GEOMETRIC CONSTRAINTS on the Space of $\mathcal{N}=2$ SCFTs

work with M. Lotito, Y. Lü, M. Martone, arXiv:1504.vwxyz

I. The problem: the relation of moduli spaces to CFTs
II. $\mathcal{N}=2$ moduli space basics
III. $\mathcal{N}=2$ SCFTs & scale-invariant moduli spaces
IV. Deformations of $\mathcal{N}=2$ SCFTs & moduli spaces
V. No dangerously irrelevant operator conjecture.
VI. Classification of rank-1 Coulomb branches.
VII. Future directions

- Will separate physical conditions moduli spaces must obey ! from the conjectural conditions used in the literature. ?

- Will present a new conjecture characterizing physical deformations of Coulomb branch geometries, & describe evidence for it.

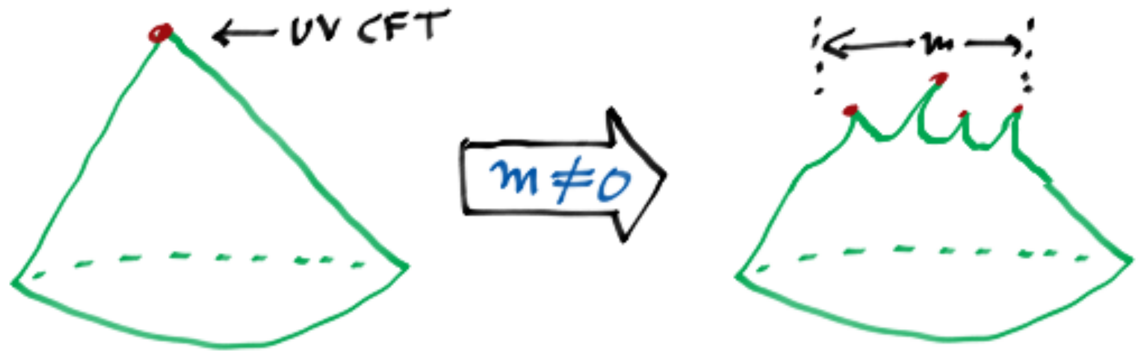
I. THE PROBLEM

How can we systematically construct the moduli space geometries of 4d $N=2$ QFTs?

- Many results (20 years of SW theory) but all are ad hoc: what possibilities are being missed?

- How do we connect moduli space geometries (= low energy effective actions) to microscopic SCFT data?

Strategy: (1) classify scale-invariant moduli spaces $\overset{?}{\Leftrightarrow}$ CFTs
then (2) classify possible deformations of " " $\overset{?}{\Leftrightarrow}$ def's of CFTs



Scale-invariant $\mathcal{M}$ Deformed $\mathcal{M}$

I. THE PROBLEM

I will start general, but will quickly simplify, as the general problem is out of reach.

Simplify assumptions:

- Look only at "Coulomb branch" (CB)
- "Rank" $\equiv \dim_{\mathbb{C}}(CB) = 1$
- Assume CB geometry is "regular" (SK hol. symplectic structure non-degenerate)

Will mention at end some of what happens when you (try to) lift these assumptions.

II. MODULI SPACE BASICS

"CB" Coulomb branch

- low energy theory has only massless $U(1)$ vector multiplets
- coordinates $= (u)$ are vevs of complex scalar in " "
- geometry is special Kähler, locally $\simeq \mathbb{C}^r$ ↖ complex dim.

"HB" Higgs branch

- low energy theory has only massless neutral hypermultiplets
- coordinates $= (q, \tilde{q})$ are vevs of 2 complex scalars in "
- geometry is hyperkähler, locally $\simeq \mathbb{H}^h$ ↖ quaternionic dim

"MB" Mixed branch

- both massless neutral hypers & $U(1)$ vectors
- scalars of both multiplets get vevs
- geometry is locally a direct product Higgs × Coulomb $\simeq \mathbb{H}^{h'} \times \mathbb{C}^{r'}$.

II. MODULI SPACE BASICS

Special Kähler geometry $\quad CB \simeq \mathbb{C}^r \ni u^i$

- $\exists$ "special coordinates" = holomorphic section of rank-2r complex vector bundle over CB w/ $Sp(2r,\mathbb{Z})$ structure group.

$$\begin{pmatrix} a_D^i \\ a_i \end{pmatrix} \longrightarrow M \begin{pmatrix} a_D^i \\ a_i \end{pmatrix} \qquad M \in Sp(2r,\mathbb{Z})$$

$\Leftrightarrow$ low energy EM duality

- Integrability cond.: $\dfrac{\partial a_D^i}{\partial a_j} = \dfrac{\partial a_D^j}{\partial a_i} \equiv \tau^{ij}(u)$ $\Leftrightarrow$ low energy $U(1)^r$ cplgs

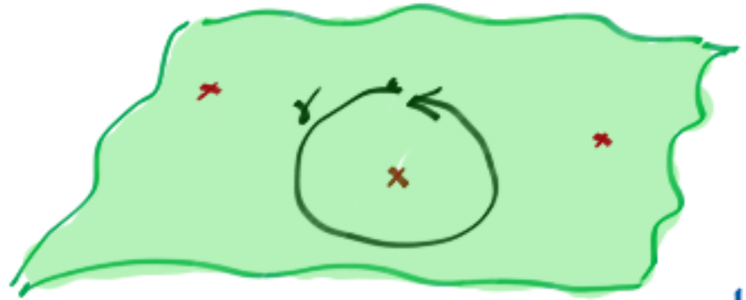
- Positivity cond.: $\operatorname{Im} \tau^{ij} > 0$

- Kähler pot'l $\mathcal{K} = \operatorname{Im}(\bar{a}_i a_D^i) \Rightarrow$ metric $g_{i\bar{j}} = \operatorname{Im} \tau^{ij}$

$\Leftrightarrow$ low energy vectorplet scalar kinetic terms

II. MODULI SPACE BASICS

Rank 1 case: $\mathcal{CB} \simeq \mathbb{C}$



$\gamma \to$ monodromy

$M_\gamma \in Sp(2,\mathbb{Z}) = SL(2,\mathbb{Z})$

$M \in SL(2,\mathbb{Z}) \Rightarrow$

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\begin{cases} a,b,c,d \in \mathbb{Z} \\ ad-bc=1 \end{cases}$

generated by

$S \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $T \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

& $-\mathbb{1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

- $M_\gamma = \mathbb{1}$ unless γ encloses a singularity.
- Integrability constraint is trivial.
- So only impose

$$M_\gamma : \tau \mapsto \frac{a\tau + b}{c\tau + d} \quad \& \quad \operatorname{Im}\tau > 0.$$

$\Rightarrow$ Not very constrained ...

III. $N=2$ SCFTs & SCALE-INVARIANT MODULI SPACES

- Unitary, positive-energy irreps of $N=2$ superconformal algebra [Dobrev & Petkova '85][Dolan & Osborn '02]:

$R \in \frac{1}{2}\mathbb{N}$ is $SU(2)_R$-spin.

Name		Lorentz spins	Dimension	$U(1)_r$	
general	$\mathcal{A}^{\Delta}_{R,r}$		$\Delta > 2R+2+\lvert r+j-\tilde{j}\rvert+j+\tilde{j}$		
1/4 BPS	$\mathcal{C}_{R,r}$		$\Delta = 2R+2-r+2j$	$r > \tilde{j}-j$	
1/2 BPS	$\hat{\mathcal{C}}_R$		$\Delta = 2R+2+j+\tilde{j}$	$r = \tilde{j}-j$	$\hat{\mathcal{C}}_1 \sim$ supercurrent
1/4 BPS	$\mathcal{B}_{R,r}$	$j=0$	$\Delta = 2R+r$	$r > \tilde{j}+1$	chiral
1/2 BPS	$\mathcal{D}_R$	$j=0$	$\Delta = 2R+r$	$r = \tilde{j}+1$	$\supset$ free vector
1/2 BPS	$\hat{\mathcal{B}}_R$	$j=\tilde{j}=0$	$\Delta = 2R$	$r=0$	$\begin{cases} \hat{\mathcal{B}}_0 = \mathbb{1} \\ \hat{\mathcal{B}}_{1/2} = \text{free hypers} \\ \hat{\mathcal{B}}_1 = \text{flavor curr.} \end{cases}$

III. $\mathcal{N}=2$ SCFTs & SCALE-INVARIANT MODULI SPACES

- Complex scalar primaries of $\mathcal{B}_{R,r}$ $(j=\tilde{j}=0)$ & $\hat{\mathcal{B}}_R$ multiplets form a chiral ring of OPE algebra.

$$(j=\tilde{j}=0)\left\{\begin{array}{llll} \mathcal{B}_{0,r} \equiv \mathcal{E}_r & \Longleftarrow \chi \text{ ring of CB} & \Rightarrow R=0: & SU(2)_R\text{-neutral} \\ \mathcal{B}_{R,r} \quad R \geq 1/2 & \Longleftarrow \quad '' \quad '' \quad \text{MB} & & \\ \hat{\mathcal{B}}_R & \Longleftarrow \quad '' \quad '' \quad \text{HB} & \Rightarrow r=0: & U(1)_r\text{-neutral} \end{array}\right.$$

(& $\mathcal{D}_0$ ~ free parts of CB χ ring)

$\hat{\mathcal{B}}_1^A \supset$ flavor current algebra F $\Rightarrow$ $A \in \mathrm{adj}(F)$

OPE selection rule $\Rightarrow$ $\mathcal{E}_r$ is flavor-neutral

$$\boxed{\text{Can have flavor action on HB, but not on CB}} \; !$$

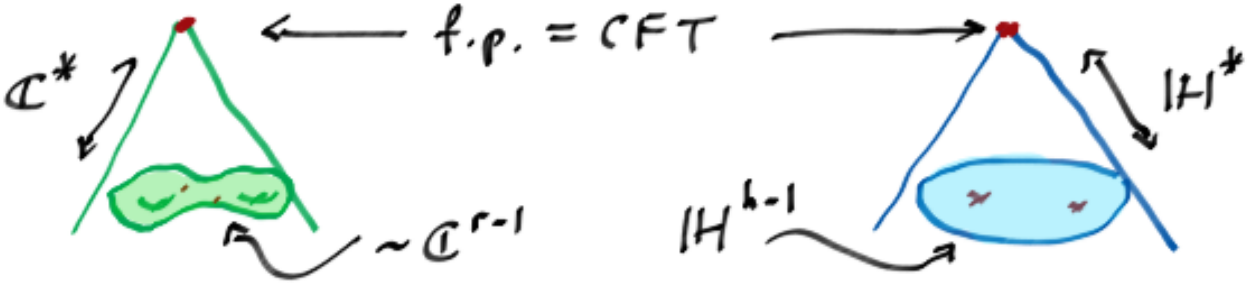
physical conditions

- Unitarity bound $\Rightarrow$ $\boxed{\begin{array}{lll} \Delta(\mathcal{E}_r) = r \geq 1 & \Rightarrow & \Delta(u) \geq 1 \\ \Delta(\hat{\mathcal{B}}_R) = 2R \in \mathbb{N} & \Rightarrow & \Delta(q) \geq 1 \end{array}}$!

III. $N=2$ SCFTs & SCALE-INVARIANT MODULI SPACES

How does this compare to moduli space geometry?

- Scale invariance + $\begin{cases} \text{hyperkähler} & \Rightarrow \text{hyperkähler cone} \\ \text{special Kähler} & \Rightarrow \text{special Kähler cone} \end{cases}$



$\left.\begin{array}{lll} \mathbb{C}^\times \text{ action} & \simeq \text{dilatations} + U(1)_r & \checkmark \\ \mathbb{H}^\times \text{ action} & \simeq \quad \text{"} \quad + SU(2)_R & \checkmark \end{array}\right\}$ basis for ident. w/ CB χring, HB χring

Conjectures (Tachikawa '12, Beem et al '14, implicit)

- $(\mathcal{E}, \mathcal{B}, \hat{\mathcal{B}})$ χ ring $\Leftrightarrow$ (CB, MB, HB) coord. ring
- CB χ ring freely generated $\Rightarrow$ $\overline{CB} \simeq \mathbb{C}^r$ globally ?
- HB always has non-trivial / faithful F action.

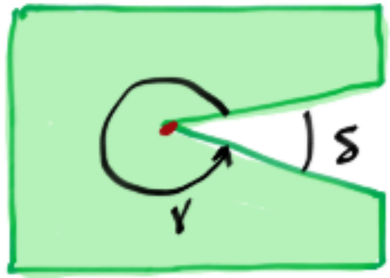
$\Updownarrow$

Basic question in CFT: When does a CFT have a moduli space of vacua (spont. breaking scale inv.)? Which φ can have $\langle\varphi\rangle \neq 0$?

III. $\mathcal{N}=2$ SCFTs & SCALE-INVARIANT MODULI SPACES

• Case of rank-1 special Kähler cones

$\simeq$



$\mathbb{C}$ w/ deficit $\measuredangle\ \delta$

Special Kähler $\Rightarrow$ $M_\gamma \in SL(2,\mathbb{Z})$ & $\operatorname{Im}\tau > 0$ $\Rightarrow$ allowed set

$$\left\{ \begin{array}{c|ccccccc|c|cc} \delta & \pi/3 & \pi/2 & \frac{2\pi}{3} & \pi & \frac{4\pi}{3} & \frac{3\pi}{2} & \frac{5\pi}{3} & 2\pi(\text{cusp}) & \multicolumn{2}{c}{\text{negative}} \\ \Delta(u) & 6 & 4 & 3 & 2 & 3/2 & 4/3 & 6/5 & \mathbb{N} & \underbrace{\frac{4}{\mathbb{N}} \quad \frac{6}{\mathbb{N}}}_{\text{violates } \Delta(u)\geq 1 \text{ unitarity}} \end{array} \right\}$$

• $\exists$ technical "regularity" assumption:

hol. symplectic form $\Omega \neq 0$ on $X \to CB$

$\Leftrightarrow$ ($\exists \neq 0$ section of canonical line bundle of X.)

Implicit conjecture:

$\boxed{\text{CB geometries are regular.}}$?

Rank-1 & regular $\Rightarrow$ $\Delta(u) \geq 1$ & "Kodaira classification"

BUT $\not\Leftarrow$: $\exists$ irreg. CB w/ $\Delta(u) \geq 1$ (& in Kodaira cl.)

III. $\mathcal{N}=2$ SCFTs & SCALE-INVARIANT MODULI SPACES

∘ Kodaira classification [Kodaira '64 '66]

Name	curve: $y^2 = \dots$	$\Delta(u)$	ord(D_x)	M_0	δ
II^*	$x^3 + u^5$	6	10	ST	$\pi/3$
III^*	$x^3 + u^3 x$	4	9	S	$\pi/2$
IV^*	$x^3 + u^4$	3	8	$-(ST)^{-1}$	$2\pi/3$
I_0^*	$x^3 + u^2 x + g u^3$	2	6	$-I$	π
IV	$x^3 + u^2$	$3/2$	4	$-ST$	$4\pi/3$
III	$x^3 + ux$	$4/3$	3	S^{-1}	$3\pi/2$
II	$x^3 + u$	$6/5$	2	$(ST)^{-1}$	$5\pi/3$
$I^*_{n>0}$	$x^3 + ux^2 + \Lambda^{-2n} u^{n+3}$	2	$n+4$	$-T^{-n}$	2π(cusp)
$I_{n>0}$	$(x-1)(x^2 + \Lambda^{-n} u^n)$	1	n	T^{-n}	2π(cusp)

$$\left.\begin{array}{l} I^*_{n\geq 0} = SU(2) \text{ gauge theories} \\ I_{n>0} = U(1) \quad '' \quad '' \end{array}\right\} \text{w/ } \beta_0 \propto n \Rightarrow \left\{\begin{array}{l} n=0 \Rightarrow \text{scale invariant} \\ n>0 \Rightarrow \text{IR-free} \end{array}\right.$$

• Compare to (known) rank-1 $\mathcal{N}=2$ CFTs & IR-free gauge theories, find:

$\boxed{\exists \text{ multiple examples for each Kodaira entry}}$ ☆

E.g. $\left\{\begin{array}{l} SU(2) \text{ w/ 4 fund. hypers} \\ SU(2) \text{ w/ 1 adj. hyper} \end{array}\right\} \Leftrightarrow$ same I_0^* sing., different theories

IV. DEFORMATIONS OF CFTs & MODULI SPACES

- A deformation of a CFT corresponds to an RG flow of QFTs.

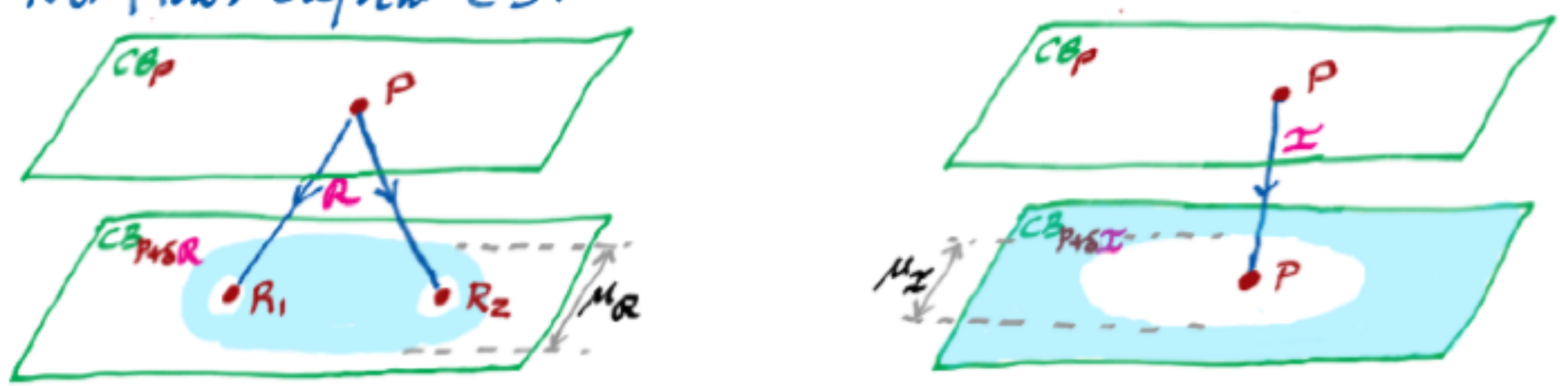
Conjecture: $\boxed{\text{Deformations by } N{=}2 \text{ operators do not spontaneously break } N{=}2 \text{ SUSY.}}$?

Fixed points have moduli spaces which deformations may lift or deform.

- By analyticity, separation of scales, $N{=}2$ non-renorm. thms in eff. act. ⇒

$\boxed{\begin{array}{l}\text{CB is not lifted,}\\ \text{HB may be lifted.}\end{array}}$! For this reason, focus on CB f.n.o.

How RG flows deform CB:



- Moduli spaces 'map' RG flows: record cross-over scales, not just fixed points.
- Relevant deformations ⇒ deform "local" vicinity of UV f.p. $\in \mathcal{M}$
- Irrelevant " ⇒ " "asymptotic" regions of $\mathcal{M}$.

IV. DEFORMATIONS OF CFTs & MODULI SPACES

- Now compare to $N=2$ SCFT data.

- Classify all possible local $N=2$ SUSY deformations of an $N=2$ SCFT

Find (new result): (following [Green et al. '10])

deformation op. $\delta_{n,m} \sim Q^n \tilde{Q}^m X$	$X \in$ SCA rep $(j=\tilde{j}=0)$ (R, r, Δ)	$\delta_{n,m}$ charges R_δ	r_δ	Δ_δ
$\delta_{0,0} = X$	$\hat{\mathcal{B}}_0 = 1$	0	0	0
$\delta_{0,2} = \tilde{Q}^2 X$	$\hat{\mathcal{B}}_R$, $R \geq 1$	$R-1$	-1	≥ 3
$\delta_{0,4} = \tilde{Q}^4 X$	$\mathcal{B}_{R,r}$ $r > 1$	R	$r-2$	> 3
$\delta_{2,2} = Q^2 \tilde{Q}^2 X$	$\hat{\mathcal{B}}_R$ $R \geq 2$	$R-1$	0	≥ 6
$\delta_{2,4} = Q^2 \tilde{Q}^4 X$	$\mathcal{B}_{R,r}$ $R \geq 1, r > 1$	$R-1$	$r-1$	> 6
$\delta_{4,4} = Q^4 \tilde{Q}^4 X$	$\mathcal{A}^{\Delta}_{R,r}$ $r \neq 0$	R	r	> 6

$\Rightarrow$ • Relevant defs: $\tilde{Q}^2 \hat{\mathcal{B}}_1^A$ $\Delta_\delta = 3$, $\in$ adj(F)

$\tilde{Q}^4 \mathcal{E}_r$ $3 < \Delta_\delta < 4$, F-singlet

• Marginal ($\Delta_\delta = 4$) defs: $\tilde{Q}^2 \hat{\mathcal{B}}_{3/2}$ $\Rightarrow$ marginally irrelevant [GKSTW '10]

$\tilde{Q}^4 \mathcal{E}_2$ $\Rightarrow$ exactly marginal $\Leftrightarrow$ "gauge coupling"

IV. DEFORMATIONS OF CFTs & MODULI SPACES

∘ Relevant operators. 3 kinds for $\mathcal{N}=2$ SCFT:

(NB: Breaks SUSY unless $[m,\bar{m}]=0$)

1) Semi-simple mass: $\delta S = m_A \int d^4x\, \tilde{Q}^2 \hat{B}_1^A + c.c.$ *

$\Rightarrow \Delta(m_A)=1$ & breaks $F \to U(1)^{\mathrm{rank}(F)} \rtimes \mathrm{Weyl}(F)$ $\Rightarrow$ $\boxed{\text{residual symm. on CB}}$!

2) U(1) mass: $\delta S = m \int d^4x\, \tilde{Q}^2 \hat{B}_1$ w/ $F=U(1)$

If also $\Delta(u)=1$ ($\Rightarrow \exists$ free vectmplet) $\Rightarrow$ $\boxed{\text{shifts but does not deform CFT singularity}}$

3) Chiral term: $\delta S = \mu \int d^4x\, \tilde{Q}^4 \mathcal{E}_r + h.c.$ $(1<r<2)$

$\Rightarrow \Delta(\mu) = 2-r$. Since $\mathcal{E}_r$ in CB χring $\Rightarrow$ $\boxed{\text{If } \exists \text{ CB condit. w/ } 1<\Delta(u)<2, \text{ then } \exists \text{ rel. deformation}}$

☆ { Precisely these kinds of deformations of CB's are found in known examples [cf. Gaiotto, Seiberg, Tachikawa '10].

IV. DEFORMATIONS OF CFTs & MODULI SPACES

- **Deformation topology:** Restrict to rank-1 CBs $\simeq \mathbb{C}$

  Using properties of special Kähler geometry (non-positivity of scalar curvature, discreteness of $SL(2,\mathbb{Z})$ monodromies), can show that:

  Mass- & chiral-term deformations must split singularities

  with one exception (U(1) IR-free g.t.):

  U(1) mass-term when $\Delta(u)=1$ only shifts singularity.

- Further constraints from SK geometry:
  (1) $[M_\infty] \Leftrightarrow$ Kodaira type
  (2) $SL(2,\mathbb{Z})$: $M_\infty = M_2 M_1$ (difficult)
  (3) $\mathrm{ord}_0(D_x)$: degree of discriminant polynomial invariant

IV. DEFORMATIONS OF CFTs & MODULI SPACES

$\boxed{\text{Assume: REGULAR, RANK-1}}$

Kodaira ⇒ Describe splitting by its "Deformation Pattern" = the list of Kodaira types that the sing. splits to.

E.g. $II^* \to \underbrace{\{I_1^6, I_4\}}_{\text{Deformation pattern}}$

Discriminant ⇒ 383 distinct possible deformation patterns.

$SL(2,\mathbb{Z})$ ⇒ $\gtrsim 200$ distinct deform. patterns w/ correct $\mathcal{M}_\infty$.

KODAIRA SINGULARITIES			
Kodaira	$\Delta(u)$	ord(D_x)	gauge
II^*	6	10	—
III^*	4	9	—
IV^*	3	8	—
I_0^*	2	6	$SU(2)$
IV	3/2	4	—
III	4/3	3	—
II	6/5	2	—
IRF $I_{n>0}^*$	2	$n+6$	$SU(2)$
IRF $I_{n>0}$	1	n	$U(1)$

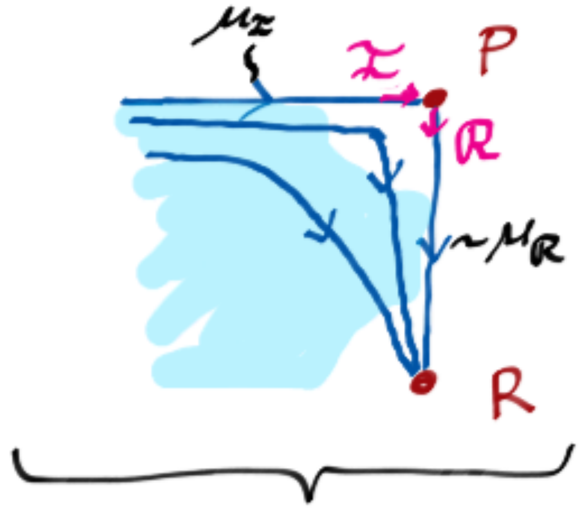
- The maximal deformation always allowed

$$\text{sing} \to \{I_1^{\text{ord}(D_x)}\} \qquad \text{e.g. } II^* \to \{I_1^{10}\}.$$

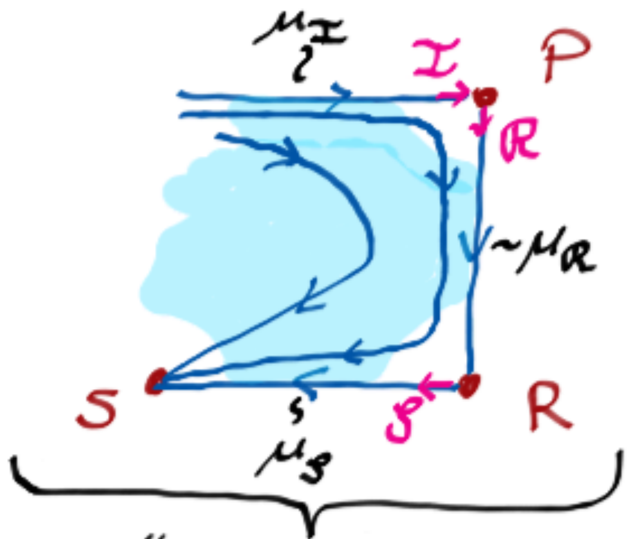
The CB SK structure is explicitly constructed for all maximal deformations [PCA, Plesser, Seiberg, Witten '95] [Minahan, Nemeschansky '96]

Reminder of dangerously irrelevant operators:

- 2 RG flow topologies:



I is "safely irrelevant"

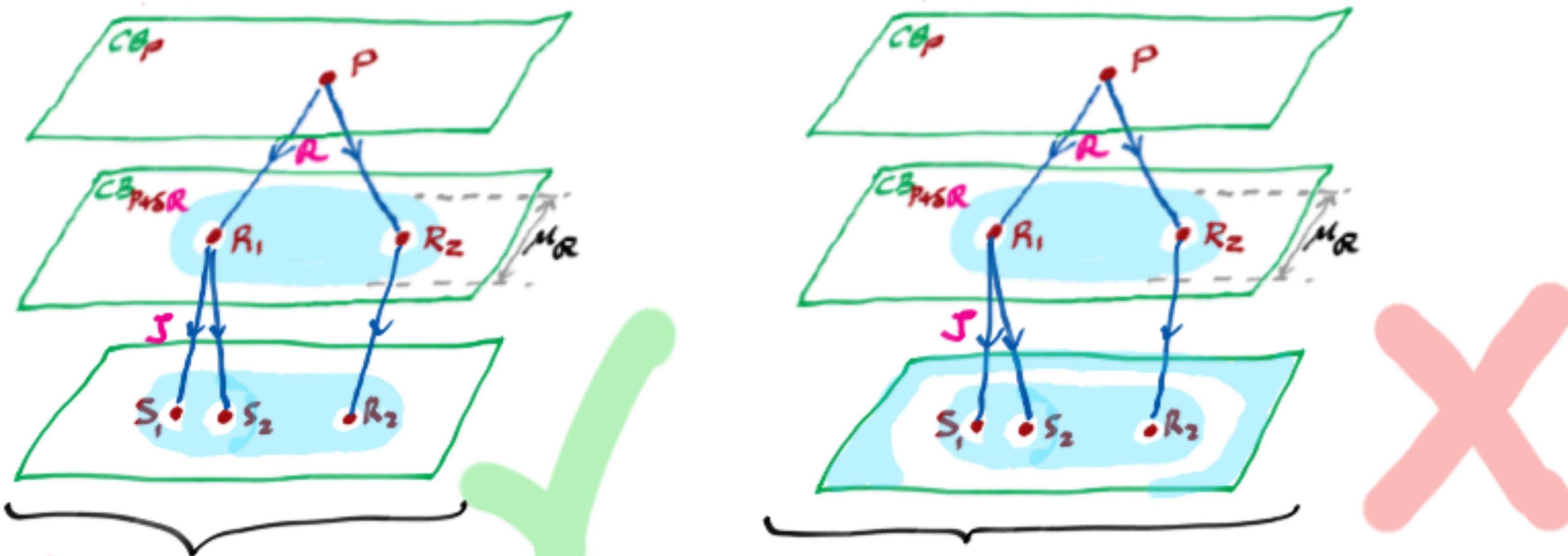


I is "dangerously irrelevant"

☆ Fact @ rank 1: Every sub-maximal deformation can be enlarged to the maximal deformation by adding more relevant parameters while remaining special Kähler.

V. NO DANGEROUSLY IRRELEVANT OPERATORS CONJECTURE

- What this means for RG flows:



J is relevant at P ✓

$\exists$ dangerously irrd. op at P ✗

- This is *evidence* for the

⇒ Conjecture: $\boxed{\text{No dangerously irrelevant operators in } N=2 \text{ field theories}}$

- Implies:
  - No accidental semi-simple flavor symmetry in IR.
  - Under generic deformation, <u>IR sing.s admit no further splittings</u>.

V. NO DANGEROUSLY IRRELEVANT OPERATORS CONJECTURE

∴ Which Kodaira sings. correspond to CFTs / IR-free theories which admit no further splitting?

- II, III, IV, I_0^* ⇒ all have $1 < \Delta(u) \leq 2$
  ∴ admit chiral term deformation.

- IV^*, III^*, II^* ⇏ no evidence for (or against) existence of non-deformable SCFTs w/ these singularities.

KODAIRA SINGULARITIES

Kodaira	$\Delta(u)$	$\mathrm{ord}(D_x)$	gauge
II^*	6	10	—
III^*	4	9	—
IV^*	3	8	—
I_0^*	2	6	$SU(2)$
IV	3/2	4	—
III	4/3	3	—
II	6/5	2	—
IRF $I_{n>0}^*$	2	$n+6$	$SU(2)$
IRF $I_{n>0}$	1	n	$U(1)$

IR-FREE SINGULARITIES CONJECTURE ⇒ $\boxed{\text{Under generic deformation, IR singularities all correspond to non-deformable IR-free theories.}}$

↑ dispensable: if not, then ∃ a few more allowed CB geometries

V. NO DANGEROUSLY IRRELEVANT OPERATORS CONJECTURE

- Rank-1 IR-free theories are:
  - U(1) gauge theories w/ charged massless matter (= Kodaira type I_n)
  - SU(2) gauge theories w/ enough massless matter (= Kodaira type I_n^*)
- Which of these admit no splittings ($\sim$ relevant deformations)?

$\ast$ U(1) w/ 1 hypermultiplet of charge $\sqrt{n} \Leftrightarrow I_n$, $n \in \mathbb{N}$.

$\ast$ SU(2) w/ ½-hypermultiplet in spin-$\frac{4n-1}{2}$ irrep $\Leftrightarrow I^*_m$ $\quad m = \frac{n(16n^2-1)}{3} - 4$

- The U(1)-flavor mass term for a single hypermultiplet charged under a U(1) gauge group shifts but does not split the I_n singularity.
- An SU(2) spin-3/2 "half"-hypermultiplet has no ($N=2$) mass term! So the I_1^* singularity is not deformed.

VI. CLASSIFICATION OF RANK-1 COULOMB BRANCHES

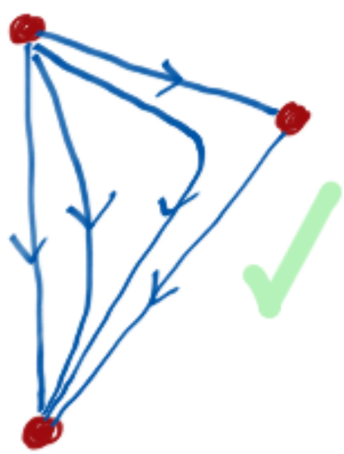
- Assuming the "IR-free" & "no dangerously irrelevant" conjectures gives us 2 powerful constraints on possible CB deformations

(1) Consistency with Dirac quantization

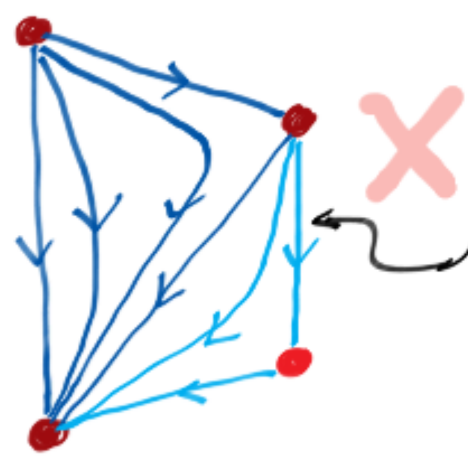
E.g. $I_0^* \to \{I_1^4, I_2\}$ $\Rightarrow$ $\exists$ hypers on CB w/ charges 1 & $\sqrt{2}$
$\Rightarrow$ incommensurate $\therefore$ inconsistent #

but $I_0^* \to \{I_2^3\}$ $\Rightarrow$ hyper charge $\sqrt{2}$ $\Rightarrow$ commensurate
and $I_0^* \to \{I_1^2, I_4\}$ $\Rightarrow$ " " 1 & $\sqrt{4}$ $\Rightarrow$ "
$\therefore$ consistent ✓

(2) Consistency with RG flow



Consistent with no dangerously-irrelevant operators



"Extra" relevant deformation $\Rightarrow$ $\exists$ dangerously-irrelevant op.

VI. CLASSIFICATION OF RANK-1 COULOMB BRANCHES

∘ These constraints narrow the possibilities considerably:

EASY	# DEFORMATION PATTERNS	383
	$\cup$	$\downarrow$
	# IR-FREE SINGULARITIES	118
	$\cup$	$\downarrow$
	# DIRAC QUANTIZATION	26
	$\cup$	$\downarrow$
~	# $SL(2,\mathbb{Z})$ MONODROMIES	16
	$\cup$	$\downarrow$
HARD	# CONSTRUCT SK GEOMETRIES	15
	$\cup$	$\downarrow$
	# RG FLOWS	12

$\nearrow$ Depends on MN ansatz
But sol'n always exists.

$\Rightarrow$ One equivalence
$I_0^* \to \{I_1^2, I_4\} \simeq \{I_2^3\}$.

$\Rightarrow$ Allows extraction of flavor symmetry

VI. CLASSIFICATION OF RANK-1 COULOMB BRANCHES

RESULT

Kodaira Sing.	Deformation pattern	Flavor symm.	
II^*	$\{I_1^{10}\}$	E_8	
	$\{I_1^6\, I_4\}$	C_5	
III^*	$\{I_1^9\}$	E_7	
	$\{I_1^5\, I_4\}$	$C_3 \oplus C_1$	
IV^*	$\{I_1^8\}$	E_6	
	$\{I_1^4\, I_4\}$	$C_2 \oplus U_1$	
	$\{I_1, I_1^*\}$	U_1	$\Leftarrow$ NEW! Does it exist in S-class?
I_0^*	$\{I_1^6\}$	D_4	$\leftarrow$ $SU(2)$ $N_f=4$
	$\{I_1^2, I_4\} \simeq \{I_2^3\}$	C_1	$\leftarrow$ $SU(2)$ $N=2^*$
IV	$\{I_1^4\}$	A_2	
III	$\{I_1^3\}$	A_1	
II	$\{I_1^2\}$	$\emptyset$	

KODAIRA SINGULARITIES

	Kodaira	$\Delta(u)$	$\mathrm{ord}(D_x)$	gauge
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	II	$6/5$	2	—
IRF	$I_{n>0}^*$	2	$n+6$	$SU(2)$
	$I_{n>0}$	1	n	$U(1)$

VIII. FUTURE DIRECTIONS

1) IRREGULAR rank-1 geometries (IRREGULAR ≡ holomorphic symplectic form degenerates)

- Now organized by the order of the irregularity, $i \in \mathbb{N}$
  - ($i=0 \Leftrightarrow$ regular)
  - $i=$ fixed $\Rightarrow$ get similar finite, tightly-constrained, classification as for $i=0$.
  - Don't know if terminates, i.e. if $\exists\, i_*$ s.t. no solutions for $i > i_*$.

- Qualitatively new behaviors allowed (but not required):
  - 

multiple CB branches
  - 

higher-genus CBs

would-be counterexamples to freely-generated CB ring conjecture

  - 

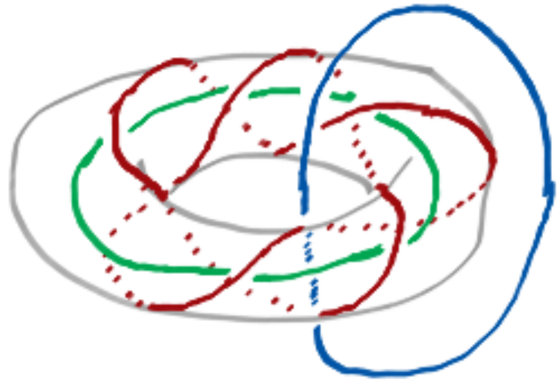
N=2 SCFTs w/o CB ??

VII. FUTURE DIRECTIONS

2.) RANK-2 CBs

- First step: analog of Kodaira classification, but for scale-invt & IR-free $\dim_{\mathbb{C}}=2$ SK geometries.
  - Quite intricate: >25 known (not ≃ Kod × Kod′)
  - Can be reduced to classifying reps of π_1(knot) in "$Sp(4,\mathbb{Z})$" subject to rep ⊂ {elliptic} ∪ {parabolic}

knot ≃



torus knot + links

- Next step: constructing SK deformations, ...

3.) Flat directions from CFT data

- What are the conditions on the CFT local operator algebra that allow a scalar operator get a v.e.v.?
- If moduli space exists, how to derive its effective action from the CFT operator algebra?

THANKS!