathbf{x}_1,\cdots,\mathbf{x}_m)}{D(\mathbf{x}_1,\cdots,\mathbf{x}_m)} \in H^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(\mathbf{b}_1)), \qquad N \in \langle D \rangle \cap \mathbb{C}[\mathbf{x}_1, \cdots, \mathbf{x}_m]$
- Generic $D(\mathbf{x}_1, \cdots, \mathbf{x}_m)$ with deg $[\mathbf{b_1}]_-$,
  $$N(\mathbf{x}_1, \cdots, \mathbf{x}_m) = \sum_{\deg \mathbf{m} = [\mathbf{b}_1]_+} c_{\mathbf{m}} \cdot \mathbf{m}(\mathbf{x}_1, \cdots, \mathbf{x}_m)$$
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- Regular $\Rightarrow N(\mathbf{x}_1, \cdots, \mathbf{x}_m)|_{x \in \mathcal{I}_{\mathbf{h}}} = \sum_{\deg \mathbf{m} = [\mathbf{b}_1]_+} c_{\mathbf{m}} \, \mathbf{m}|_{x \in \mathcal{I}_{\mathbf{h}}} = 0$
- Solve the linear system to get the subspace of the available linear combinations for N, each gives $q_1 = N/D$ for a fix D.
- Choosing different D, Linearly independent check and compare to $h^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(\mathbf{b_1}))$

Smoothness Check of X

- $\mathcal{M} \hookrightarrow \mathcal{A}$, $\Theta_{\mathcal{M}/\mathcal{A}} = \mathrm{d}^K \mathcal{N}_{\mathcal{M}/\mathcal{A}} = \mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_K \neq 0$ on $\mathcal{M}$. Bertini
- $X \hookrightarrow \mathcal{M}$, $\Theta_{X/\mathcal{M}} = \mathrm{d}^L \mathcal{N}_{X/\mathcal{M}} = \mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}q_L \neq 0$ on X.
  In terms of d and $\mathbf{x_r} \in \mathbb{P}^{n_r}$, $(K+L)$-form :

$$\Theta_{X/\mathcal{A}} = \mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_K \wedge \mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}q_L$$

$$\Theta_{X/\mathcal{A}} = 0 \Longleftrightarrow \Theta_{X/\mathcal{M}} = 0$$

- $\{p_1 = \cdots = p_K = q_1 = \cdots = q_L = 0, \quad \Theta_{X/\mathcal{A}} = 0\}$

$$X = \left[\begin{array}{c||c|c} \mathbb{P}^1 & 3 & -1 \\ \mathbb{P}^4 & 2 & 3 \end{array} \right], \quad \mathcal{M} = \left[\begin{array}{c||c} \mathbb{P}^1 & 3 \\ \mathbb{P}^4 & 2 \end{array} \right].$$

$$\{p_1 = q_1 = 0, \quad \mathrm{d}p_1 \wedge \mathrm{d}q_1 = 0\}$$

$\Longrightarrow X$ is smooth!

Reducedness

The coordinate Ring $R(X)$ may not be reduced.

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$$X = \left[\begin{array}{c||cc|c} \mathbb{P}^1 & 0 & 2 & 0 \\ \mathbb{P}^4 & 1 & 1 & 3 \\ \mathbb{P}^1 & 1 & 4 & -3 \end{array}\right], \quad \mathcal{M} = \left[\begin{array}{c||cc} \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^4 & 1 & 1 \\ \mathbb{P}^1 & 1 & 4 \end{array}\right].$$

- $h^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(0,3,-3)) = h^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(0,1,-1)) = 1,$

$$p_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = x_3^0\, P_{11}(\mathbf{x}_2) + x_3^1\, P_{12}(\mathbf{x}_2)$$

$$q_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \alpha \left(\frac{P_{12}(\mathbf{x}_2)}{x_3^0}\right)^3 \equiv s^3$$

- $R(X) = R(\mathcal{M})/\langle s^3 \rangle \quad \Longrightarrow \quad R(X') = R(\mathcal{M})/\langle s \rangle$

$$X' = \left[\begin{array}{c||cc|c} \mathbb{P}^1 & 0 & 2 & 0 \\ \mathbb{P}^4 & 1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 4 & -1 \end{array}\right], \quad \mathcal{M} = \left[\begin{array}{c||cc} \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^4 & 1 & 1 \\ \mathbb{P}^1 & 1 & 4 \end{array}\right].$$

$\Longrightarrow \quad X \sim X'$ is not a Calabi-Yau

Outline

Ineffective Splitting & Identities

Splitting

- Splitting transition: deformation & blow-up

$$\left[\begin{array}{c\|cccc} \mathbb{P}^n & 1 & \cdots & 1 & 0 \\ A & \mathbf{u}_1 & \cdots & \mathbf{u}_{n+1} & C \end{array}\right] \longleftrightarrow \left[\begin{array}{c\|cc} A & \sum_i^{n+1} \mathbf{u}_i & C \end{array}\right]$$

$$\left[\begin{array}{c\|cc} \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^2 & 3 & 0 \\ \mathbb{P}^2 & 0 & 3 \end{array}\right]_{\chi=0} \longleftrightarrow \left[\begin{array}{c\|c} \mathbb{P}^2 & 3 \\ \mathbb{P}^2 & 3 \end{array}\right]_{\chi=-162}$$

- Effective vs. Ineffective splitting: Euler number change or not. The two configuration related by ineffective splitting are equivalent.

  Remark: In gCICY, this criteria still holds only when these two gCICY configuration matrixes are smooth.

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Identities Candelas, Dale, Lutken, Schimmrigk

- List of identities in standard CICY also applied to gCICY.
  Only involve holomorphic line bundles on two diff configuration matrices.

$$\left[\begin{array}{c\|c} \mathbb{P}^1 & 1 \\ \mathbb{P}^1 & 1 \end{array}\right] = \mathbb{P}^1 \Longrightarrow \left[\begin{array}{c\|cc} \mathbb{P}^1 & 1 & a \\ \mathbb{P}^1 & 1 & b \\ A & \mathbf{0} & C \end{array}\right] = \left[\begin{array}{c\|c} \mathbb{P}^1 & a+b \\ A & C \end{array}\right],$$

Empty set & Multiple components

Empty set

- Connected smooth X, $\mathcal{O}_X$ has unique global section: constant function.
- In gCICY, the trivial line bundle may appear as a non-trivial column.

$$X = \left[\begin{array}{c||c|c} \mathbb{P}^1 & 1 & -1 \\ \mathbb{P}^1 & 1 & 1 \end{array} \right] \; ; \quad \mathcal{M} = \left[\begin{array}{c||c} \mathbb{P}^1 & 1 \\ \mathbb{P}^1 & 1 \end{array} \right] = \mathbb{P}^1 \, .$$

$h^*(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(-1,1)) = (1,0)$, Vanishing locus of X is empty.

- If a big configuration contains it as a sub, it is empty set.

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Multiple components

- gCICY may contains multiple components, multiple copies of CY.

$$\left[\begin{array}{c||ccc|c} \vdots & \ddots & \mathbf{0} & \ddots & \ddots \\ \mathbb{P}^1 & \cdots & n & \cdots & \cdots \\ \vdots & \ddots & \mathbf{0} & \ddots & \ddots \end{array} \right] ,$$

Outline

Codim $(1,1)$ gCICYs 3-folds

$$X_{(1,1)} = \left[\begin{array}{c||c|c} \mathbb{P}^{n_1} & a^1 & b^1 \\ \mathbb{P}^{n_2} & a^2 & b^2 \\ \vdots & \vdots & \vdots \\ \mathbb{P}^{n_m} & a^m & b^m \end{array}\right], \quad \mathcal{M} = \left[\begin{array}{c||c} \mathbb{P}^{n_1} & a^1 \\ \mathbb{P}^{n_2} & a^1 \\ \vdots & \vdots \\ n_m & a^m \end{array}\right]$$

- 5 possible ambient spaces
  $\mathbb{P}^4\times\mathbb{P}^1$, $\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^1$, $\mathbb{P}^3\times\mathbb{P}^1\times\mathbb{P}^1$, $\mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$, $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$

Codim $(1,1)$ gCICYs 3-folds

$$X_{(1,1)} = \left[\begin{array}{c||c|c} \mathbb{P}^{n_1} & a^1 & b^1 \\ \mathbb{P}^{n_2} & a^2 & b^2 \\ \vdots & \vdots & \vdots \\ \mathbb{P}^{n_m} & a^m & b^m \end{array}\right], \quad \mathcal{M} = \left[\begin{array}{c||c} \mathbb{P}^{n_1} & a^1 \\ \mathbb{P}^{n_2} & a^1 \\ \vdots & \vdots \\ n_m & a^m \end{array}\right]$$

- 5 possible ambient spaces
  $\mathbb{P}^4 \times \mathbb{P}^1$, $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$, $\mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
- $h^0(\mathcal{M}, \mathcal{L}|_{\mathcal{M}}) \neq 0, \quad \mathcal{L}|_{\mathcal{M}} = \mathcal{O}_{\mathcal{M}}(b^1, \ldots, b^m)$.

$$0 \longrightarrow \mathcal{N}^\vee \otimes \mathcal{L} \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}|_{\mathcal{M}} \longrightarrow 0$$

$\mathcal{N}^\vee \otimes \mathcal{L} = \mathcal{O}_{\mathcal{A}}(-a^1 + b^1, \ldots, -a^m + b^m), \quad \mathcal{L} = \mathcal{O}_{\mathcal{A}}(b^1, \ldots, b^m)$
$h^*(\mathcal{A}, \mathcal{N}^\vee \otimes \mathcal{L})$, $h^*(\mathcal{A}, \mathcal{L})$ can be evaluated by Bott-Borel-Weyl formula

- Calabi-Yau condition: $c_1^r(X_{(1,1)}) = (n_r + 1) - a^r - b^r = 0$

Negative bound in codim (1,1)

$$H^0(\mathcal{A}, \mathcal{N}^\vee \otimes \mathcal{L}) \to H^0(\mathcal{A}, \mathcal{L}) \to H^0(\mathcal{M}, \mathcal{L}) \to H^1(\mathcal{A}, \mathcal{N}^\vee \otimes \mathcal{L}) \to H^1(\mathcal{A}, \mathcal{L}) \to \ldots$$

- $X \hookrightarrow \mathcal{A}$ is not algebraic, $h^0(\mathcal{A}, \mathcal{L}) = 0$.

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$$H^0(\mathcal{A},\mathcal{N}^\vee \otimes \mathcal{L}) \to H^0(\mathcal{A},\mathcal{L}) \to H^0(\mathcal{M},\mathcal{L}) \to H^1(\mathcal{A},\mathcal{N}^\vee \otimes \mathcal{L}) \to H^1(\mathcal{A},\mathcal{L}) \to \dots$$

- $X \hookrightarrow \mathcal{A}$ is not algebraic, $h^0(\mathcal{A},\mathcal{L}) = 0$.
- $h^1(\mathcal{A},\mathcal{N}^\vee \otimes \mathcal{L}) \neq 0$, Bott-Borel-Weyl formula, Künneth formula $\Longrightarrow$ $\exists!\, i = 1 \mid n_1 = 1, \quad (b^1 - a^1) \leq -2, \quad \forall i \neq 1 \;\; b^i > 0, \;\; (b^i - a^i) \geq 0$. The negative entries only appear in one $\mathbb{P}^1$.

Negative bound in codim (1,1)

$$H^0(\mathcal{A},\mathcal{N}^\vee\otimes\mathcal{L})\to H^0(\mathcal{A},\mathcal{L})\to H^0(\mathcal{M},\mathcal{L})\to H^1(\mathcal{A},\mathcal{N}^\vee\otimes\mathcal{L})\to H^1(\mathcal{A},\mathcal{L})\to\dots$$

- $X\hookrightarrow\mathcal{A}$ is not algebraic, $h^0(\mathcal{A},\mathcal{L})=0$.
- $h^1(\mathcal{A},\mathcal{N}^\vee\otimes\mathcal{L})\neq 0$, Bott-Borel-Weyl formula, Künneth formula $\Longrightarrow$ $\exists!\, i=1 \mid n_1=1, \quad (b^1-a^1)\leq -2, \quad \forall i\neq 1\ \ b^i>0, \ (b^i-a^i)\geq 0$. The negative entries only appear in one $\mathbb{P}^1$.
- $h^1(\mathcal{A},\mathcal{N}^\vee\otimes\mathcal{L})>h^1(\mathcal{A},\mathcal{L})$.

$$h^1(\mathcal{A},\mathcal{L})=(-b^1-1)\prod_{i=2}^m\binom{b^i+n_i}{n_i},\ h^1(\mathcal{A},\mathcal{N}^\vee\otimes\mathcal{L})=(a^1-b^1-1)\prod_{i=2}^m\binom{b^i-a^i+n_i}{n_i}.$$

$$\frac{(a^1-b^1-1)}{(-b^1-1)}>R,\qquad R\equiv\frac{\prod_{i=2}^m\binom{b^i+n_i}{n_i}}{\prod_{i=2}^m\binom{b^i-a^i+n_i}{n_i}}$$

- CY condition: $b^1=2-a^1, b^i+a^i=n^i+1$
  - $R>4, \quad b^1=-1.$
  - $R=4, \quad b^1\geq -2.$
  - $R=2, \quad b^1<-1.$ Equivalent to certain CICY.
  - $R=1, \quad b^1<-2.$ Copies of CICY.

Classification of codim (1,1)

X	R	i	χ	$(h^{1,1}(X), h^{1,2}(X))$	Infinite Class
$\left[\begin{array}{c\|\|c\|c} \mathbb{P}^4 & 2 & 3 \\ \mathbb{P}^1 & 3 & -1 \end{array}\right]$	7	N/A	-88	$(2, 46)$	N/A
$\left[\begin{array}{c\|\|c\|c} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & 2+i & -i \end{array}\right]$	2	$i \in \mathbb{Z}_{>0}$	-168	$(2, 86)$	Type III
$\left[\begin{array}{c\|\|c\|c} \mathbb{P}^4 & 0 & 5 \\ \mathbb{P}^1 & 2+i & -i \end{array}\right]$	1	$i \in \mathbb{Z}_{>0}$	$-200(i+2)$	$(i+2, 101(i+2))$	Type I

Table: 3 cases in $\mathbb{P}^4 \times \mathbb{P}^1$. They are smooth.

X	R	i	χ	$(h^{1,1}(X), h^{1,2}(X))$	Infinite Class
$\left[\begin{array}{c\|\|c\|c} \mathbb{P}^2 & 1 & 2 \\ \mathbb{P}^2 & 1 & 2 \\ \mathbb{P}^1 & 2+i & -i \end{array}\right]$	4	$i = 1, 2$	$-78, \ -60$	$(3, 42), \ (3, 33)$	N/A
$\left[\begin{array}{c\|\|c\|c} \mathbb{P}^2 & 0 & 3 \\ \mathbb{P}^2 & 1 & 2 \\ \mathbb{P}^1 & 2+i & -i \end{array}\right]$	2	$i \in \mathbb{Z}_{>0}$	-144	$(3, 75)$	Type III
$\left[\begin{array}{c\|\|c\|c} \mathbb{P}^2 & 0 & 3 \\ \mathbb{P}^2 & 0 & 3 \\ \mathbb{P}^1 & 2+i & -i \end{array}\right]$	1	$i \in \mathbb{Z}_{>0}$	$-162(i+2)$	$(2(i+2), 83(i+2))$	Type I

Table: 3 cases in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$. They are smooth.

Classification of codim (1,1)

X	R	i	χ	$(h^{1,1}(X), h^{1,2}(X))$	Infinite Class
$\left[\begin{array}{c\|\|c\|c} \mathbb{P}^3 & 2 & 2 \\ \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^1 & 3 & -1 \end{array}\right]$	10	N/A	-56	$(3, 31)$	N/A
$\left[\begin{array}{c\|\|c\|c} \mathbb{P}^3 & 2 & 2 \\ \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & 3 & -1 \end{array}\right]$	20	N/A	-104	$(3, 55)$	N/A
$\left[\begin{array}{c\|\|c\|c} \mathbb{P}^3 & 1 & 3 \\ \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & 2+i & -i \end{array}\right]$	4	$i = 1, 2$	$-72, \ -48$	$(3, 39), \ (3, 27)$	N/A
$\left[\begin{array}{c\|\|c\|c} \mathbb{P}^3 & 1 & 3 \\ \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^1 & 2+i & -i \end{array}\right]$	2	$i \in \mathbb{Z}_{>0}$	-144	$(3, 75)$	Type III
$\left[\begin{array}{c\|\|c\|c} \mathbb{P}^3 & 0 & 4 \\ \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & 2+i & -i \end{array}\right]$	2	$i \in \mathbb{Z}_{>0}$	-168	$(2, 86)$	Type II
$\left[\begin{array}{c\|\|c\|c} \mathbb{P}^3 & 0 & 4 \\ \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^1 & 2+i & -i \end{array}\right]$	1	$i \in \mathbb{Z}_{>0}$	$-168(i+2)$	$(i+2, 86(i+2))$	Type I

Table: 6 cases in $\mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$. They are smooth.

Also can have 6 cases in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and 5 cases in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Infinite Class

- Type I : Copies of CY
- Type II: Equivalant to CICY by ineffective splitting

$$\left[\begin{array}{c||c|c} \mathbb{P}^1 & 2+i & -i \\ \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^3 & 0 & 4 \end{array}\right], \qquad i \in \mathbb{Z}_{>0} \,.\cong \quad \left[\begin{array}{c||c} \mathbb{P}^1 & 2 \\ \mathbb{P}^3 & 4 \end{array}\right]$$

- Type III: Equivalant to CICY by ineffective splitting and identities

$$\left[\begin{array}{c||cc} \mathbb{P}^1 & 1 & a \\ \mathbb{P}^n & 1 & nb \\ Y & 0 & M \end{array}\right] \to \left[\begin{array}{c||c} \mathbb{P}^1 & a+b \\ \mathbb{P}^{n-1} & nb \\ Y & M \end{array}\right]$$

$$\left[\begin{array}{c||c|c} \mathbb{P}^1 & 2+i & -i \\ \mathbb{P}^4 & 1 & 4 \end{array}\right] \to \left[\begin{array}{c||cc|c} \mathbb{P}^1 & 0 & 2+i & -i \\ \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^4 & 1 & 0 & 4 \end{array}\right] \to \left[\begin{array}{c||c|c} \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^3 & 0 & 4 \\ \mathbb{P}^1 & 2+i & -i \end{array}\right] \to \left[\begin{array}{c||c} \mathbb{P}^3 & 4 \\ \mathbb{P}^1 & 2 \end{array}\right].$$

Remark: By comparing the Hodge number, Chern class and intersection number, there are 8 new manifolds with following Hodege number: Wall's

$$(h_{1,1}, h_{2,1}) = (2,46), (3,31), (3,39), (3,27), (3,42), (3,33), (4,20), (5,29)$$

Codim $(2,1)$ gCICYs 3-folds

$$X_{(2,1)} = \left[\begin{array}{c||cc|c} \mathbb{P}^{n_1} & a_1^1 & a_2^1 & b^1 \\ \mathbb{P}^{n_2} & a_1^2 & a_2^2 & b^2 \\ \vdots & \vdots & \vdots & \vdots \\ \mathbb{P}^{n_N} & a_1^m & a_2^m & b^m \end{array}\right], \quad \mathcal{M} = \left[\begin{array}{c||cc} \mathbb{P}^{n_1} & a_1^1 & a_2^1 \\ \mathbb{P}^{n_2} & a_2^2 & a_2^2 \\ \vdots & \vdots & \vdots \\ \mathbb{P}^{n_N} & a_1^m & a_2^m \end{array}\right].$$

- $\mathbb{P}^5\times\mathbb{P}^1$, $\mathbb{P}^4\times\mathbb{P}^2$, $\mathbb{P}^3\times\mathbb{P}^3$, $\mathbb{P}^4\times\mathbb{P}^1\times\mathbb{P}^1$, $\mathbb{P}^3\times\mathbb{P}^2\times\mathbb{P}^1$, $\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2$, $\mathbb{P}^3\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$, $\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^1$, $\mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$, $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$.
- $h^0(\mathcal{M},\mathcal{O}_{\mathcal{M}}(\mathbf{b}))\neq 0$, Negative b^i only appear in two $\mathbb{P}^1$ factors or one $\mathbb{P}^2$.
- Vanishing first Chern class & $h^*(X,\mathcal{O})=\{1,0,0,1\}$.

$$X_{(2,1)} = \left[\begin{array}{c||cc|c} \vdots & \vdots & \vdots & \vdots \\ \mathbb{P}^2 & 0 & 3 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{array}\right] \Rightarrow X'_{(2,1)} = \left[\begin{array}{c||cc|c} \vdots & \vdots & \vdots & \vdots \\ \mathbb{P}^2 & 0 & 3+n & -n \\ \vdots & \vdots & \vdots & \vdots \end{array}\right], \; 0\leq n\leq 4$$

Codim $(2, 1)$ gCICYs 3-folds Distribution

Embedding projective spaces	# of classes of generalized configuration matrices	# of spaces with positive χ	# of spaces with non-positive χ
$\mathbb{P}^5 \times \mathbb{P}^1$	168	0	28
$\mathbb{P}^4 \times \mathbb{P}^2$	210	0	6
$\mathbb{P}^4 \times \mathbb{P}^1 \times \mathbb{P}^1$	1,197	3	226
$\mathbb{P}^3 \times \mathbb{P}^2 \times \mathbb{P}^1$	1,800	2	261
$\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$	550	0	12
$\mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	4,410	17	528
$\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$	5,235	9	511
$\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	12,180	16	754
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	8,442	10	350
Total	34,192	57	2,676

Table: The distribution of codimension $(2, 1)$ spaces embedded in products of projective spaces. Remainning singularity check.

New Hodge Data

$(h^{1,1}(X), h^{1,2}(X))$	X
$(1,91)$	$\left[\begin{array}{c\|\|cc\|c} \mathbb{P}^2 & 1 & 1 & 1 \\ \mathbb{P}^2 & 0 & 3 & 0 \\ \mathbb{P}^1 & 0 & 0 & 2 \\ \mathbb{P}^1 & 1 & 2 & -1 \end{array}\right]$
$(1,109)$	$\left[\begin{array}{c\|\|cc\|c} \mathbb{P}^2 & 1 & 0 & 2 \\ \mathbb{P}^2 & 0 & 3 & 0 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 1 & 3 & -2 \end{array}\right]$
$(2,98)$	$\left[\begin{array}{c\|\|cc\|c} \mathbb{P}^2 & 1 & 0 & 2 \\ \mathbb{P}^1 & 0 & 2 & 0 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 0 & 2 & 0 \\ \mathbb{P}^1 & 1 & 3 & -2 \end{array}\right]$
$(6,18)$	$\left[\begin{array}{c\|\|cc\|c} \mathbb{P}^2 & 0 & 1 & 2 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 3 & -2 \end{array}\right], \left[\begin{array}{c\|\|cc\|c} \mathbb{P}^2 & 0 & 1 & 2 \\ \mathbb{P}^1 & 0 & 3 & -1 \\ \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \end{array}\right], \left[\begin{array}{c\|\|cc\|c} \mathbb{P}^2 & 0 & 1 & 2 \\ \mathbb{P}^1 & 0 & 0 & 2 \\ \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 1 & 3 & -2 \\ \mathbb{P}^1 & 1 & 0 & 1 \end{array}\right]$
$(10,19)$	$\left[\begin{array}{c\|\|cc\|c} \mathbb{P}^2 & 0 & 0 & 3 \\ \mathbb{P}^2 & 1 & 1 & 1 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 1 & 3 & -2 \end{array}\right], \left[\begin{array}{c\|\|cc\|c} \mathbb{P}^2 & 0 & 0 & 3 \\ \mathbb{P}^2 & 1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 5 & -4 \end{array}\right], \left[\begin{array}{c\|\|cc\|c} \mathbb{P}^2 & 0 & 0 & 3 \\ \mathbb{P}^2 & 1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 2 & 3 & -3 \end{array}\right], \left[\begin{array}{c\|\|cc\|c} \mathbb{P}^2 & 0 & 0 & 3 \\ \mathbb{P}^2 & 2 & 2 & -1 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \end{array}\right]$
$(9,13)$	$\left[\begin{array}{c\|\|cc\|c} \mathbb{P}^3 & 2 & 0 & 2 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 1 & 3 & -2 \end{array}\right], \left[\begin{array}{c\|\|cc\|c} \mathbb{P}^3 & 2 & 0 & 2 \\ \mathbb{P}^1 & 0 & 3 & -1 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 & 0 \end{array}\right]$
$(9,15)$	$\left[\begin{array}{c\|\|cc\|c} \mathbb{P}^3 & 1 & 0 & 3 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 1 & 3 & -2 \end{array}\right]$
$(10,14)$	$\left[\begin{array}{c\|\|cc\|c} \mathbb{P}^2 & 1 & 0 & 2 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 1 & 3 & -2 \\ \mathbb{P}^1 & 1 & 0 & 1 \end{array}\right], \left[\begin{array}{c\|\|cc\|c} \mathbb{P}^2 & 1 & 0 & 2 \\ \mathbb{P}^1 & 0 & 3 & -1 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 1 & 0 & 1 \end{array}\right], \left[\begin{array}{c\|\|cc\|c} \mathbb{P}^2 & 1 & 0 & 2 \\ \mathbb{P}^1 & 0 & 0 & 2 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 1 & 3 & -2 \end{array}\right]$

Table 13: The Hodge pairs and configuration matrices of novel codimension (2,1) examples. These new Hodge pairs do not appear in the regular CICY list [2], Kreuzer-Skarke list [29] or elsewhere in the known literature [58].

Outline

Fibration structure

$$X = \left[\begin{array}{c||cc} \mathcal{A}_1 & 0 & \mathcal{F} \\ \mathcal{A}_2 & \mathcal{B} & \mathcal{T} \end{array} \right].$$

Fibration of the manifold $F = [\mathcal{A}_1||\mathcal{F}]$ over the base $B = [\mathcal{A}_2||\mathcal{B}]$

$$X = \left[\begin{array}{c||cc|cc} \mathbb{P}^5 & 3 & 1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 & 1 & -1 \\ \hline \mathbb{P}^1 & 1 & 1 & -1 & 1 \end{array} \right], \quad F \text{ is } K3, \quad B \text{ is } \mathbb{P}^1.$$

Check $\mathcal{O}_{\mathcal{M}'}(-1,1)$ has global section on $\mathcal{M}' = \left[\begin{array}{c||ccc} \mathbb{P}^5 & 3 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 & 1 \end{array} \right]$,
$h^0(\mathcal{M}', \mathcal{O}_{\mathcal{M}'}(1,-1)) = 2 > 0$.

$$X = \left[\begin{array}{c||cc|cc} \mathbb{P}^5 & 3 & 1 & 1 & 1 \\ \hline \mathbb{P}^1 & 1 & 1 & 1 & -1 \\ \mathbb{P}^1 & 1 & 1 & -1 & 1 \end{array} \right], \quad F \text{ is Elliptic curve}, \quad B \text{ is } \mathbb{P}^1 \times \mathbb{P}^1.$$

M-theory on CY 4-folds and instantons

M-theory on CY_4, Y_4, M5-brane contribute to non-purterbative superpotential

$$\text{necessary condition} \quad \chi(D, \mathcal{O}_D) = 1$$

$$Y_4 = \left[\begin{array}{c||c|c} \mathbb{P}^3 & 1 & 3 \\ \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & 3 & -1 \\ \mathbb{P}^1 & 1 & 1 \end{array}\right]^{h^{1,1}=4,\, h^{3,1}=68,\, h^{2,2}=332}_{\chi=480} \qquad F \text{ is } T^2 \,, \quad B_3 \text{ is } \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

M-theory on CY 4-folds and instantons

M-theory on CY_4, Y_4, M5-brane contribute to non-purterbative superpotential

$$\text{necessary condition} \quad \chi(D, \mathcal{O}_D) = 1$$

$$Y_4 = \left[\begin{array}{c||c|c} \mathbb{P}^3 & 1 & 3 \\ \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & 3 & -1 \\ \mathbb{P}^1 & 1 & 1 \end{array}\right]^{h^{1,1}=4,\, h^{3,1}=68,\, h^{2,2}=332}_{\chi=480}$$

F is $K3$, B_2 is $\mathbb{P}^1 \times \mathbb{P}^1$

M-theory on CY 4-folds and instantons

M-theory on CY_4, Y_4, M5-brane contribute to non-purterbative superpotential

$$\text{necessary condition} \quad \chi(D, \mathcal{O}_D) = 1$$

$$Y_4 = \left[\begin{array}{c||c|c} \mathbb{P}^3 & 1 & 3 \\ \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & 3 & -1 \\ \mathbb{P}^1 & 1 & 1 \end{array}\right]^{h^{1,1}=4,\, h^{3,1}=68,\, h^{2,2}=332}_{\chi=480} \qquad F \text{ is } K3, \quad B_2 \text{ is } \mathbb{P}^1 \times \mathbb{P}^1$$

- $D = \mathcal{O}(1, -1, 3, 1), \; h^*(D, \mathcal{O}_D) = \{1, 0, 0, 0\}.$

M-theory on CY 4-folds and instantons

M-theory on CY_4, Y_4, M5-brane contribute to non-purterbative superpotential

$$\text{necessary condition} \quad \chi(D, \mathcal{O}_D) = 1$$

$$Y_4 = \left[\begin{array}{c||c|c} \mathbb{P}^3 & 1 & 3 \\ \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & 3 & -1 \\ \mathbb{P}^1 & 1 & 1 \end{array}\right]^{h^{1,1}=4,\, h^{3,1}=68,\, h^{2,2}=332}_{\chi=480} \qquad F \text{ is } K3, \quad B_2 \text{ is } \mathbb{P}^1 \times \mathbb{P}^1$$

- $D = \mathcal{O}(1, -1, 3, 1), \; h^*(D, \mathcal{O}_D) = \{1, 0, 0, 0\}$.
  Not a section of these Fiber, also non-trivial base dependence.
- Non-trivial instanton superpotential in M-theory, also in dual F-theory/Type IIB theories
- Y_4 is also $K3$ fibered, $\tau : B_3 \xrightarrow{\mathbb{P}^1} B_2$, $\tau(D) \in \mathbb{P}^1 \times \mathbb{P}^1$.
  $\Longrightarrow$ World-sheet instanton in 4D Het theory.

Outlook

Mathematics:

- Fully classification and singularity check for codim (2,1) gCICY, and other types of gCICY.
- Computability and simple algebraic construction
- Topological data calculated on some of the gCICY which is singular.

Physical Application:

- Discrete Symmetries, Torsion and Wilson Lines
- Relationship to GLSMs

Thank you for your attention!