BPS-State Counting:

Quiver Invariant, Abelianisation & Mutation

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Outline

Warm-Up

- A topology Exercise

Rudiments

- index and Wall-Crossing
- BPS Quivers

Quiver Invariants

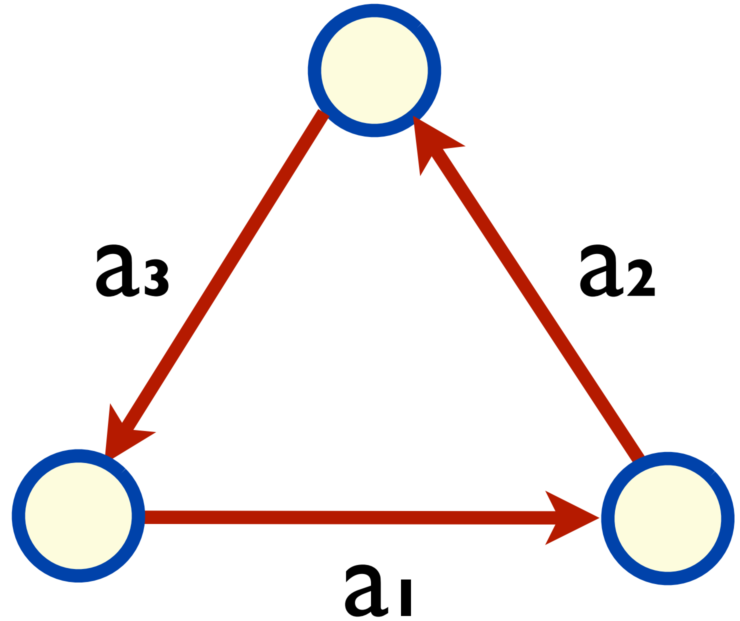
- Characterisation of the Higgs Moduli Spaces

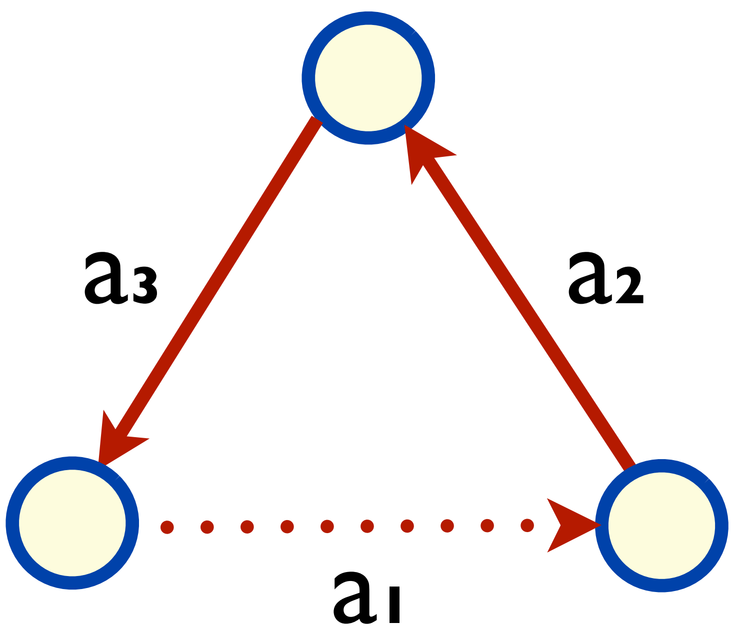
Non-Abelian Quivers

- Abelianisation
- Mutation

Warm Up

a topology exercise

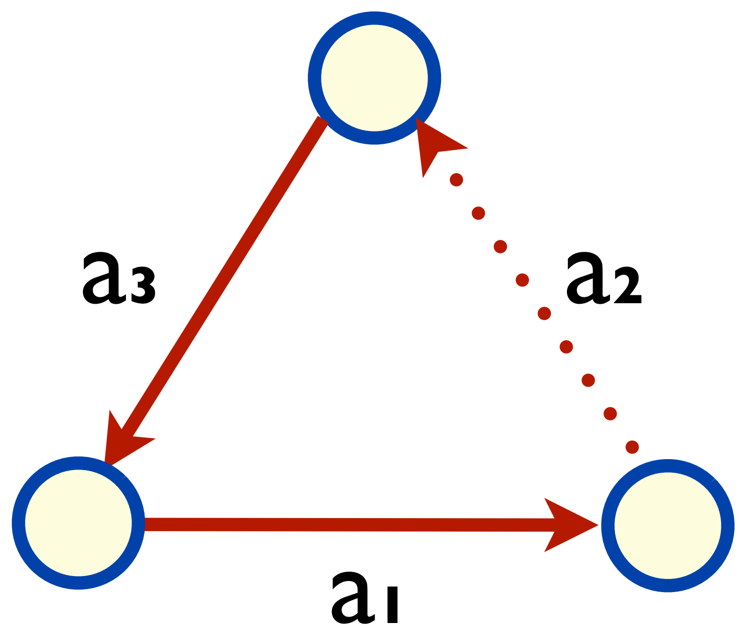




Chamber 1

$$\mathbb{P}^{a_2-1} \times \mathbb{P}^{a_3-1}$$

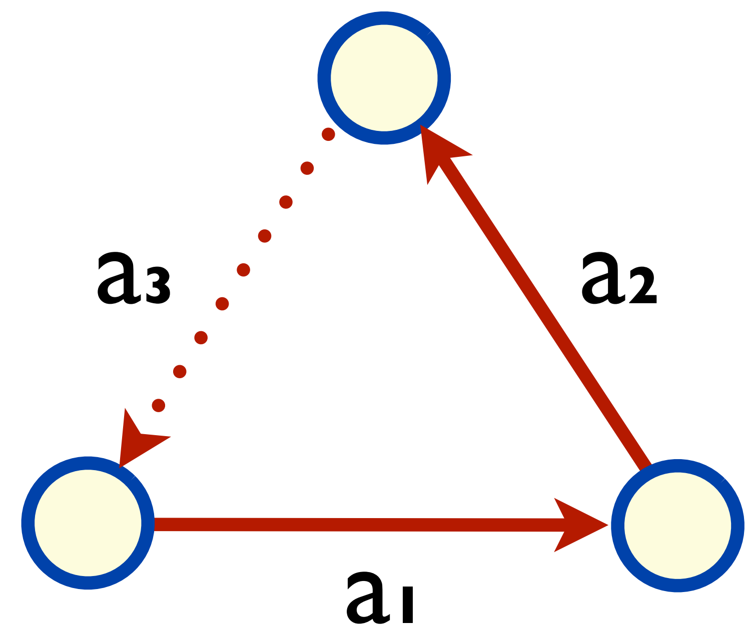
$$\mathcal{O}(1,1)^{\oplus a_1}$$



Chamber 2

$$\mathbb{P}^{a_3-1} \times \mathbb{P}^{a_1-1}$$

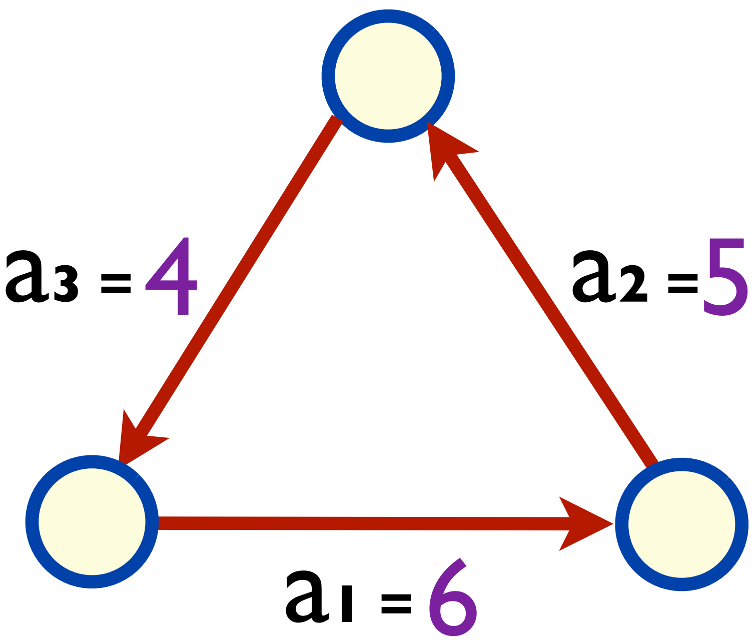
$$\mathcal{O}(1,1)^{\oplus a_2}$$

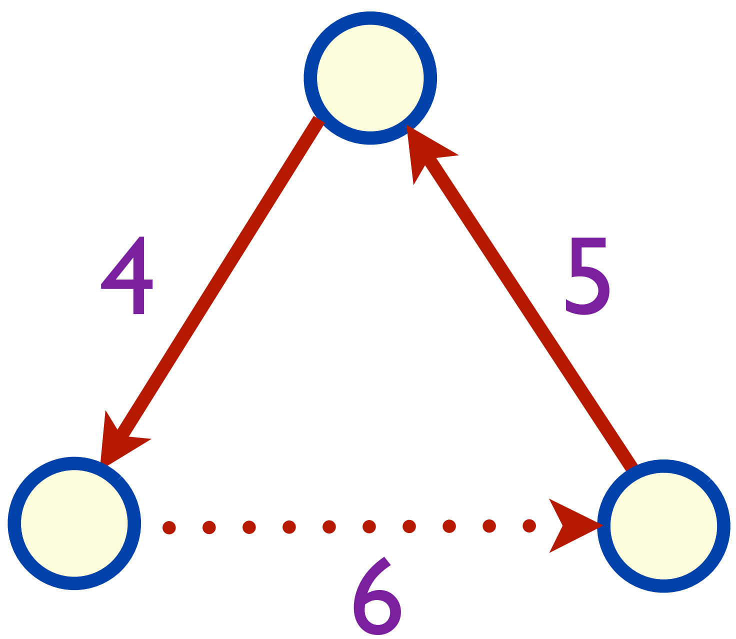


Chamber 3

$$\mathbb{P}^{a_1-1} \times \mathbb{P}^{a_2-1}$$

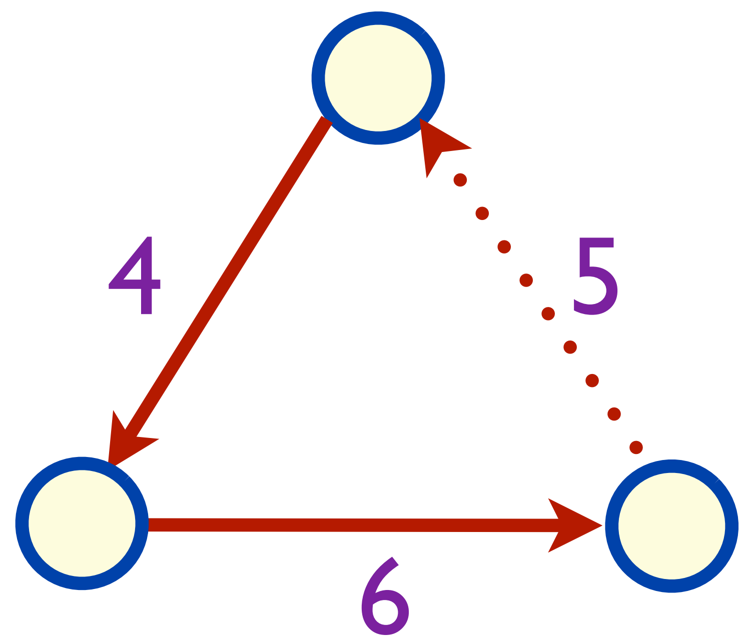
$$\mathcal{O}(1,1)^{\oplus a_3}$$





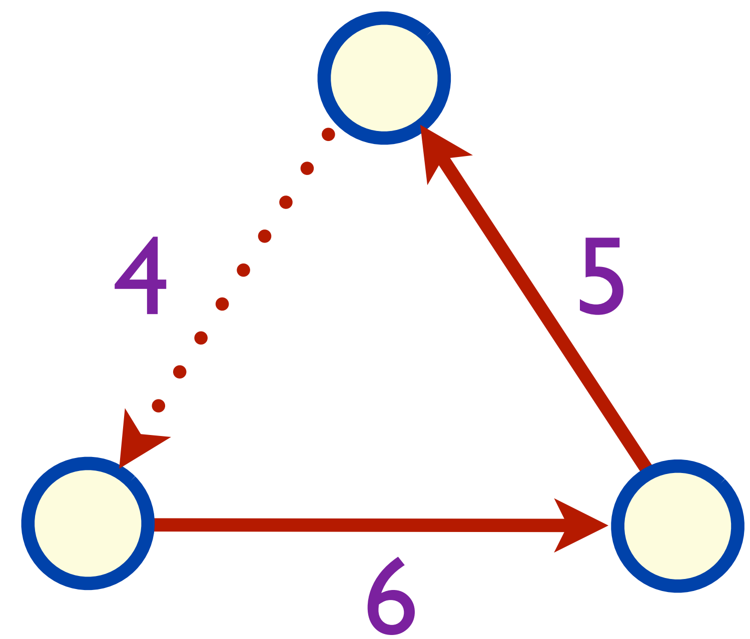
Chamber 1

$$\mathbb{P}^4 \times \mathbb{P}^3$$
$$\mathcal{O}(1,1)^{\oplus 6}$$



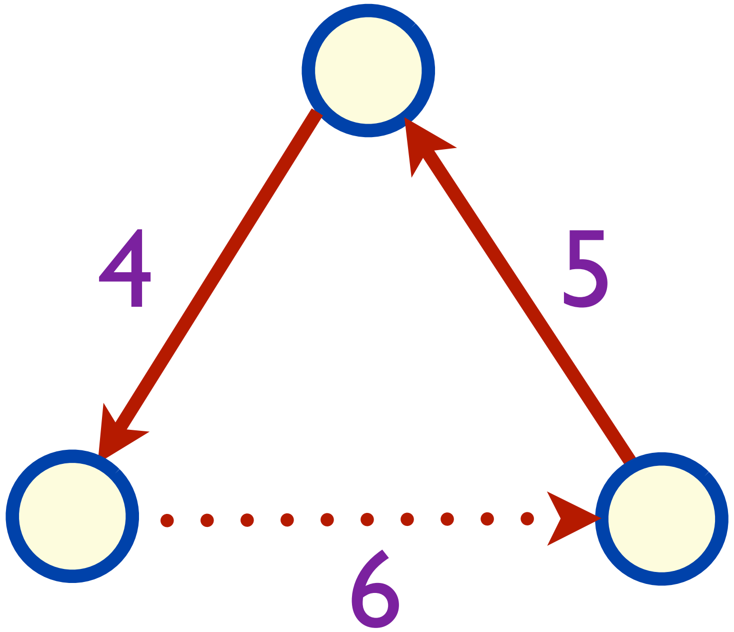
Chamber 2

$$\mathbb{P}^3 \times \mathbb{P}^5$$
$$\mathcal{O}(1,1)^{\oplus 5}$$

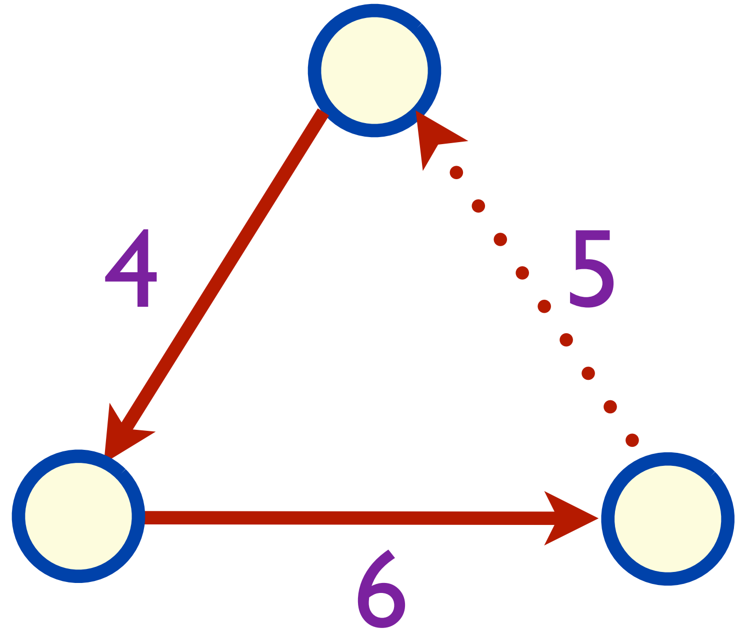


Chamber 3

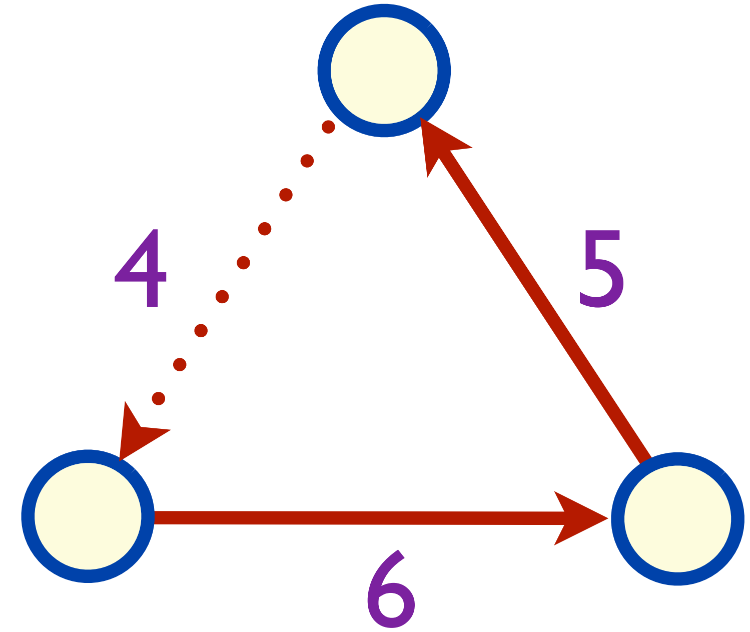
$$\mathbb{P}^5 \times \mathbb{P}^4$$
$$\mathcal{O}(1,1)^{\oplus 4}$$



CHAMBER 1



CHAMBER 2



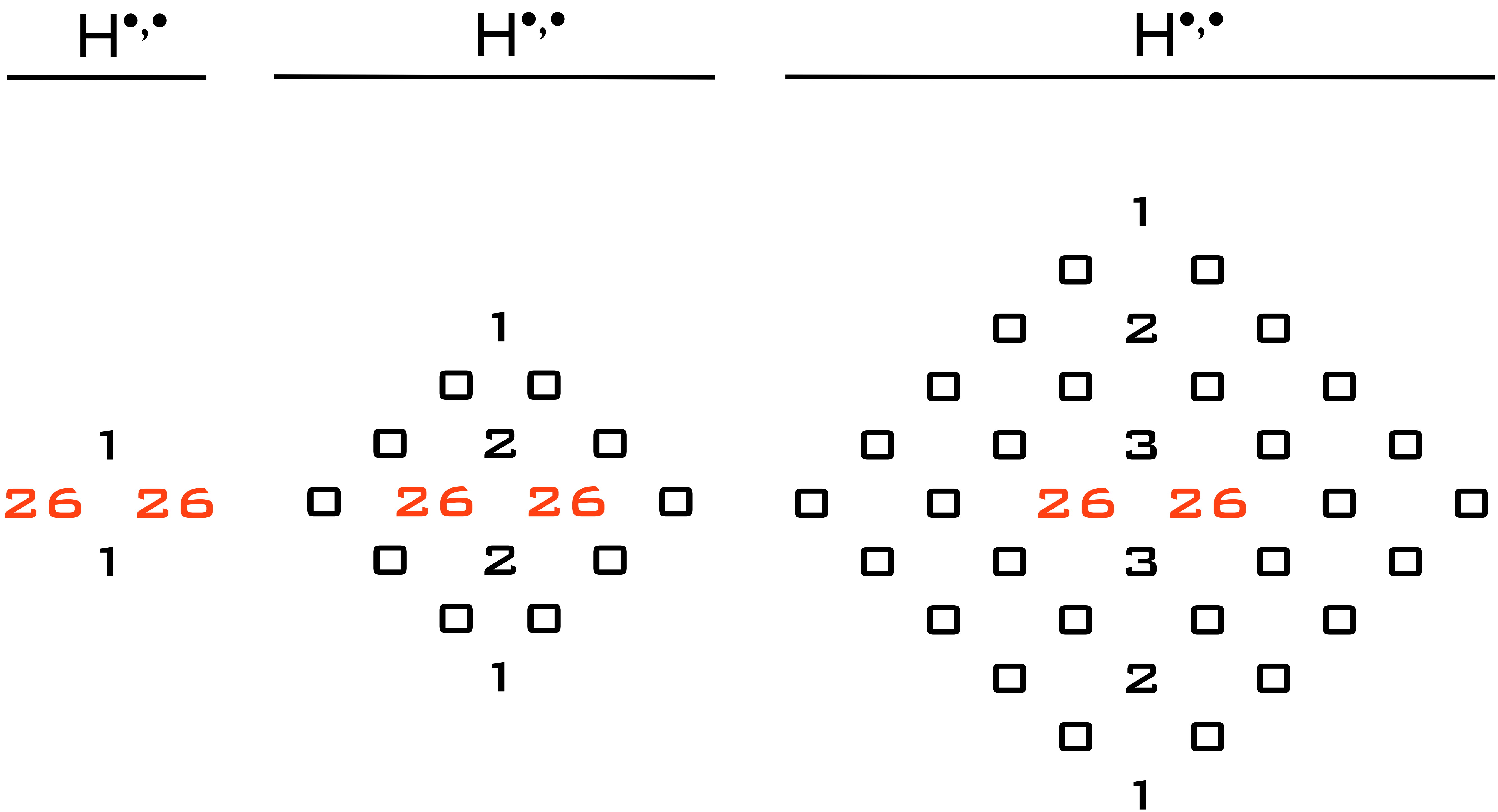
CHAMBER 3

$1 \cdot y_{-1}$
$52 \cdot y_0$
$1 \cdot y_1$

$1 \cdot y_{-3}$
$0 \cdot y_{-2}$
$2 \cdot y_{-1}$
$52 \cdot y_0$
$2 \cdot y_1$
$0 \cdot y_2$
$1 \cdot y_3$

$1 \cdot y_{-5}$
$0 \cdot y_{-4}$
$2 \cdot y_{-3}$
$0 \cdot y_{-2}$
$3 \cdot y_{-1}$
$52 \cdot y_0$
$3 \cdot y_1$
$0 \cdot y_2$
$2 \cdot y_3$
$0 \cdot y_4$
$1 \cdot y_5$

Cahmber 1 Chamber 2 Chamber 3



Rudiments

BPS Index

$\mathcal{N}$=2 Basics

- We consider $\mathcal{N}$=2 Abelian gauge theories.
- States have integer charges: $\gamma \in \mathbb{Z}^{2r} \equiv \Gamma$
- Poincare extends to $\mathcal{N}$=2 super-Poincare: M gets bounded by |Z|.
- M=|Z| CASE: "short" repre, $S_j=[j]\otimes r_{hh}$, where $r_{hh}=2[0]\oplus[1/2]$ is the 4-dim$^\ell$ irrep of the odd alg.
- M>|Z| CASE: "long" repre, $L_j=[j]\otimes r_{hh}\otimes r_{hh}$

BPS Index

- For the Hilbert space $\mathcal{H}^1_\gamma = \left[\bigoplus_{j \in \frac{1}{2}\mathbb{Z}_{\geq 0}} S_j{}^{\oplus \mathbf{n}_j(\gamma)} \right] \oplus \left[\bigoplus_{l \in \frac{1}{2}\mathbb{Z}_{\geq 0}} L_l{}^{\oplus \mathbf{m}_l(\gamma)} \right]$,

  define the BPS index as:

$$\Omega(\gamma) := \sum_{j \in \frac{1}{2}\mathbb{Z}_{\geq 0}} (-1)^{2j}(2j+1)\, \mathbf{n}_j(\gamma)$$
$$= \mathrm{Tr}'_{\mathcal{H}^1_\gamma} (-1)^{2J_3}$$

- Only genuine short reps contribute to $\Omega(\gamma)$.

- The little super-algebra contains su(2)$_R$ and hence one can define the refined index as:

$$\Omega(\gamma; y) = \mathrm{Tr}'_{\mathcal{H}^1_\gamma} (-1)^{2J_3} y^{2I_3 + 2J_3} \xrightarrow{y=1} \Omega(\gamma) = \mathrm{Tr}'_{\mathcal{H}^1_\gamma} (-1)^{2J_3}$$

Rudiments

Wall-crossing

Wall-Crossing

- $\Omega(\gamma)$ is invariant under arbitrary deformations of $\mathcal{H}^1_\gamma$, but may change under deformations of the theory.

- The index is ill-defined when $\mathcal{H}^1_\gamma$ mixes with the multi-ptl spectrum, i.e., if γ can split into γ_1 and γ_2 s.t.
$\gamma_1 + \gamma_2 = \gamma \;, \quad Z_1/Z_2 \in \mathbb{R}^+$.

- Thus, in the parameter space, there appears a wall, across which the BPS index jumps.

Wall-Crossing

- Generic BPS one-particle states as loose bound states of charge centers, balanced by classical forces.

  [Lee, Yi `98; Bak, Lee, Lee, Yi `99; Gauntlett, Kim, Park, Yi `99; Stern, Yi `00; Gauntlett, Kim, Lee, Yi `00]

- The equilibrium distances become infinite as one approaches the wall [Denef `02] :

$$R = \frac{\langle \gamma_1, \gamma_2 \rangle}{2} \frac{|Z_1 + Z_2|}{\mathrm{Im}[\bar{Z}_1 Z_2]}$$

Rudiments

BPS Quivers

BPS Quivers

- BPS states ~ D-branes wrapping various cycles.
- Low-energy D-brane dynamics by a $\mathcal{D}$=4, $\mathcal{N}$=1 quiver gauge theory reduced to the eff. particle world-line.
- E.g. IIB on CY_3: one-particle BPS states seen as a D3-brane wrapping a SLag.
- Two pictures arise for the same BPS bound state of branes:

  (1) Set of particles at equilibrium
  (2) Fusion of D-branes

  related via quiver quantum mechanics [Denef `02]

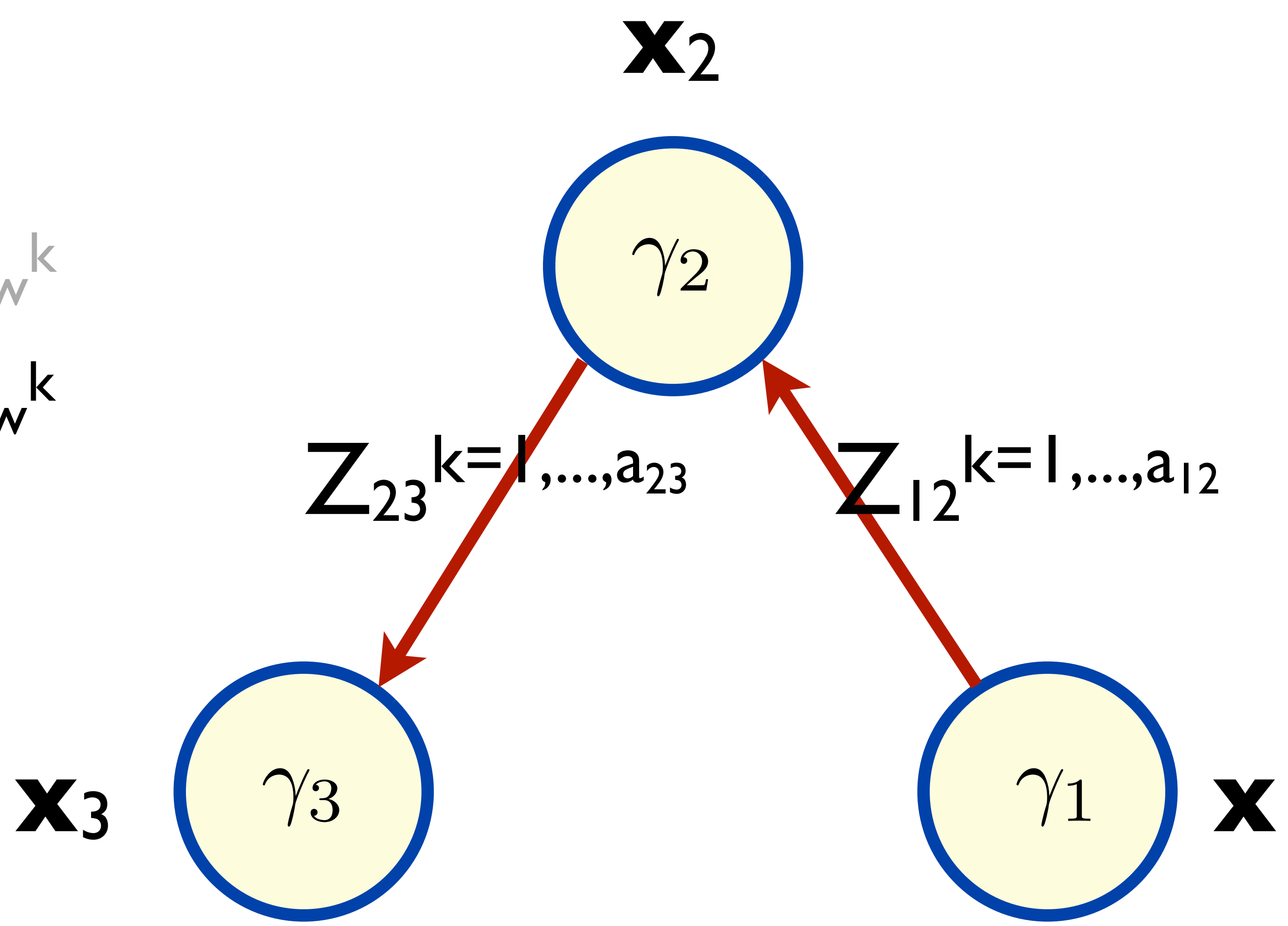
BPS Quivers

- U(1) vectors include $\mathbf{x}_v=(x_v^1, x_v^2, x_v^3)$ and bi-fund. chirals include $Z_{vw}^{k=1,\ldots,a_{vw}}$, where $a_{vw} = \langle\gamma_v, \gamma_w\rangle$

- Two phases

(1) Coulomb: $\mathbf{x}_v$, Z_{vw}^k

(2) Higgs: $\mathbf{x}_v$, Z_{vw}^k



BPS Index

- For large $\mathbf{x}_v$-$\mathbf{x}_w$, chirals are massive and eff. dynamics leads to

$$\mathcal{K}_v \equiv \sum_{w \neq v} \frac{\langle \gamma_w, \gamma_v \rangle}{|\mathbf{x}_w - \mathbf{x}_v|} - \theta_v(u) = 0 \quad \text{for } \forall v, \text{ with } \theta_v = 2\,\mathrm{Im}[e^{-i\alpha} Z_{\gamma_v}(u)]$$

- By studying the sol^n space $\mathcal{M} = \{\mathbf{x}_v \mid \mathcal{K}_v = 0\ ,\ \forall v\} \setminus \mathbb{R}^3$, one can obtain the COULOMB INDEX $\Omega_{\text{Coulomb}}(\{\gamma_v\}; y)$
[de Boer, El-Showk, Messamah, van Den Bleeken `09], [Manschot, Pioline, Sen `11]

- Dialing the coupling to 0, one can describe the system as QM on the variety $\mathcal{M}_H = \{Z^k_{vw} \mid D_v = \theta_v\ ,\ \forall v\} / \prod_v U(1)$.

- The HIGGS INDEX is given as:

$$\Omega_{\text{Higgs}}(\{\gamma_v\}; y) = \sum_{p,q} (-1)^{p+q-d}\, y^{2p-d}\, h^{p,q}(\mathcal{M}_H)$$

Coulomb vs Higgs

- It has been shown [Denef `02; Sen `11]:

  $$\Omega_{\text{Coulomb}} = \Omega_{\text{Higgs}}$$

- Multi-center picture has a smooth transition into the fused D-brane picture at a single point.

- The two pictures might become very different if the quivers have a loop [Denef, Moore `07]:

  $$\Omega_{\text{Coulomb}} << \Omega_{\text{Higgs}}$$

Quiver Invariants

Intrinsic Higgs States

- The Higgs phase might in general have more states than the Coulomb phase multi-center states.
- We may call these additional ones "intrinsic" Higgs states.
- Thus, the Higgs index can be written as:
  $$\Omega_{\mathrm{Higgs}} = \Omega_{\mathrm{Coulomb}} + \text{“}\,\Omega_{\mathrm{Inv}}\text{”}$$
- The intrinsic Higgs states are expected not to experience wall-crossing.

Cyclic Example

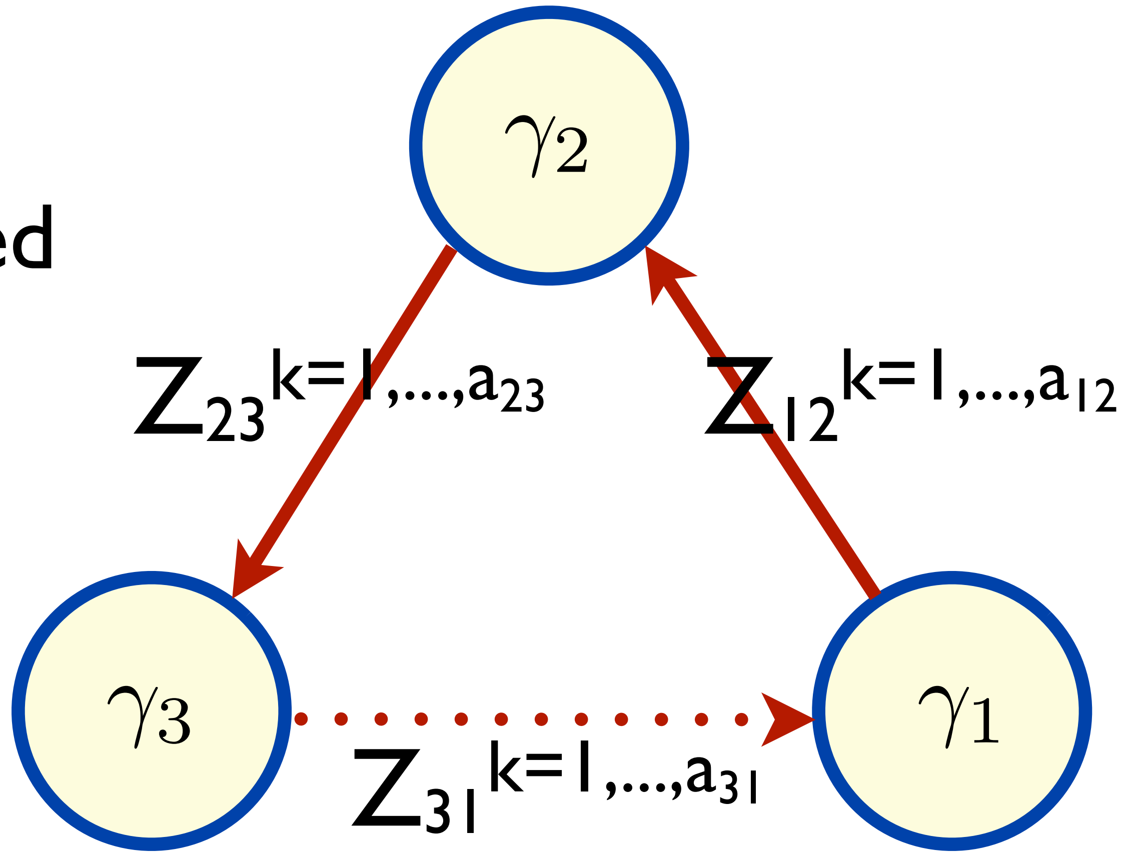
- Consider a 3-node quiver with superpotential

$$\mathcal{W}(\{Z_{12}^k\},\{Z_{23}^k\},\{Z_{31}^k\}) = \sum C_{k_1 k_2 k_3}\ Z_{12}^{k_1}\ Z_{23}^{k_2}\ Z_{31}^{k_3}$$

- There arise 3 different quiver varieties, in each of which one set of chirals vanishes.

- The moduli space is embedded by F-terms in D-term variety.

$$\boxed{\mathcal{M}_H \overset{i}{\hookrightarrow} \mathcal{A}}$$



Characterisation of Ω_{Inv}

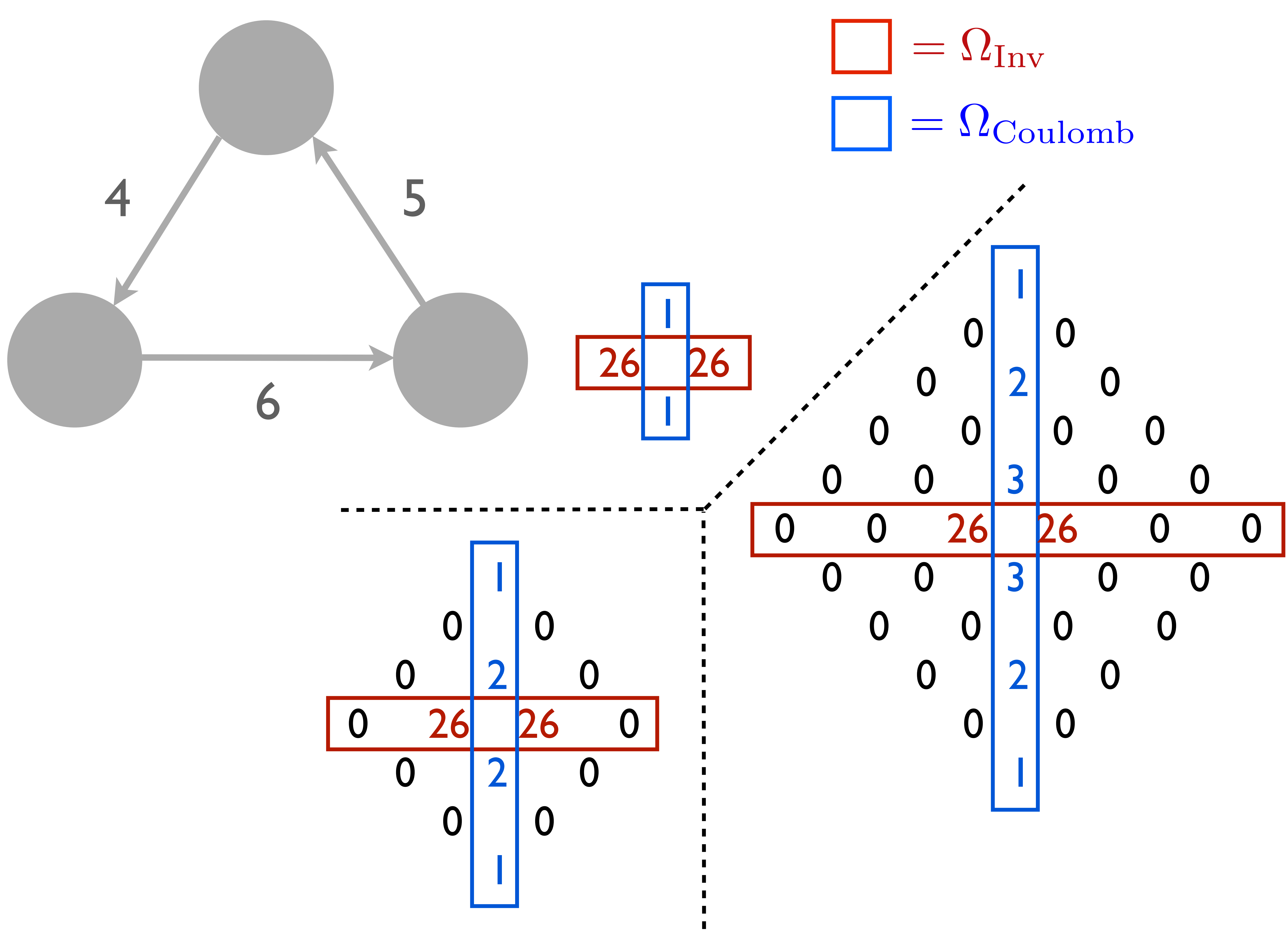
- Embedding structure $\mathcal{M}_H \overset{i}{\hookrightarrow} \mathcal{A}$

  $\Longrightarrow$ Naturally splits the Higgs phase states:

$$\underbrace{\boxed{H^\bullet(\mathcal{M}_H)}}_{\Omega_{\text{Higgs}}} = \underbrace{\boxed{i^*\left(H^\bullet(\mathcal{A})\right)}}_{\Omega_{\text{Coulomb}}} \oplus \underbrace{\boxed{\left[H^\bullet(\mathcal{M}_H)/i^*\left(H^\bullet(\mathcal{A})\right)\right]}}_{\Omega_{\text{Inv}}}$$

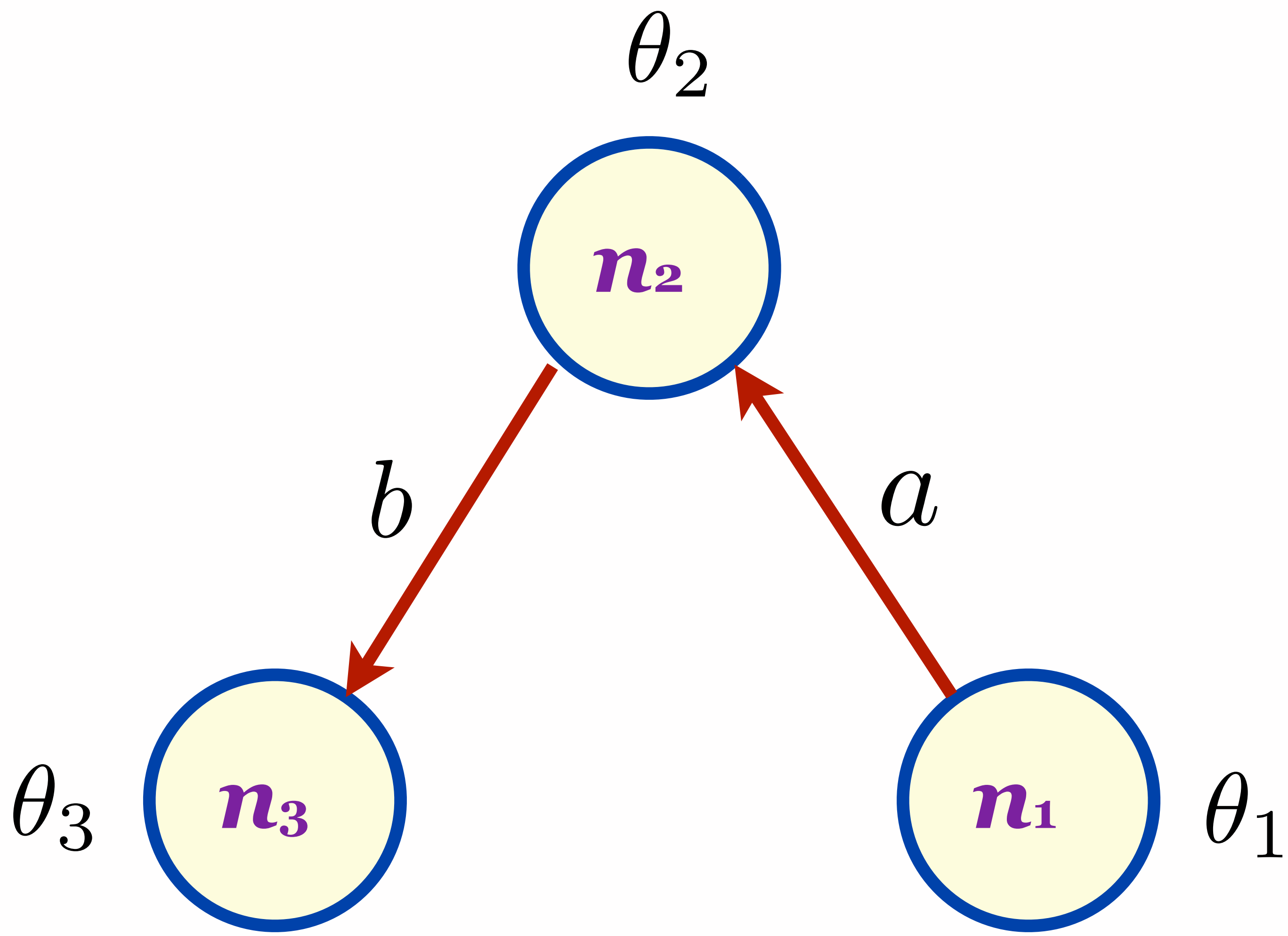
[**S.-J.L.**, Z.-L.Wang, P.Yi `12]

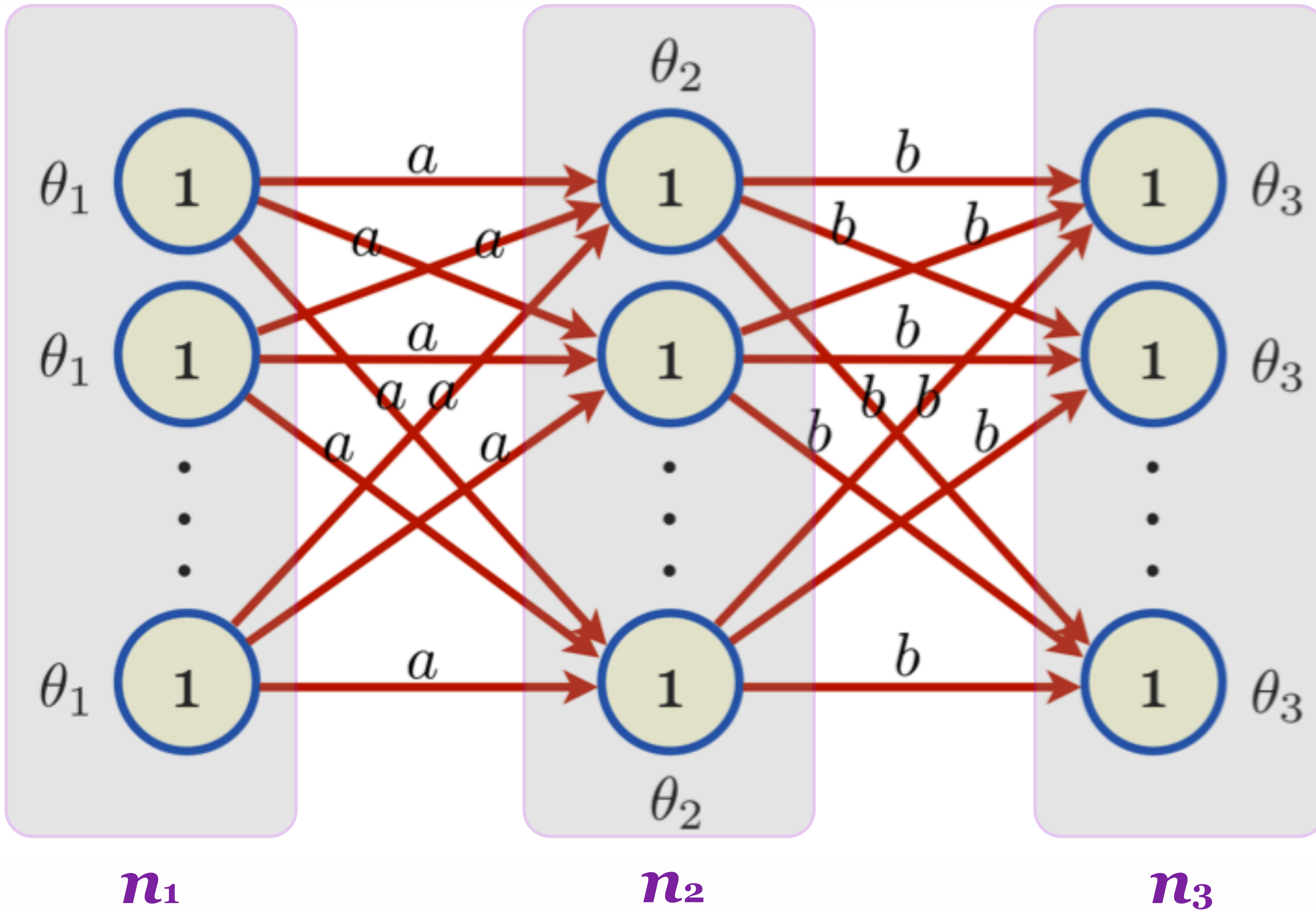
(cf.) [Bena, Berkooz, de Boer, El-Showk, van Den Bleeken `12]


$= \Omega_{\mathrm{Inv}}$
$= \Omega_{\mathrm{Coulomb}}$
4
5
6
1
26 26
1
1
0 0
0 2 0
0 26 26 0
0 2 0
0 0
1
1
0 0
0 2 0
0 0 0 0
0 0 3 0 0
0 0 26 26 0 0
0 0 3 0 0
0 0 0 0
0 2 0
0 0
1

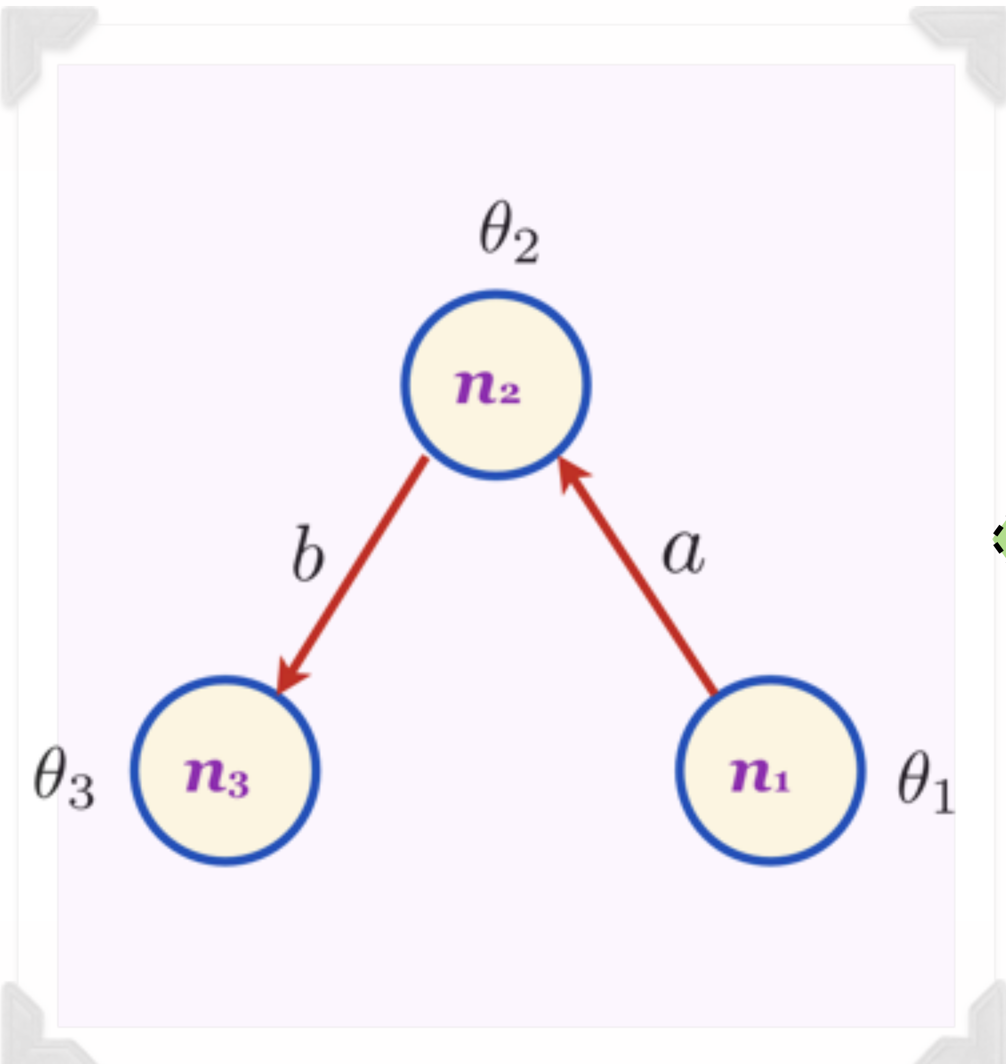
Nonabelian Quivers

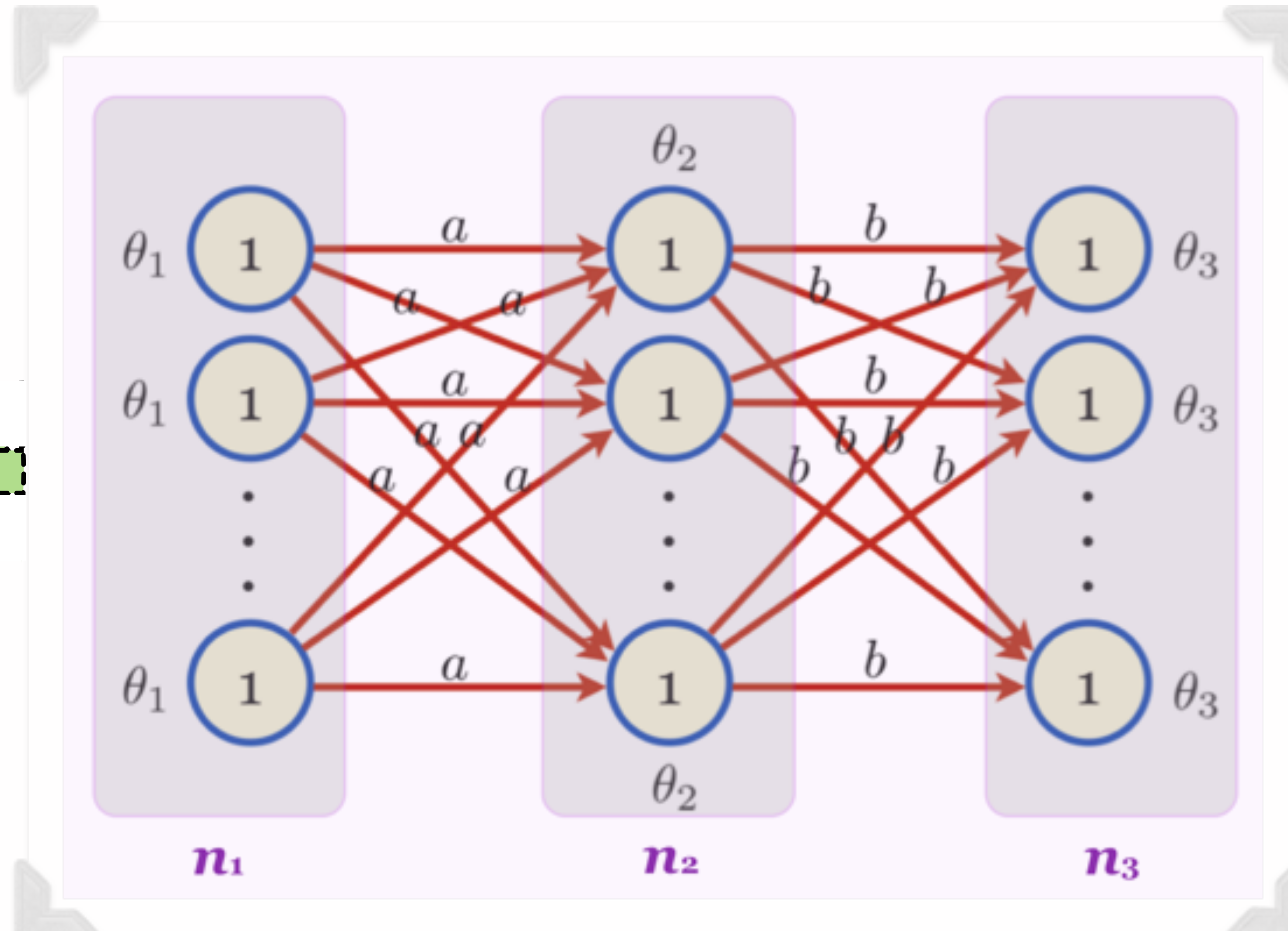
Abelianisation


θ_2
n_2
b
a
θ_3
n_3
n_1
θ_1


θ_1
θ_1
θ_1
1
a
b
θ_2
θ_2
θ_3
θ_3
θ_3
n_1
n_2
n_3

Abelianisation





X $\qquad\qquad$ $\tilde{X}$

The Prescription in a Nutshell

[Martin, `00], [Ciocan-Fontanine, Kim, Sabbah, `06]

(cf.) [Hori, Vafa, `00]

Loopless Quivers

$$
\begin{tikzcd}
Y = \mu_G^{-1}(0)/T \arrow[rr, "\iota"'] \arrow[d, "\pi"] & & \tilde{X} = \mu_T^{-1}(0)/T \\
\boxed{X = \mu_G^{-1}(0)/G} & &
\end{tikzcd}
$$

<u>Bridging</u> : $\displaystyle\int_X a = \frac{1}{|W|}\int_{\tilde{X}} \hat{a} \wedge e(\boldsymbol{\Delta})$, where

- $\pi^\star a = \iota^\star \hat{a}$
- $W = \mathrm{Weyl}(G)$
- $\boldsymbol{\Delta} = \bigoplus_{\mathrm{root}\ \alpha} \mathcal{L}_\alpha$

Loopless Quivers

$$\begin{tikzcd} Y = \mu_G^{-1}(0)/T \arrow[r, "\iota"'] \arrow[d, "\pi"] & \tilde{X} = \mu_T^{-1}(0)/T \\ \boxed{X = \mu_G^{-1}(0)/G} & \end{tikzcd}$$

$$\omega_y(\mathcal{T}X) \equiv \prod_\mu \left[x_\mu \cdot \left(\frac{y e^{-x_\mu} - y^{-1}}{1 - e^{-x_\mu}} \right) \right]$$

<u>Index</u> : $\Omega(y) = \frac{1}{|W|} \int_{\tilde{X}} \omega_y(\mathcal{T}\tilde{X}) \wedge \frac{e(\mathbf{\Delta})}{\omega_y(\mathbf{\Delta})}$,

where $\omega_y \leftarrow f_{\omega_y}(x) = \frac{x}{(1 - e^{-x})} \cdot (y e^{-x} - y^{-1})$

Quivers with a Potential

$$\begin{tikzcd} & Y = \mu_G^{-1}(0)/T \arrow[r, "\iota"'] \arrow[d, "\pi"] & \tilde{X} = \mu_T^{-1}(0)/T \\ \boxed{M} \arrow[r, "F=0"'] & \boxed{X = \mu_G^{-1}(0)/G} & \end{tikzcd}$$

<u>Index</u> : $\Omega(y) = \frac{1}{|W|} \int_{\tilde{X}} \omega_y(\mathcal{T}\tilde{X}) \wedge \frac{e(\tilde{\mathcal{N}})}{\omega_y(\tilde{\mathcal{N}})} \wedge \frac{e(\mathbf{\Delta})}{\omega_y(\mathbf{\Delta})},$

where $\omega_y \leftarrow f_{\omega_y}(x) = \frac{x}{(1-e^{-x})} \cdot (ye^{-x} - y^{-1})$

Applications

- Non-Abelian Quiver Invariant
- Partition-sum Structure of the Index

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- Non-Abelian Quiver Invariant
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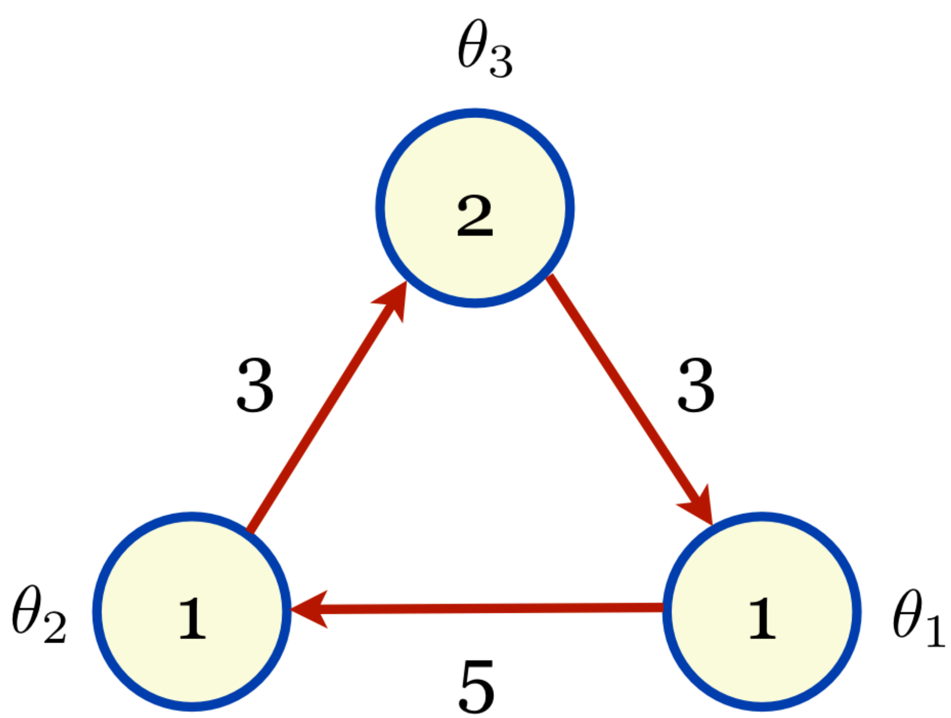

Chamber (a)

$$\Omega^{(a)}_{\text{Higgs}}(y) = 6$$

$$\Omega^{(a)}_{\text{Coulomb}}(y) = 1$$

Chamber (b)

$$\Omega^{(b)}_{\text{Higgs}}(y) = \frac{1}{y^2} + 7 + y^2$$

$$\Omega^{(b)}_{\text{Coulomb}}(y) = \frac{1}{y^2} + 2 + y^2$$

$$\Omega_{\text{inv}} = 5$$

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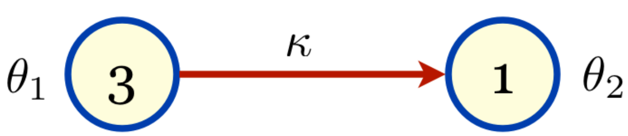
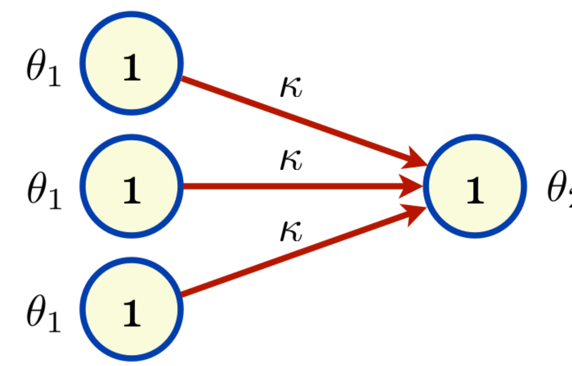
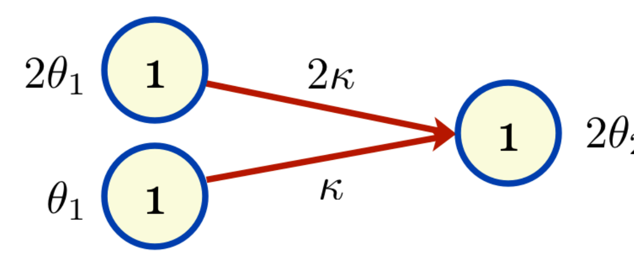


$$\mathcal{P}_1 = (\{1,1,1\};\{1\})$$



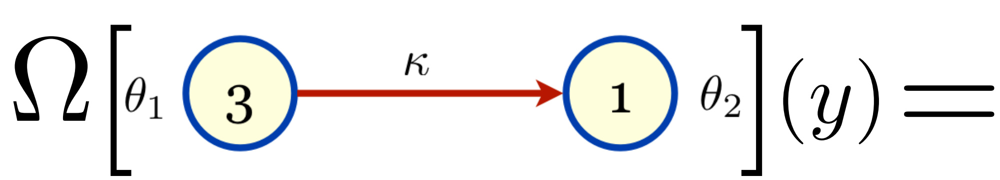
$$\mathcal{P}_2 = (\{1,2\};\{1\})$$

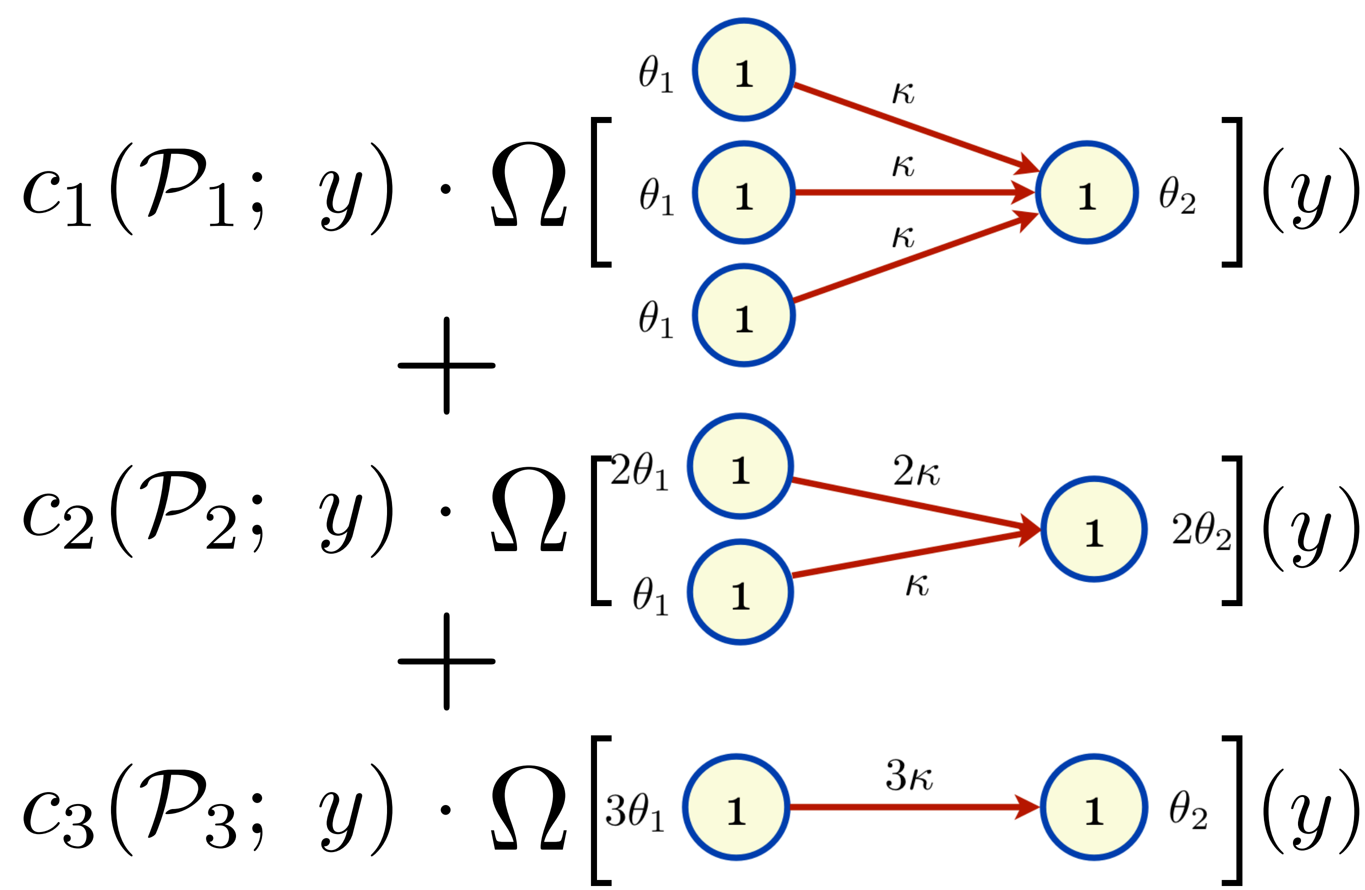


$$\mathcal{P}_3 = (\{3\};\{1\})$$

Applications

- Non-Abelian Quiver Invariant
- <u>Partition-sum Structure of the Index</u>





$$c(\mathcal{P};y) \equiv \frac{1}{|\Gamma(\mathcal{P})|}\prod_{v=1}^{N}\prod_{a_v=1}^{l_v}\frac{1}{r_{v,a_v}}\frac{y-y^{-1}}{y^{r_{v,a_v}}-y^{-r_{v,a_v}}}$$

Applications

- Non-Abelian Quiver Invariant
- Partition-sum Structure of the Index

Applications

- Non-Abelian Quiver Invariant
- Partition-sum Structure of the Index

· · · Works in principle for any quivers but practically hard · · ·

- Asymptotic behavior?
- Another path towards Non-Abelian Quivers?

Nonabelian Quivers

Mutation

Left and Right Mutations

- Relate the index of a complicated quiver to that of a simpler one via mutation:

$$\mathrm{Q} = (\{N_i\}\,;[b_{ij}])_{\zeta_i} \xrightarrow{\mu} \widehat{\mathrm{Q}} = (\{\widehat{N}_i\}\,;[\widehat{b}_{ij}])_{\widehat{\zeta}_i}$$

- With respect to a node k, either Left or Right: μ_k^L or μ_k^R
- The action on charges $\gamma_i' s$ characterises the mutation:

$$\mu_k^L(\gamma_i) = \begin{pmatrix} -\gamma_k & i = k \\ \gamma_i + [b_{ki}]_+\gamma_k & \text{otherwise} \end{pmatrix}$$

$$\mu_k^R(\gamma_i) = \begin{pmatrix} -\gamma_k & i = k \\ \gamma_i + [b_{ik}]_+\gamma_k & \text{otherwise} \end{pmatrix}$$

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$$\downarrow$$

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$$\downarrow$$

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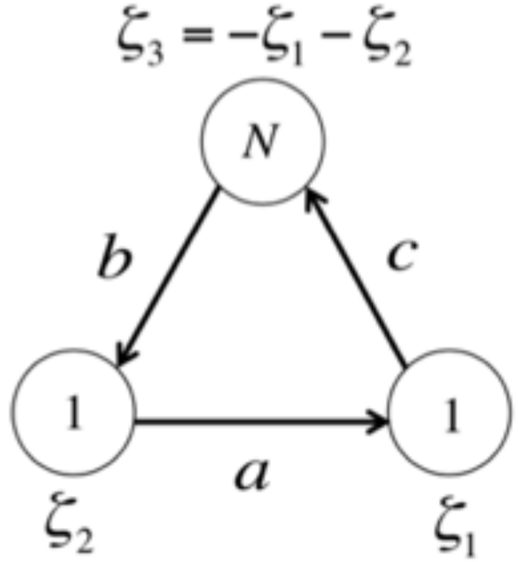
$$\mu_k^L(\gamma_i) = \begin{pmatrix} -\gamma_k & i=k \\ \gamma_i + [b_{ki}]_+\gamma_k & \text{otherwise} \end{pmatrix}$$

$$\mu_k^R(\gamma_i) = \begin{pmatrix} -\gamma_k & i=k \\ \gamma_i + [b_{ik}]_+\gamma_k & \text{otherwise} \end{pmatrix}$$

$$\mu_k(b_{ij}) = \begin{pmatrix} -b_{ij} & \text{if } i=k \text{ or } j=k \\ b_{ij} + \mathrm{sgn}(b_{ik})[b_{ik}b_{kj}]_+ & \text{otherwise} \end{pmatrix}$$

Triangle Quiver with $\vec{\mathrm{N}} = (1, 1, N)$



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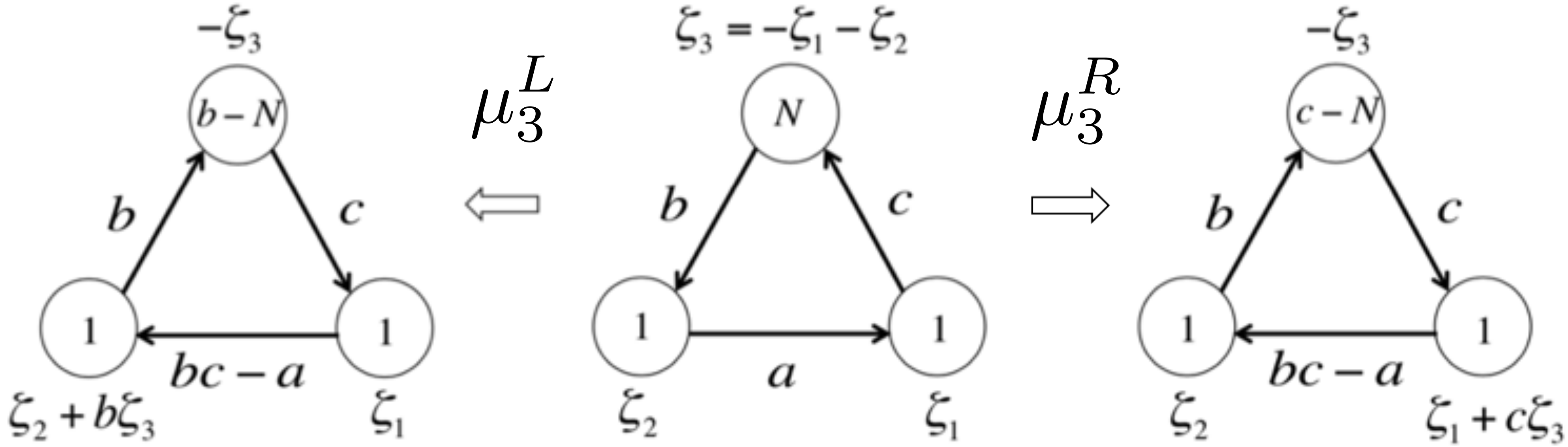
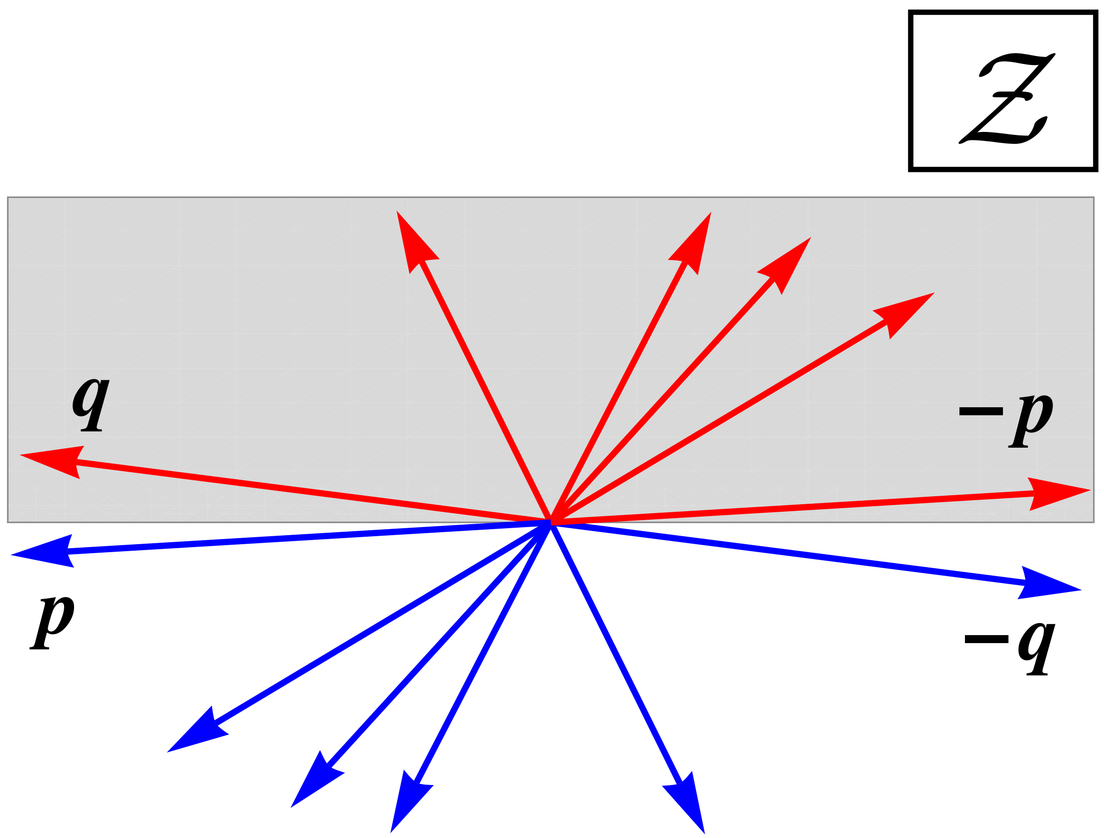
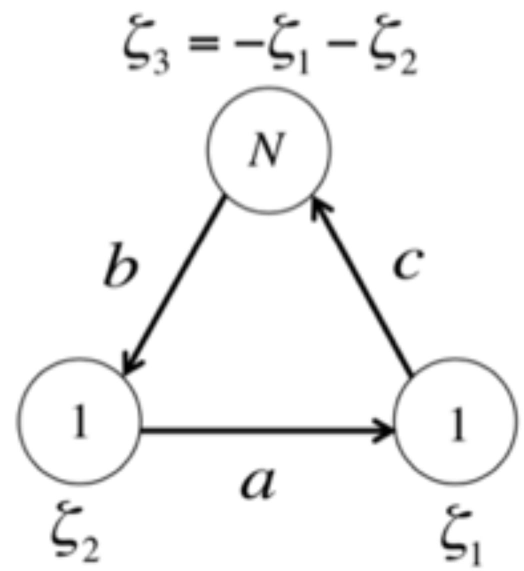


- Trade off between vectors and chirals could be made.
- Would all mutations preserve the Witten index?

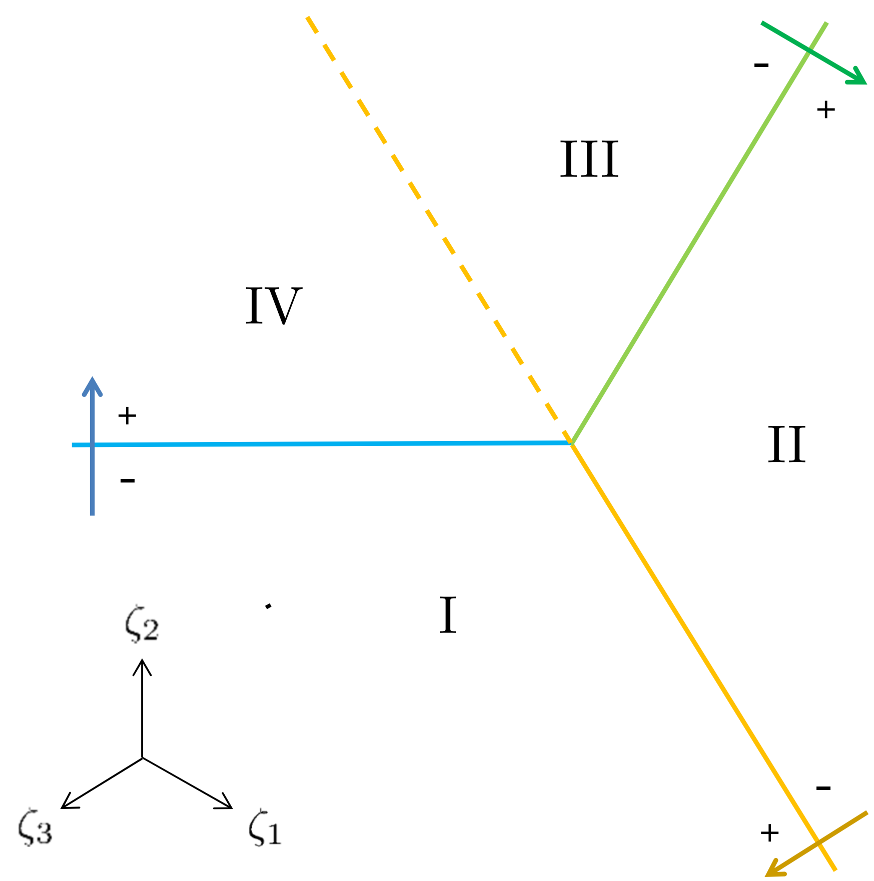
Mutation as a viewpoint change in how BPS particles are distinguished from anti-BPS particles

[Alim, Cecotti, Cordova, Espahdodi, Rastogi, Vafa `11]

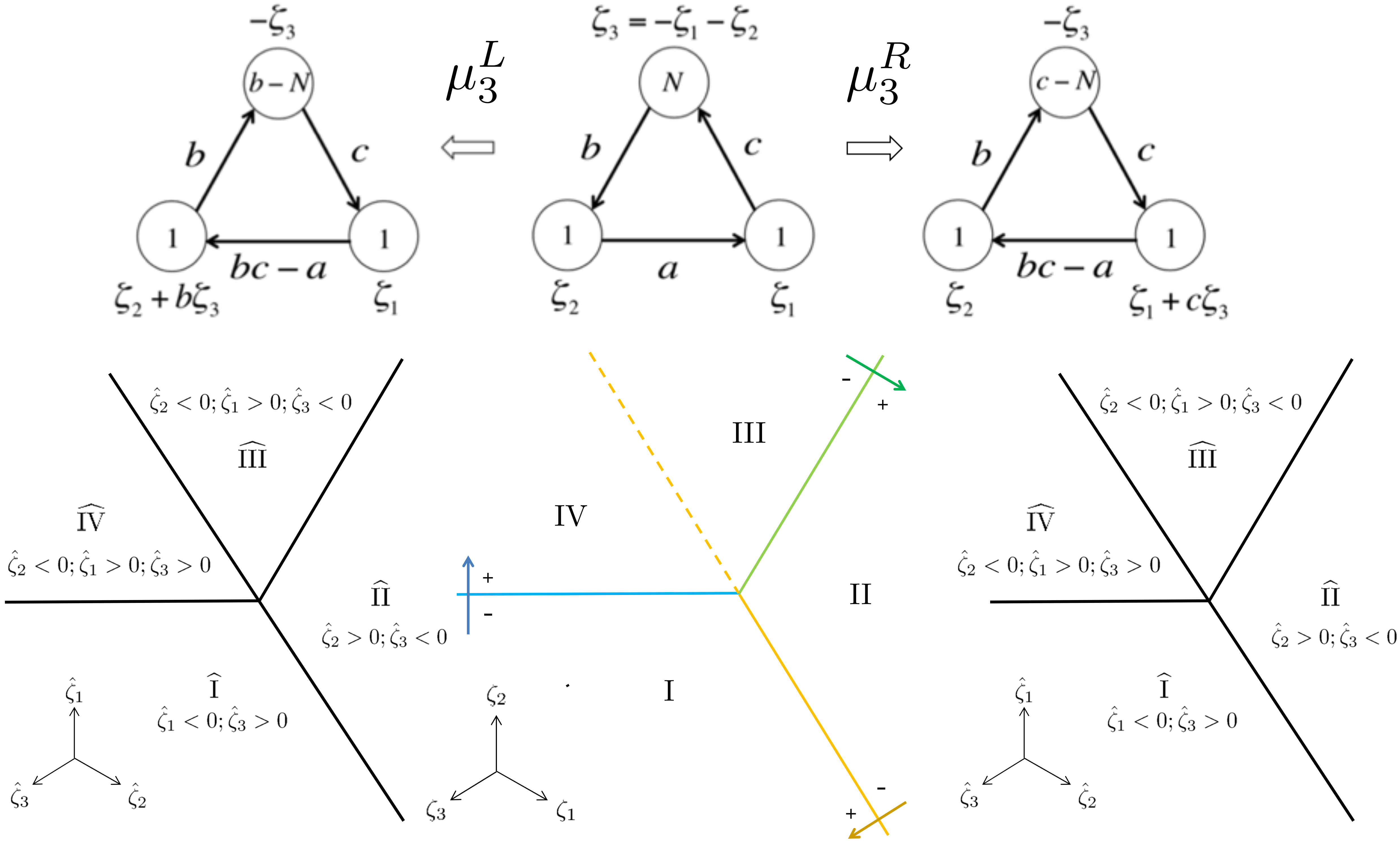




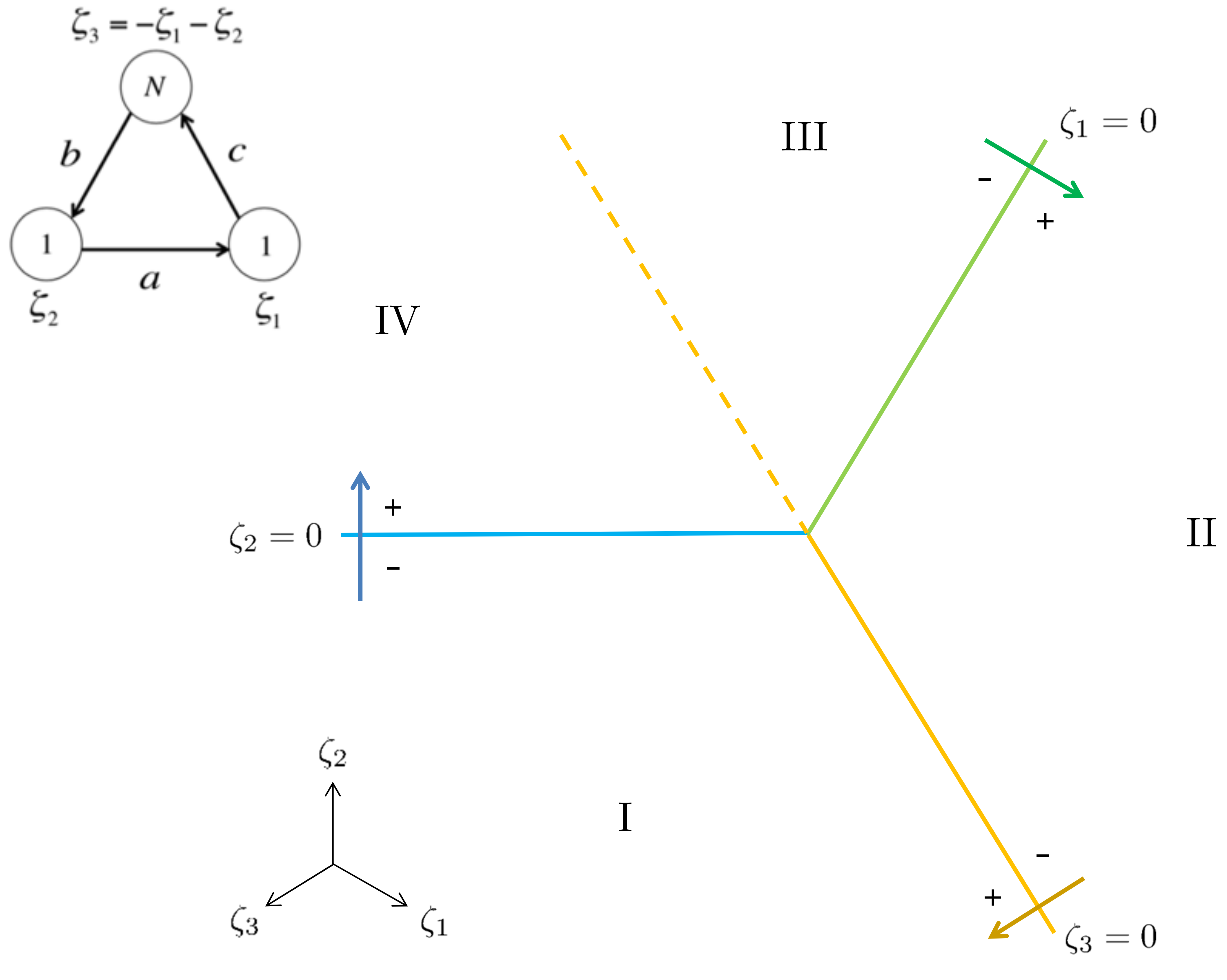
$\zeta_3 = -\zeta_1 - \zeta_2$
N
b
c
1
1
a
ζ_2
ζ_1

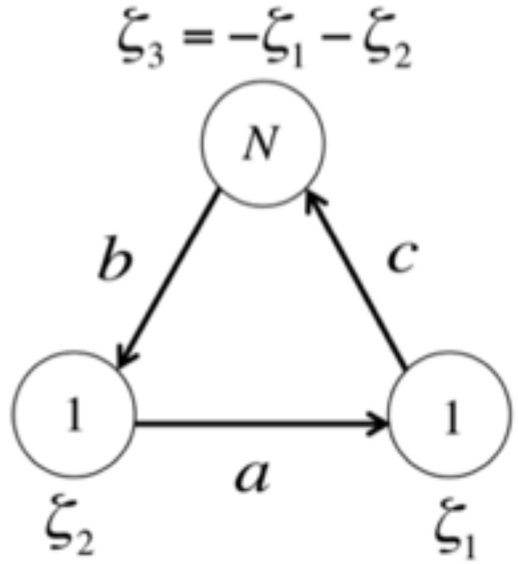


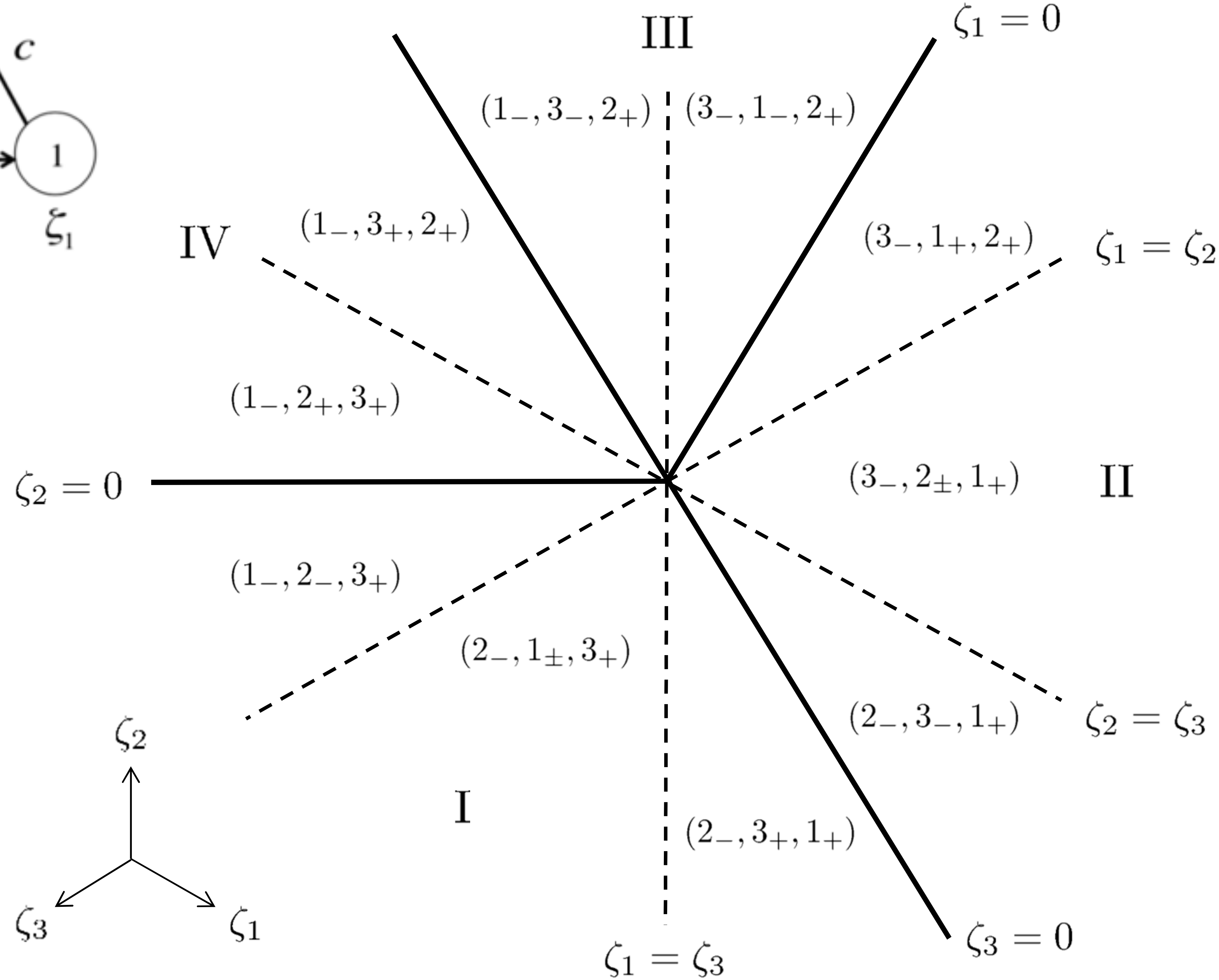
III
IV
II
I
+
-
ζ_2
ζ_3
ζ_1

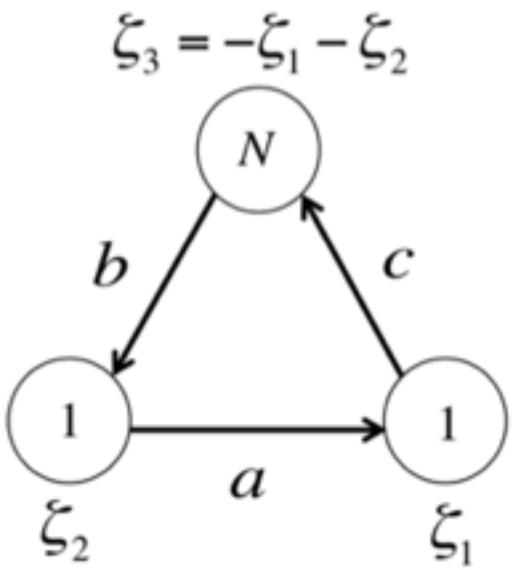


$-\zeta_3$
$b-N$
b
c
1
1
$bc-a$
$\zeta_2+b\zeta_3$
ζ_1
μ_3^L
$\zeta_3=-\zeta_1-\zeta_2$
N
b
c
1
1
a
ζ_2
ζ_1
μ_3^R
$-\zeta_3$
$c-N$
b
c
1
1
$bc-a$
ζ_2
$\zeta_1+c\zeta_3$
$\hat{\zeta}_2<0;\hat{\zeta}_1>0;\hat{\zeta}_3<0$
$\widehat{\mathrm{III}}$
$\widehat{\mathrm{IV}}$
$\hat{\zeta}_2<0;\hat{\zeta}_1>0;\hat{\zeta}_3>0$
$\widehat{\mathrm{II}}$
$\hat{\zeta}_2>0;\hat{\zeta}_3<0$
$\widehat{\mathrm{I}}$
$\hat{\zeta}_1<0;\hat{\zeta}_3>0$
$\hat{\zeta}_1$
$\hat{\zeta}_3$
$\hat{\zeta}_2$
III
IV
II
I
ζ_2
ζ_3
ζ_1
$\hat{\zeta}_2<0;\hat{\zeta}_1>0;\hat{\zeta}_3<0$
$\widehat{\mathrm{III}}$
$\widehat{\mathrm{IV}}$
$\hat{\zeta}_2<0;\hat{\zeta}_1>0;\hat{\zeta}_3>0$
$\widehat{\mathrm{II}}$
$\hat{\zeta}_2>0;\hat{\zeta}_3<0$
$\widehat{\mathrm{I}}$
$\hat{\zeta}_1<0;\hat{\zeta}_3>0$
$\hat{\zeta}_1$
$\hat{\zeta}_3$
$\hat{\zeta}_2$

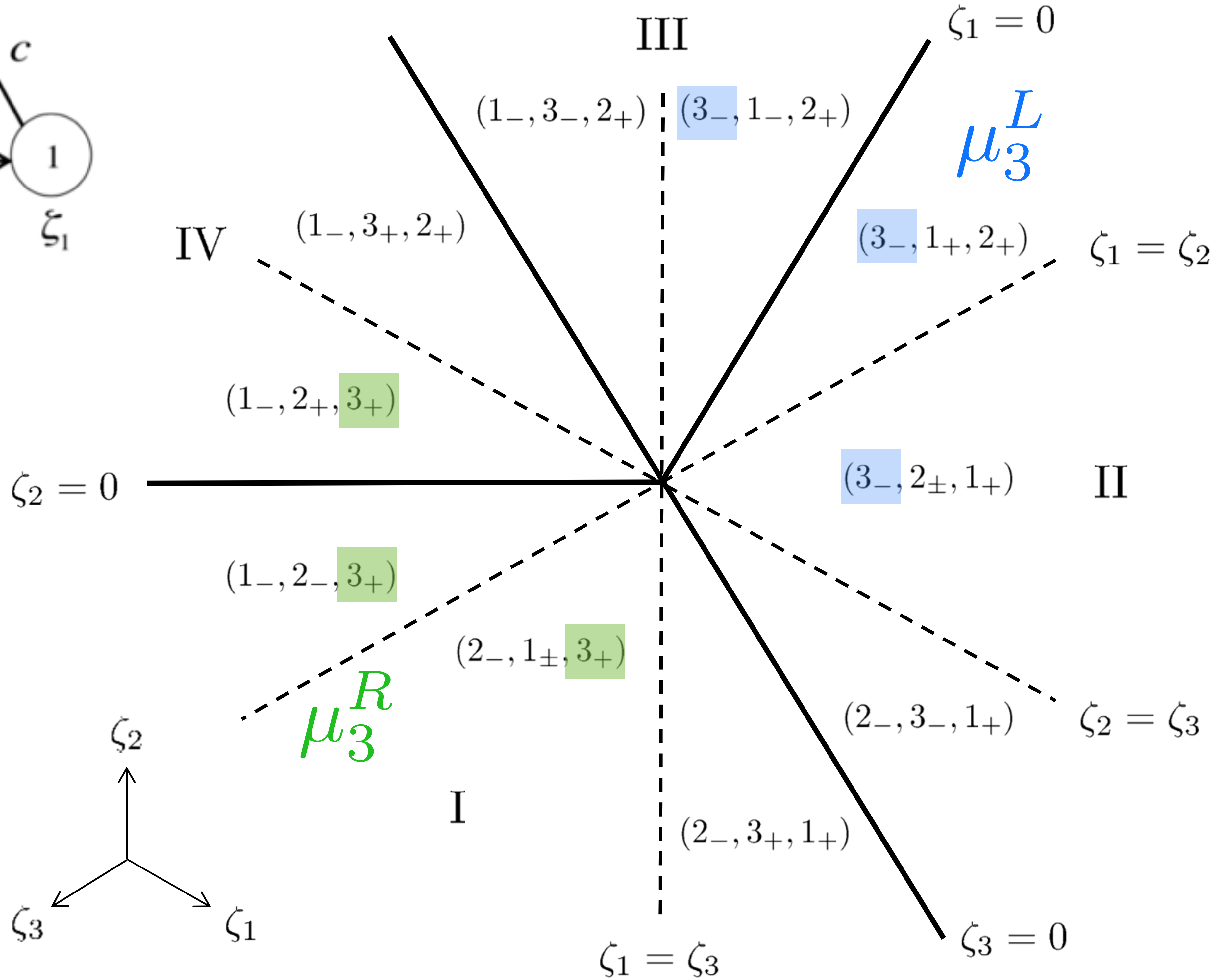

$\zeta_3 = -\zeta_1 - \zeta_2$
N
b
c
1
1
a
ζ_2
ζ_1
III
IV
II
I
$\zeta_1 = 0$
$\zeta_2 = 0$
$\zeta_3 = 0$
+
-
ζ_2
ζ_3
ζ_1


$\zeta_3 = -\zeta_1 - \zeta_2$
N
b
c
1
1
a
ζ_2
ζ_1

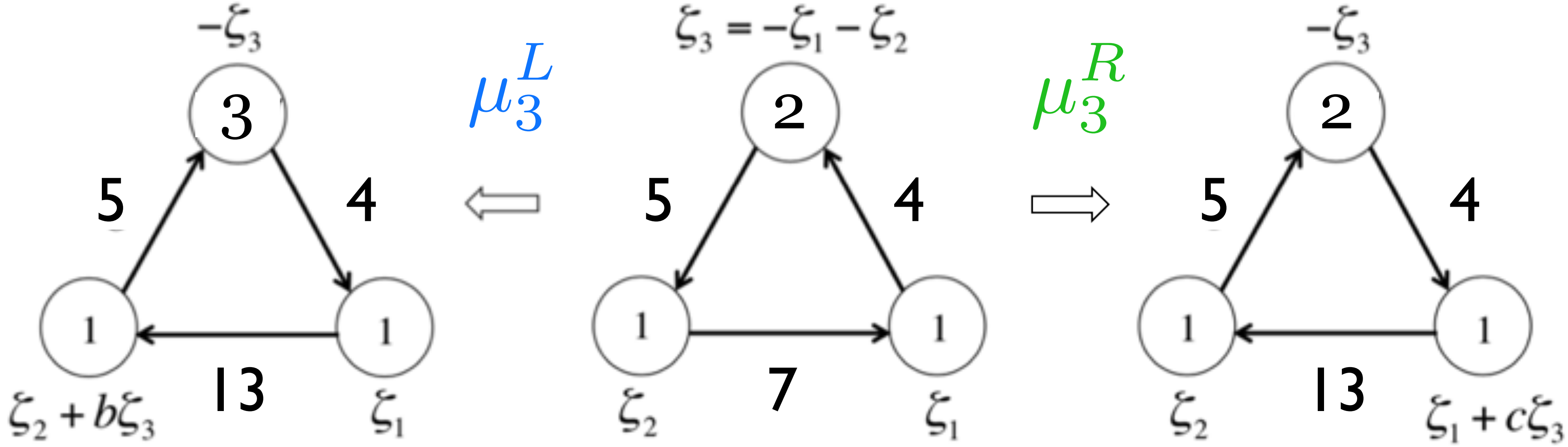

III
$\zeta_1 = 0$
$(1_-, 3_-, 2_+)$
$(3_-, 1_-, 2_+)$
IV
$(1_-, 3_+, 2_+)$
$(3_-, 1_+, 2_+)$
$\zeta_1 = \zeta_2$
$(1_-, 2_+, 3_+)$
$\zeta_2 = 0$
$(3_-, 2_\pm, 1_+)$
II
$(1_-, 2_-, 3_+)$
$(2_-, 1_\pm, 3_+)$
$(2_-, 3_-, 1_+)$
$\zeta_2 = \zeta_3$
I
$(2_-, 3_+, 1_+)$
$\zeta_3 = 0$
$\zeta_1 = \zeta_3$
ζ_2
ζ_3
ζ_1



Can we mutate with respect to node 3?

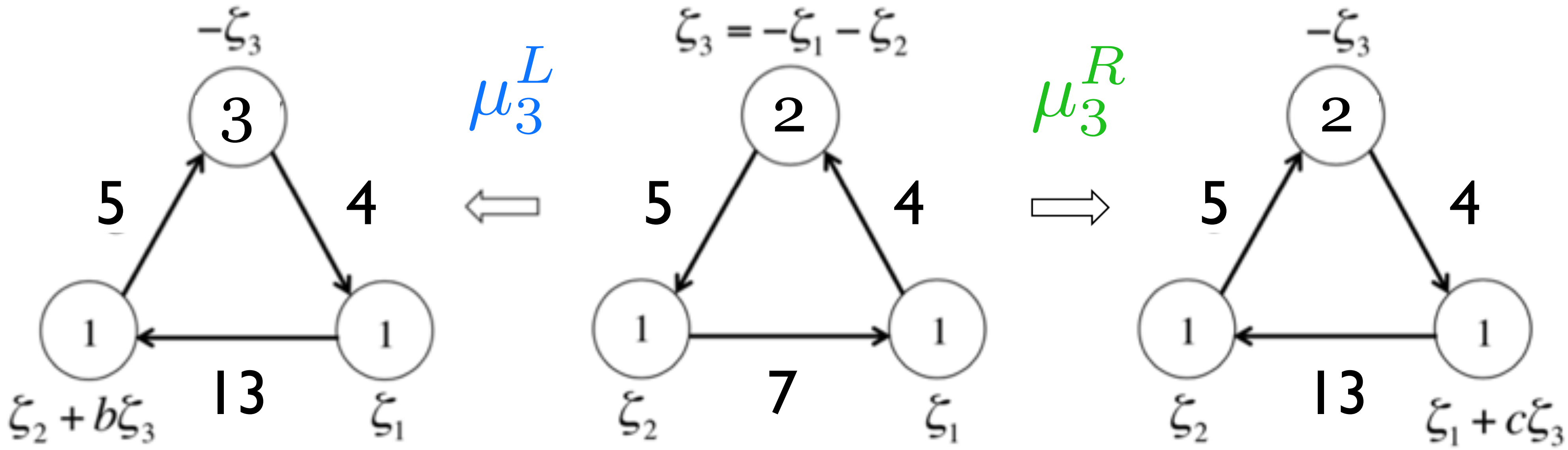


Thus, $\mu_3^L(\mathrm{Q})$ ($\mu_3^R(\mathrm{Q})$) must reproduce $\Omega_{\mathrm{Q}}(\mathrm{II})$ and $\Omega_{\mathrm{Q}}(\mathrm{III})$ ($\Omega_{\mathrm{Q}}(\mathrm{I})$ and $\Omega_{\mathrm{Q}}(\mathrm{IV})$).



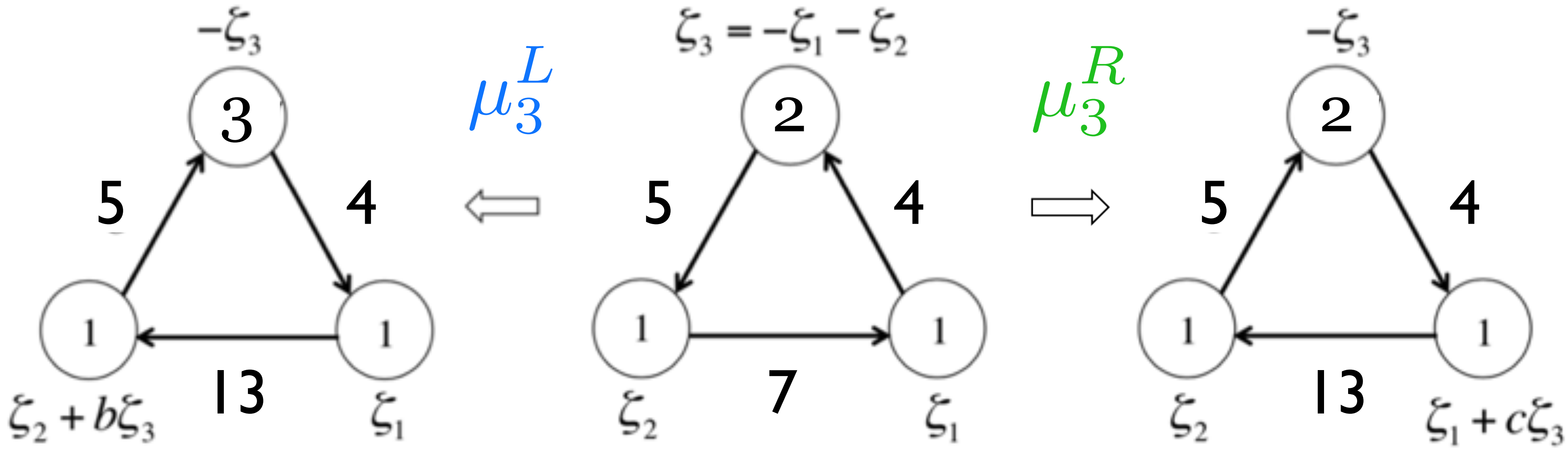
$\hat{\mathrm{Q}} \equiv \mu_3^L(\mathrm{Q})$ ($\mu_3^R(\mathrm{Q})$) must reproduce $\Omega_{\mathrm{Q}}(\mathrm{II})$ and $\Omega_{\mathrm{Q}}(\mathrm{III})$ ($\Omega_{\mathrm{Q}}(\mathrm{I})$ and $\Omega_{\mathrm{Q}}(\mathrm{IV})$).

For example, take a=7, b=5, c=4 and N=2.



$\hat{\mathrm{Q}} \equiv \mu_3^L(\mathrm{Q})$ ($\mu_3^R(\mathrm{Q})$) must reproduce $\Omega_{\mathrm{Q}}(\mathrm{II})$ and $\Omega_{\mathrm{Q}}(\mathrm{III})$ ($\Omega_{\mathrm{Q}}(\mathrm{I})$ and $\Omega_{\mathrm{Q}}(\mathrm{IV})$).

$\Omega(\widehat{\mathrm{I}})=?$	$\Omega(\mathrm{I})=?$	$\Omega(\widehat{\mathrm{I}})=?$
$\Omega(\widehat{\mathrm{II}})=?$	$\Omega(\mathrm{II})=?$	$\Omega(\widehat{\mathrm{II}})=?$
$\Omega(\widehat{\mathrm{III}})=?$	$\Omega(\mathrm{III})=?$	$\Omega(\widehat{\mathrm{III}})=?$
$\Omega(\widehat{\mathrm{IV}})=?$	$\Omega(\mathrm{IV})=?$	$\Omega(\widehat{\mathrm{IV}})=?$



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In principle one can compute all these indices via the Abelianisation.
But the toric varieties involved here are of dimension 20-ish,
meaning that one needs to deal with such high-rank lattices.

Furthermore, the analytical structure for Witten index is encoded only implicitly as one needs to extract the intersection numbers in a combinatorial manner.

Index of d=1 GLSM via Path Integral

[K.Hori, H.Kim, P.Yi `14]

(cf.) [Benini, Eager, Hori, Tachikawa `13], [Cordova, Chao `14], [Hwang, Kim, Kim, Park `14]

- Compact expression has been obtained:

$$\Omega(y;\zeta) = \frac{1}{|W|}\text{JK-Res}_\zeta\left[g(u)\mathrm{d}^r u\right]$$

where $u = x_3 + iA_0\,|_{\text{zero-mode}}$ are the zero modes of Cartan part, and the "integrand" is

$$g(u) = \prod_A g^{(A)}_{\text{vector}}(u)\prod_I g^{(I)}_{\text{chiral}}(u)$$

with $g^{(A)}_{\text{vector}}(u) = \left(\frac{1}{2\sinh\frac{z}{2}}\right)^{r_A}\prod_{\alpha\in\Delta_A}\frac{\sinh\frac{\alpha(u)}{2}}{\sinh\frac{\alpha(u)-z}{2}}$ and $g^{(I)}_{\text{chiral}}(u) = -\frac{\sinh\frac{q_I(u)+(\frac{R_I}{2}-1)z}{2}}{\sinh\frac{q_I(u)+\frac{R_I}{2}z}{2}}$

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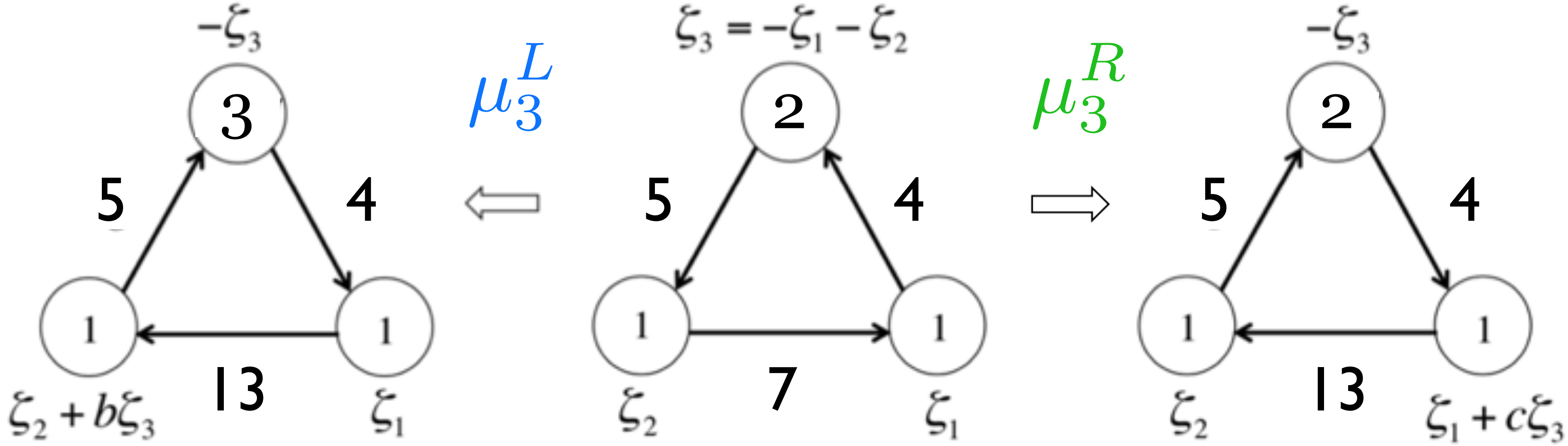
- The JK-Res is a sum over all co-dim "r" singularities in $(\mathbb{C}^\star)^r$, defined as intersection of hyperplanes via $\{Q_{i_1},\cdots,Q_{i_r}\}$:

$$\text{JK-Res}_{\zeta:\{Q_{i_1},\cdots,Q_{i_r}\}}\frac{\mathrm{d}^r u}{(Q_1\cdot u)\cdots(Q_r\cdot u)} = \begin{cases}\dfrac{1}{|\det(Q)|} & \text{if } \zeta\in\text{Span}_+\langle Q_{i_1},\cdots,Q_{i_r}\rangle\\[2ex] 0 & \text{otherwise}\end{cases}$$



$\hat{\mathrm{Q}} \equiv \mu_3^L(\mathrm{Q})$ ($\mu_3^R(\mathrm{Q})$) must reproduce $\Omega_{\mathrm{Q}}(\mathrm{II})$ and $\Omega_{\mathrm{Q}}(\mathrm{III})$ ($\Omega_{\mathrm{Q}}(\mathrm{I})$ and $\Omega_{\mathrm{Q}}(\mathrm{IV})$).

$$\boxed{\Omega(\mathrm{I}) = 50 \,,}$$

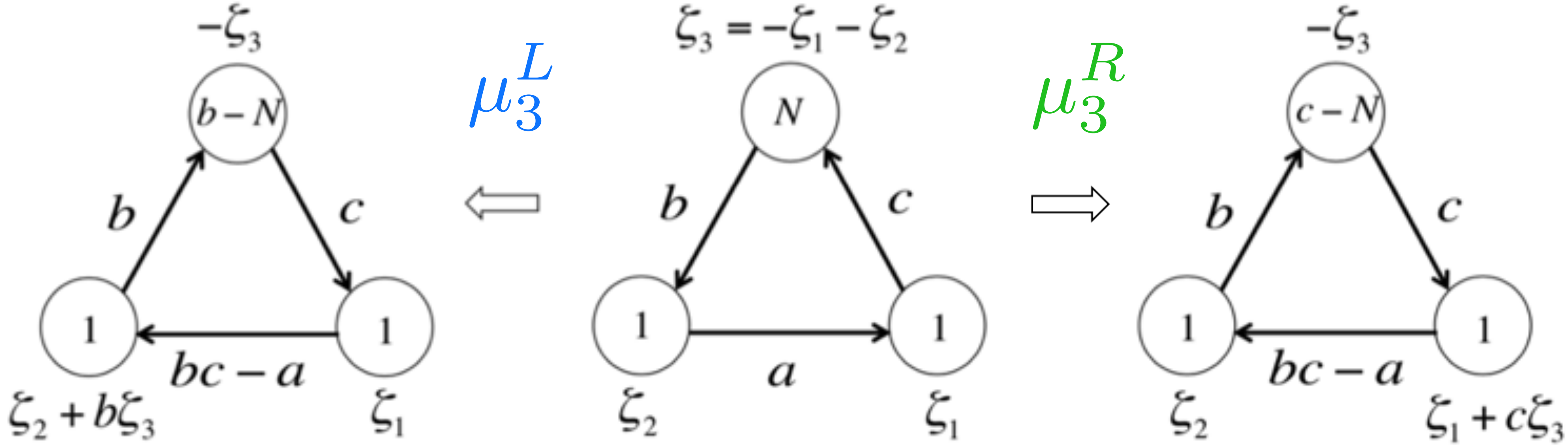
$$\boxed{\Omega(\mathrm{II}) = 1/\mathbf{y}^4 + 2/\mathbf{y}^2 + 87 + 2\mathbf{y}^2 + \mathbf{y}^4 \,,}$$

$$\boxed{\Omega(\mathrm{III}) = 1/\mathbf{y}^6 + 2/\mathbf{y}^4 + 4/\mathbf{y}^2 + 89 + 4\mathbf{y}^2 + 2\mathbf{y}^4 + \mathbf{y}^6 \,,}$$

$$\boxed{\Omega(\mathrm{IV}) = 1/\mathbf{y}^6 + 2/\mathbf{y}^4 + 4/\mathbf{y}^2 + 54 + 4\mathbf{y}^2 + 2\mathbf{y}^4 + \mathbf{y}^6 \,.}$$

$$\begin{aligned}
&\boxed{\Omega(\widehat{\mathrm{I}}) = 1/\mathbf{y}^6 + 2/\mathbf{y}^4 + 4/\mathbf{y}^2 + 89 + 4\mathbf{y}^2 + 2\mathbf{y}^4 + \mathbf{y}^6}\,, \\
&\Omega(\widehat{\mathrm{II}}) = 35 \,, \\
&\Omega(\widehat{\mathrm{III}}) = 1/\mathbf{y}^4 + 2/\mathbf{y}^2 + 37 + 2\mathbf{y}^2 + \mathbf{y}^4 \,, \\
&\boxed{\Omega(\widehat{\mathrm{IV}}) = 1/\mathbf{y}^4 + 2/\mathbf{y}^2 + 87 + 2\mathbf{y}^2 + \mathbf{y}^4 \,.}
\end{aligned}$$

$$\begin{aligned}
\Omega(\widehat{\mathrm{I}}) &= 1/\mathbf{y}^{10} + 2/\mathbf{y}^8 + 4/\mathbf{y}^6 + 6/\mathbf{y}^4 + 8/\mathbf{y}^2 \\
&\quad + 58 + 8\mathbf{y}^2 + 6\mathbf{y}^4 + 4\mathbf{y}^6 + 2\mathbf{y}^8 + \mathbf{y}^{10} \,, \\
&\boxed{\Omega(\widehat{\mathrm{II}}) = 1/\mathbf{y}^6 + 2/\mathbf{y}^4 + 4/\mathbf{y}^2 + 54 + 4\mathbf{y}^2 + 2\mathbf{y}^4 + \mathbf{y}^6,} \\
&\boxed{\Omega(\widehat{\mathrm{III}}) = 50 \,,} \\
\Omega(\widehat{\mathrm{IV}}) &= 50 \,.
\end{aligned}$$



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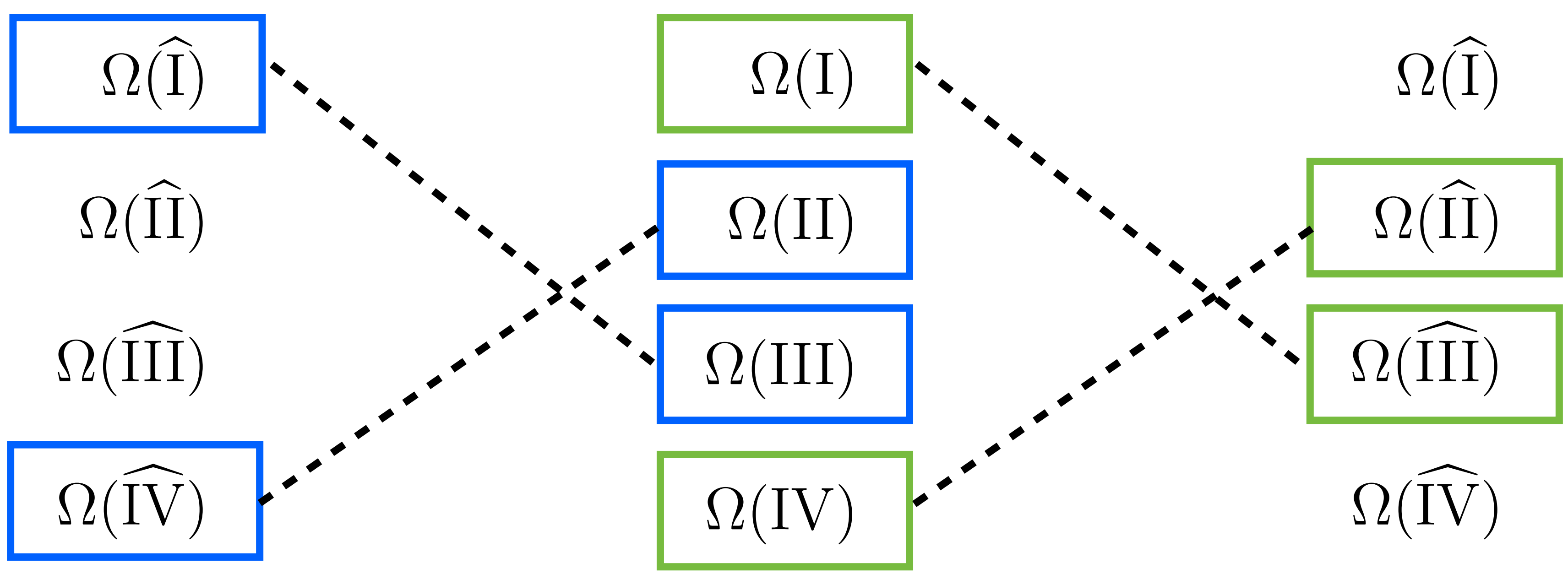

The JK-Res approach leads to the desired links even analytically.

Summary and Outlook

- d=4 N=2 BPS states were studied via d=1 N=4 Quiver GLSM
- Wall-crossing-sensitive indices have wall-crossing-safe invariants
- The *quiver invariants* of an abelian cyclic quiver theory are naturally characterised as the "middle" cohomology;

  non-abelian generalisation of the geometric interpretation?
- The moduli space geometry for a non-abelian quiver can be tackled via abelianisation and/or path integral
- Mutation of d=1 quiver theory can only be *selectively* performed to preserve Witten index
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Thank you!