Fun with 2-Group Symmetry

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1803.09336 Benini, Córdova, PH

Global Symmetry in QFT

- Global symmetry acts on operators and it leaves all correlation functions invariant. Global symmetry can have an ’t Hooft anomaly: in the presence of the background gauge field the partition function transforms by an overall phase.
- Anomaly is invariant under the RG flow. Global symmetry and its anomaly provide non-perturbative tools to study the low energy quantum dynamics, which is often strongly coupled.
- Applications to dualities and topological phases of matter.
- Important to understand the complete global symmetry and its anomaly. Continuous and discrete symmetries, higher-form symmetries, two-group symmetry…

Outline

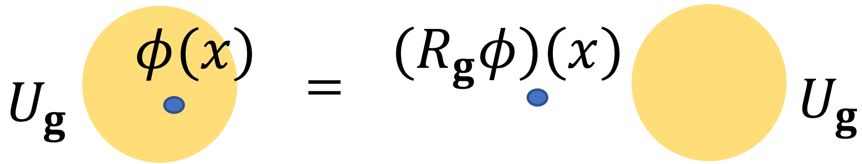
- Review 0-form and 1-form symmetries in terms of symmetry defects.
- 2-group symmetry.
- Anomaly of 2-group symmetry.
- Consistency condition on RG flow.

Ordinary 0-Form Global Symmetry

[Gaiotto,Kapustin,Seiberg,Willett]

- Generated by codimension-1 defects that obey group-law fusion

- Local operators are in representation of the symmetry group.



- The correlation functions of the symmetry defects are topological.
- For continuous symmetry described by currents, the symmetry defect is $U_{\mathbf{g}} = \exp i \oint \star j$. Topological property = current conservation $d \star j = 0$.
- 1-form gauge field coupled to codimension-1 symmetry generator.

1-Form Global Symmetry

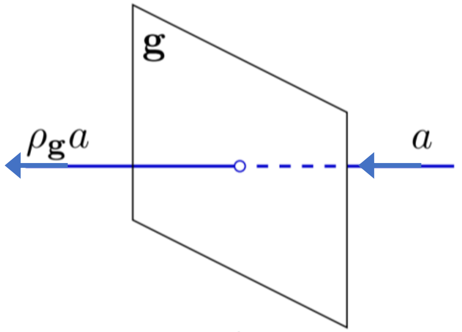
[Kapustin,Seiberg], [Gaiotto,Kapustin,Seiberg,Willett]

- Generated by codimension-2 defects that obey group-law fusion. Symmetry group must be Abelian.
- Line operators transform by some charges under the symmetry group.
- The correlation functions of the symmetry defects are topological.
- 2-form gauge field coupled to codimension-2 symmetry generator.
- Example: 4d Maxwell theory has $U(1) \times U(1)$ 1-form symmetry
$$j_2^E = F, \qquad j_2^M = \star F, \qquad d \star j_2^E = d \star j_2^M = 0\,.$$
- Example: $SU(N)$ gauge theory. The Z_N center of gauge group assigns Z_N 1-form charges to the Wilson lines. Gauging the Z_N 1-form symmetry modifies the bundle to be the Z_N quotient $SU(N)/\mathrm{Z}_N$.

2-Group Global Symmetry: Mixes 0-Form and 1-Form Symmetries

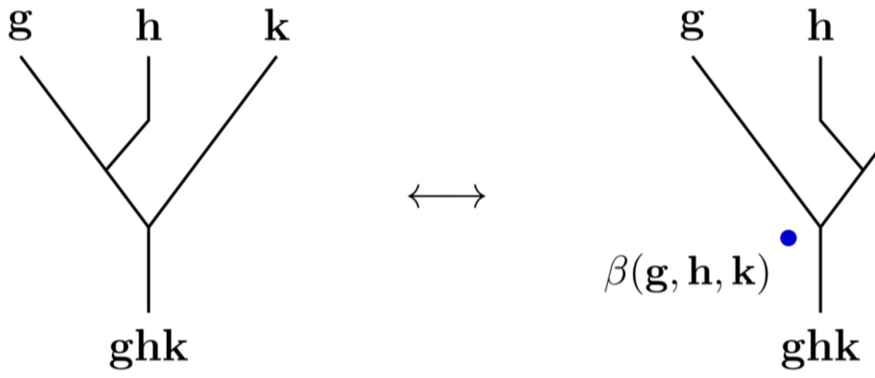
[Baez,Lauda],[Baez,Schreiber],[Kapustin,Thorngren],[Sharpe],[Córdova, Dumitrescu,Intriligator],[Delcamp,Tiwari],[Benini,Córdova,PH]…

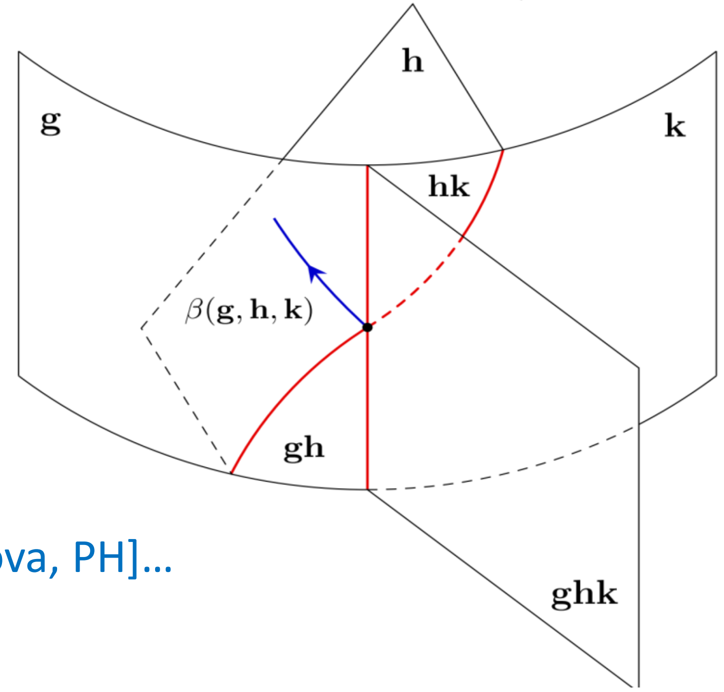
- 0-form symmetry G. 1-form symmetry A.
- 0-form symmetry acts on 1-form symmetry
$$\rho : G \to \mathrm{Aut}(A)\ .$$
- Postnikov class $[\beta] \in H^3_\rho(G, A) : G \times G \times G \to A$



New 4-junction for symmetry defects.

Non-associativity of 0-form symmetry defects.





[Benini,Córdova, PH]…

2-Group Background Gauge Field

[Baez,Lauda],[Baez,Schreiber],[Kapustin,Thorngren],[Sharpe],[Córdova, Dumitrescu,Intriligator],[Delcamp,Tiwari],[Benini,Córdova,PH]…

- Denote background X for 0-form symmetry G, and background B_2 for 1-form symmetry A. X is an 1-cocycle, B_2 is a 2-cochain that satisfies

$$\delta_\rho B_2 = X^* \beta \,.$$

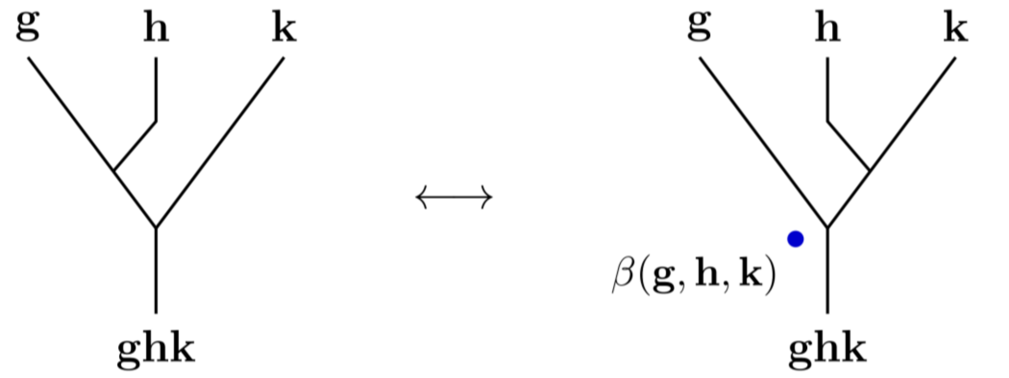
Only $[\beta] \in H^3_\rho(BG, A)$ is meaningful: $\beta \to \beta + \delta_\rho \lambda_2$, $B_2 \to B_2 + X^* \lambda_2$.

- Non-trivial $[\beta]$: cannot gauge only the 0-form symmetry.
- A 0-form gauge transform also produces a background for 1-form symmetry i.e. inserts a 1-form symmetry defect.
- Can gauge only the 1-form symmetry, with $X = 0$.
- If $[\beta] = 0$ the 2-group symmetry factorizes into 0- and 1-form symmetries.

2-Group Background Gauge Field

- If we turn off the background gauge field, then the 2-group symmetry means the correlation functions are invariant under 0-form and 1-form symmetry separately.

- If we consider correlation functions with symmetry defects, then the 2-group symmetry implies a particular rule for fusing the 0-form symmetry defects.



2-Group Symmetry from Gauging a Subgroup in Mixed Anomaly (Green-Schwarz)

[Tachikawa],[Córdova, Dumitrescu,Intriligator]

- Two massless Dirac fermions in 4d, $U(1)_X \times U(1)_Y$ 0-form symmetry:

Weyl	ψ^1	ψ^2	ψ^3	ψ^4
$U(1)_X$	1	-1	0	0
$U(1)_Y$	*k*	*k*	-*k*	-*k*

Mixed anomaly: $k \int_{5d} X \frac{dX}{2\pi} \frac{dY}{2\pi}$,

where k is an integer.

- Next we promote Y to be dynamical y. Emergent $U(1)$ 1-form symmetry generated by $\exp\left(i\oint dy\right)$. New background B_2 couples as $\int_{4d} B_2 dy/2\pi$. Impose constraint on B_2 to maintain gauge invariance:

$$k\int_{5d} X \frac{dX}{2\pi}\frac{dy}{2\pi} + \int_{5d} dB_2 \frac{dy}{2\pi} = 0 \Rightarrow dB_2 + kX\frac{dX}{2\pi} = 0\,.$$

2-Group Symmetry from Gauging a Subgroup in Mixed Anomaly (Green-Schwarz)

- Gauging $U(1)_Y$ extends $U(1)_X$ by the emergent 1-form symmetry to become a 2-group symmetry: $G = U(1)_X$, $A = U(1)$, $\rho = 1$, and the Postnikov class β represented by $-\frac{k}{2\pi} X dX$.
- Analogous to Green-Schwarz mechanism.
- The condition $dB_2 + kX\frac{dX}{2\pi} = 0$ modifies the gauge transformations

$$X \to X + d\lambda_0$$

$$B_2 \to B_2 + d\lambda_1 - k\lambda_0 \frac{dX}{2\pi} \; .$$

Non-trivial background X for 0-form symmetry also enforces a background B_2 for the 1-form symmetry.

2-Group Symmetry is Not An Anomaly for 0-Form Symmetry

- Require the action $\int B_2 \star j_2 + X \star j_1 + \cdots$ to be invariant under the 2-group gauge transformation implies the conservation of 0-form symmetry current j_1 is violated by a non-trivial operator j_2, the 1-form symmetry current:

$$d \star j_1 = j_2 \left(\frac{kdX}{2\pi}\right), \qquad d \star j_2 = 0.$$

- Partition function transforms under a 0-form gauge transformation by an operator insertion instead of a phase.
- Not an ’t Hooft anomaly of the 0-form symmetry. 2-group symmetry cannot be ``canceled’’ by inflow. [Córdova,Dumitrescu,Intriligator],[Benini,Córdova, PH]

Example: QED3 with 2 Fermions of Charge 2

[Benini,Córdova,PH]

- Wilson line of charge 1 is unbreakable and transforms under $A = \mathrm{Z}_2$ 1-form symmetry corresponds to the Z_2 center in the gauge group.
- Two free fermions have at least $U(2)$ 0-form symmetry, neglecting charge conjugation. After gauging $U(1)$, the basic monopole operator is dressed with 2 fermion zero modes, and thus the central $\mathrm{Z}_2 \subset U(2)$ symmetry that flips the sign of the two fermions does not act on any local operators.
- Faithful 0-form symmetry $G = U(2)/\mathrm{Z}_2 \cong SO(3) \times U(1)$. The $U(1)$ is identified with the magnetic symmetry.

Example: QED3 with 2 Fermions of Charge 2

- Background X for G that is not a background for $U(2)$: non-trivial $X^*w_2(G)$

$$w_2(G) = w_2\big(SO(3)\big) + w_2(U(1))$$

is the Z_2 obstruction to lifting the bundle to a $U(2)$ bundle.

- The Z_2: $\psi \to -\psi$ in the quotient $G = U(2)/\mathrm{Z}_2$ can be identified with a Z_4 gauge rotation, since the fermions have charge 2. Backgrounds with non-trivial $w_2(G)$ modifies the gauge bundle by a Z_4 quotient. [Benini,PH,Seiberg]
- The Z_4 quotient requires background B_2 for Z_2 1-form symmetry

$$\delta B_2 = \mathrm{Bock}\big(X^* w_2(G)\big) = X^*\mathrm{Bock}\big(w_2(G)\big).$$

- 2-group symmetry with Postnikov class

$$[\beta] = \mathrm{Bock}\big(w_2(G)\big) = \mathrm{Bock}\Big(w_2\big(SO(3)\big)\Big)\ .$$

Enhanced 2-Group Symmetry at Low Energy

- QED3 with 2 fermions of charge 2 can be obtained from the theory with charge 1 by gauging the Z_2 subgroup magnetic symmetry.
- In the theory with charge 1, the $U(1)$ magnetic symmetry is conjectured to enhance to $SU(2)$ at low energies, and the UV 0-form symmetry $U(2)$ is conjectured to enhance to $O(4)$.

  [Xu,You], [PH,Seiberg],[Benini,PH,Seiberg],[Wang,Nahum,Metlitski,Xu,Senthil],[Córdova,PH,Seiberg]
- In the theory with charge 2, the same conjecture implies there is an enhanced 2-group symmetry at low energies with $G_{\mathrm{IR}} = O(4)/\mathrm{Z}_2$ 0-form symmetry, Z_2 1-form symmetry and the Postnikov class

$$[\beta_{\mathrm{IR}}] = \mathrm{Bock}\big(w_2(G_{\mathrm{IR}})\big) = \mathrm{Bock}\Big(w_2\big(PO(4)\big)\Big).$$

Anomaly for 2-Group Symmetry in the UV

- QED3 with two fermions of charge 2 has action $\sum_j i\bar{\psi}_j \gamma D_{2a} \psi_j + \frac{4}{4\pi} ada$, where we regularized the massless fermions. The theory is parity invariant.
- For non-trivial 2-group background, the gauge bundle has a Z_4 quotient

$$\oint \frac{da}{2\pi} = \frac{1}{4} \oint Y_2 \bmod \mathrm{Z}\,, \qquad Y_2 = 2\widetilde{B_2} - \left(X^* \widetilde{w_2(G)}\right) \in Z^2(M, \mathrm{Z}_4)\,,$$

where tildes denote a lift to Z_4 cochains. The 2-group constraint implies $\delta Y_2 = 0$ and lift-independence.

- The theory is not well-defined but has an anomaly for 2-group symmetry

$$\int_{4d} \frac{4}{4\pi} dada = \frac{\pi}{4} \int_{4d} Y_2 Y_2 \ .$$

Mass deformation

- Give large positive masses to charge-2 fermions. The theory flows to $U(1)_4$.
- The microscopic Z_2 1-form symmetry is enhanced to Z_4.
- The IR theory $U(1)_4$ couples to the UV 2-group background using the background for the emergent Z_4 1-form symmetry in the IR:

$$Y_2 = 2\widetilde{B_2 - \left(X^* w_2(G)\right)}\,.$$

- The Z_4 1-form symmetry has an 't Hooft anomaly, [Kapustin,Seiberg],[Gaiotto, Kapustin,Seiberg,Willett]

$$\frac{\pi}{4}\int_{4d}(Y_2)^2 = \int_{4d}\left(\frac{\pi}{4}X^* w_2(G)^2 - \pi B_2\left(X^* w_2(G)\right) + \pi (B_2)^2\right),$$

where we omit tildes and use the continuous notation. Matches the anomaly in the UV.

Anomaly of 2-Group Symmetry

[Kapustin,Thorngren], [Benini,Córdova,PH]

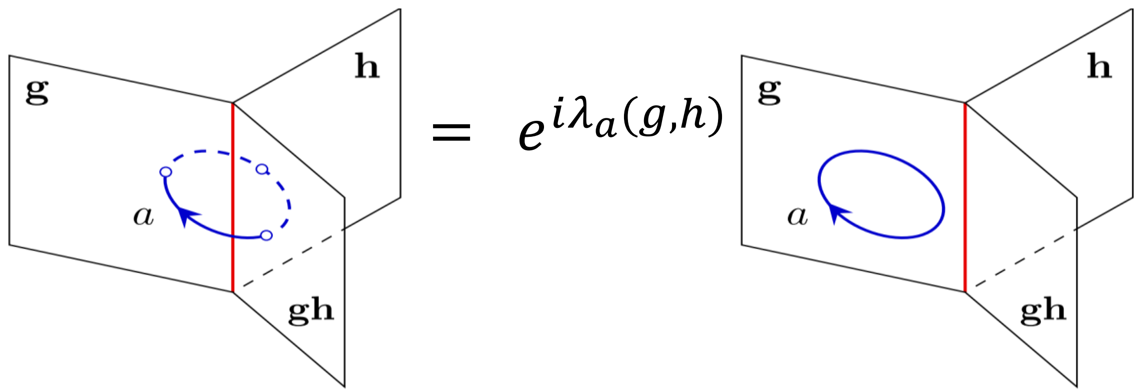
- The anomaly of 2-group symmetry in $3d$ has the structure
$$\int_{4d} X^*\omega - \langle X^*\lambda, B_2\rangle + q(\mathrm{P}\, B_2)\,, \qquad \omega \in C^4\big(BG, U(1)\big), \lambda \in C^2\big(BG, \hat{A}\big)\,.$$
- The anomaly must be a well-defined $4d$ bulk term. This means it is independent of the $5d$ extension and therefore closed:
$$\delta\omega = \langle \lambda, \beta\rangle, \qquad \delta_\rho \lambda = \langle \beta, \star\rangle + q(\mathrm{P}_1 \beta)\,.$$
- Anomaly is defined up to an additional $3d$ local counterterm:
$$S_{3d} = \int_{3d} -\langle X^*\eta, B_2\rangle + X^*\nu\,, \eta \in C^1\big(BG, \hat{A}\big), \nu \in C^3\big(BG, U(1)\big)\,.$$

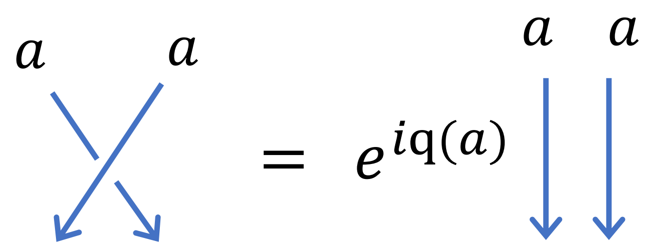
This shifts $\lambda \to \lambda + \delta_\rho \eta, \omega \to \omega + \langle \eta, \beta\rangle + \delta\nu$. Non-trivial Postnikov class $[\beta]$ allows more counterterms to cancel the 0-form symmetry anomaly ω.

Anomaly of 2-Group Symmetry

[Benini,Córdova,PH]

- 0-form symmetry anomaly ω : the 0-form symmetry defect does not obey the pentagon identity for the fusion of four 0-form symmetry defects, but up to a phase.
- 0-form/1-form mixed anomaly λ : when the 1-form symmetry defect encircles 3-junction of 0-form symmetry defects, it produces a phase.
- 1-form symmetry anomaly q: when two 0-form symmetry defects braid each other once, it produces a phase.





2-Group Symmetry and RG Flow

- Consider RG flow starting from the UV theory coupled to background for 2-group symmetry (we cancel the anomaly by inflow from a bulk).
- The IR theory should also couple to the same background since the partition function is invariant under RG flows.
- The UV background field should be consistent with the IR symmetry. Does it give a constraint on the RG flow?
- The anomaly for the UV symmetry should match in the IR since the bulk is the same in the UV and in the IR.

Intrinsic and Extrinsic Symmetries

- Intrinsic symmetry: the true global symmetry that acts on the theory.
- Extrinsic symmetry: symmetry that may not act faithfully.
- Extrinsic symmetry can be the UV symmetry acting on operators that decouple along the RG flow, and thus it does not act in IR theory.
- A theory can couple to the background for an extrinsic symmetry using the backgrounds for the intrinsic symmetry.

Intrinsic and Extrinsic Symmetries

- Example: coupling to $U(1)$ gauge field X by the Z_N 1-form symmetry background $B_2 = \alpha dX$, where $\alpha \in \mathrm{R}/\mathrm{Z}$ and we normalize $\oint B_2 \in \frac{2\pi}{N}\mathrm{Z}$.
- Example: $U(1)$ Maxwell theory in 4d has intrinsic $U(1)_E \times U(1)_M$ 1-form symmetries with the mixed anomaly $\frac{1}{2\pi}\int B_2^E dB_2^M$.

[Gaiotto,Kapustin,Seiberg,Willett]

It can couple to the background $B_2^E = B_2^M = \pi w_2$ where the basic electric and magnetic lines are attached with $\pi\int w_2$ and are fermions: "all-fermion electrodynamics". Reproduce the gravitional anomaly

[Kravec,McGreevy,Swingle]...

$$\frac{1}{2\pi}\int_{5d} B_2^E dB_2^M = \frac{1}{2\pi}\int_{5d} w_2 w_3 \ .$$

Intrinsic and Extrinsic Symmetries

[Benini,Córdova,PH]

- 2-group background for 0-form and 1-form extrinsic symmetries G', A' coupled through the intrinsic 2-group symmetry G, A.
- Homomorphisms $f_0: G' \to G,\ \ f_1: A' \to A$,
$$X = f_0(X'), \qquad B_2 = f_1(B_2') - (X')^* \eta, \qquad \eta \in H^2_\rho(G', A).$$
- $\rho:\ G \to \mathrm{Aut}(A), \rho':\ G' \to \mathrm{Aut}(A')$ compatible with f_0, f_1.
- Relate the Postnikov classes $[\beta] \in H^3_\rho(G, A), [\beta'] \in H^3_{\rho'}(G', A')$:
$$\delta_\rho B_2 = X^* \beta, \qquad \delta_{\rho'} B_2' = (X')^* \beta'.$$
- Postnikov classes $[\beta] \in H^3_\rho(G, A), [\beta'] \in H^3_{\rho'}(G', A')$ satisfy
$$f_0^*[\beta] = f_1([\beta'])\ .$$

Constraint on RG from 2-Group Symmetry

[Benini,Córdova,PH]

- Consider RG flows that preserve the symmetry. The UV symmetry is the extrinsic symmetry, and the IR symmetry is intrinsic:
$$\left(f_0^{\mathrm{UV}\to\mathrm{IR}}\right)^*\left[\beta^{\mathrm{IR}}\right] = f_1^{\mathrm{UV}\to\mathrm{IR}}\left(\left[\beta^{\mathrm{UV}}\right]\right).$$
- If the UV has non-trivial 2-group symmetry but the IR does not $\left[\beta^{\mathrm{IR}}\right]=0$, then the IR theory must have an accidental 1-form symmetry. (or some line operators decouple.)
- If the IR has non-trivial 2-group symmetry but the UV does not $\left[\beta^{\mathrm{UV}}\right]=0$, then the IR theory must have an accidental 0-form symmetry. (or some local operators decouple.)

Constraint on RG from 2-Group Symmetry

- When the IR theory is a 3d TQFT, it has trivial 2-group symmetry $\left[\beta^{\mathrm{IR}}\right] = 0$ if

(1) The IR TQFT is Abelian, or [Barkeshli,Bonderson,Cheng,Wang],[Benini,Córdova,PH]

(2) The IR 0-form symmetry does not permute the lines (conjecture).

- In such cases, if the UV has non-trivial 2-group symmetry, then there must be an emergent 1-form symmetry in the IR.

(Example: QED3 with 2 fermions of charge 2 flows to $U(1)_4$)

Constraint on UV Completion

- If the theory has trivial 2-group symmetry $[\beta] = 0$, then the full 1-form symmetry cannot be realized in any UV completion that has non-trivial 2-group symmetry.

- If the theory has non-trivial 2-group symmetry $[\beta] \neq 0$, then the full 0-form symmetry cannot be realized in any UV completion that has trivial 2-group symmetry $\left[\beta^{\mathrm{UV}}\right] = 0$.

Constraint on Symmetry Breaking

- The UV symmetry is spontaneously broken to a subgroup in the IR.
- The extrinsic symmetry is the IR symmetry, and the intrinsic symmetry is the UV symmetry:

$$\left(f_0^{\mathrm{IR}\to\mathrm{UV}}\right)^*[\beta^{\mathrm{UV}}] = f_1^{\mathrm{IR}\to\mathrm{UV}}([\beta^{\mathrm{IR}}])\,,$$

where f_0, f_1, are inclusion maps.

- If the 1-form symmetry is completely broken, then either the UV has trivial 2-group symmetry (i.e. trivial $[\beta^{\mathrm{UV}}]$), or the 0-form symmetry is also spontaneously broken to a subgroup.
- In $G^{\mathrm{UV}} = U(1), A^{\mathrm{UV}} = U(1)$ it can be shown from Goldstone modes.

[Córdova,Dumitrescu,Intriligator]

More Examples of 2-group: Finite Group Gauge Theory

- Gauging a 0-form finite symmetry G in an (untwisted) finite group H gauge theory leads to an extension of the gauge group
$$1 \to H \to K \to G \to 1 \,,$$

Where G acts on the Wilson lines by $\rho: G \to \mathrm{Out}(H)$.

- The extensions are classified by $H^2_\rho\big(G, Z(H)\big)$: different backgrounds for $Z(H)$ 1-form symmetry $B_2 \to B_2 + X^*\eta$ for $\eta \in H^2_\rho(G, Z(H))$.
- Obstruction to the existence of an extension K is described by $[\beta] \in H^3_\rho(G, Z(H))$. 2-group symmetry with Postnikov class $[\beta]$.
- Example: D_{16} or Q_{16} gauge theory has a Z_2 symmetry that combines with a Z_2 center 1-form symmetry to be 2-group symmetry.

Conclusion

- 2-group symmetry is a mixture of 0-form and 1-form symmetries, where the mixing is described by the Postnikov class.
- If the Postnikov class is non-trivial, one cannot gauge the 0-form symmetry without gauging the 1-form symmetry. This is kinematic and is not an ’t Hooft anomaly for the 0-form symmetry.
- 2-group symmetry can occur in simple examples such as QED4 and QED3 with two Dirac fermions of charge 2, and 3d gapped TQFTs.
- We discuss the structure of the ’t Hooft anomaly for 2-group symmetry. And we derive a new consistency condition on the RG flows using the 2-group symmetry that constrains the emergent symmetries.

Thank You