Quantum Information, Machine Learning and Knot Theory

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Based on

- V. Balasubramanian, J. R. Fliss, R.G. Leigh & OP, JHEP 1704 (2017) 061, arXiv:1611.05460.
- V. Balasubramanian, M. DeCross, J. R. Fliss, Arjun Kar, R.G. Leigh & OP, arXiv:1801.0113.
- V. Jejjala, A. Kar & OP, arXiv:1902.05547.

Part 1: Quantum Information

Entanglement Entropy in Qubits: Brief Overview

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- This is to be contrasted against unentangled product states like

$$|0\rangle \otimes |0\rangle, \;\; |0\rangle \otimes |1\rangle, \;\; |+\rangle \otimes |+\rangle \text{ etc.}$$

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$$|GHZ\rangle = \frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big).$$

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- On the contrary, the W-state is not separable:

$$\mathrm{Tr}_3|W\rangle\langle W| = \frac{1}{3}|00\rangle\langle 00| + \frac{2}{3}|\Psi^+\rangle\langle\Psi^+|, \qquad |\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}.$$

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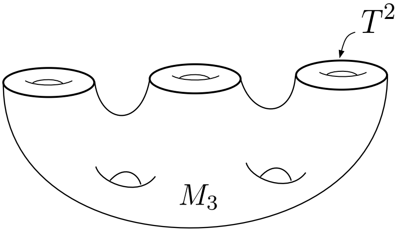
- We will consider the theory for gauge groups $U(1)$ and $SU(2)$.

Which states?

- The states we will consider are created by performing the Euclidean path integral of Chern-Simons theory on 3-manifolds M_n with boundary consisting of n copies of T^2.



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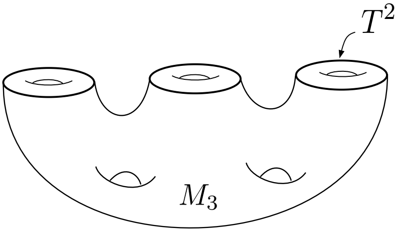

- For a given M_n of this form, the path-integral of Chern-Simons theory on M_n defines a state

$$|\Psi\rangle \in \mathcal{H}(T^2) \otimes \mathcal{H}(T^2) \otimes ... \otimes \mathcal{H}(T^2)$$

$$\Psi[A_{(0)}] = \int_{A|_\Sigma = A_{(0)}} [DA]\ e^{iS_{CS}[A]}$$

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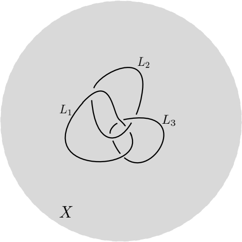
- We start with a closed 3-manifold (i.e., a compact 3-manifold without boundary) X, and an n-component link in X

$$\mathcal{L}^n = L_1 \cup L_2 \cup \cdots L_n$$



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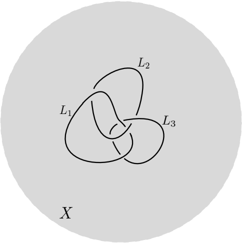

- Let us take X to be the 3-sphere S^3 for simplicity.

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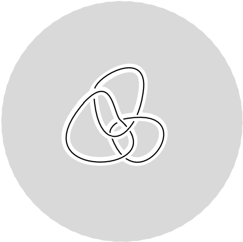

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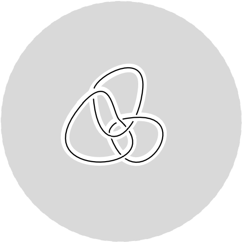

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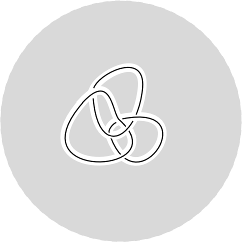

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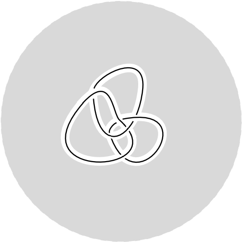

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- The path-integral of Chern-Simons theory on the link-complement assigns to a link $\mathcal{L}^n$ in S^3 a state $|\mathcal{L}^n\rangle \in \mathcal{H}(T^2)^{\otimes n}$.

The Hilbert space on a Torus

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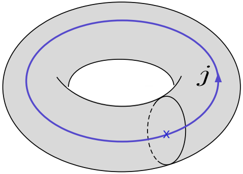

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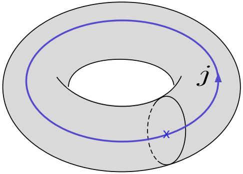

- The Hilbert space is finite dimensional for compact groups. (For $SU(2)$, the basis is labelled by spins $j = 0, \frac{1}{2}, \cdots \frac{k}{2}$.)

Back to Link complements

- Now we can write the state prepared by path integration on the link complement $S^3 - \mathcal{L}^n$ in this basis as:

$$|\mathcal{L}^n\rangle = \sum_{j_1,\cdots,j_n} C_{\mathcal{L}^n}(j_1, j_2, \cdots j_n)|j_1\rangle \otimes |j_2\rangle \cdots \otimes |j_n\rangle$$

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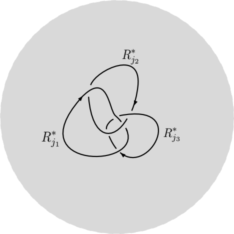
$$C_{\mathcal{L}^n}(j_1, \cdots, j_n) = \left\langle \mathrm{Tr}_{R^*_{j_1}}(e^{\oint_{L_1} A}) \cdots \mathrm{Tr}_{R^*_{j_n}}(e^{\oint_{L_n} A}) \right\rangle_{S^3}$$



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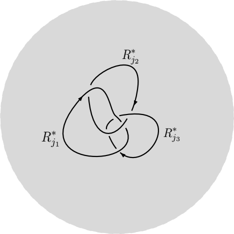

- These are called colored link invariants. (For $G = SU(2)$ they are called colored Jones polynomials.)

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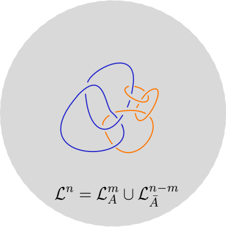

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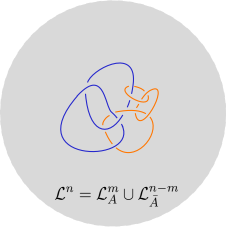

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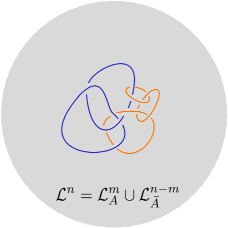

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- The entanglement entropy is given by the Von Neumann entropy of this density matrix:

$$S_{EE} = -\mathrm{Tr}_{\mathcal{L}_A}(\rho_A \ln \rho_A)$$

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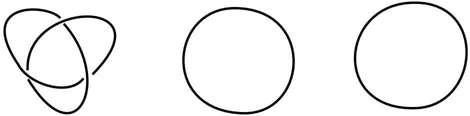

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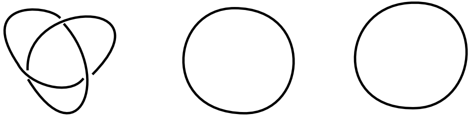

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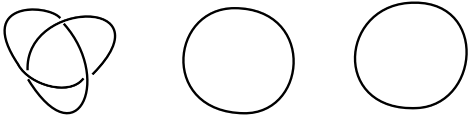

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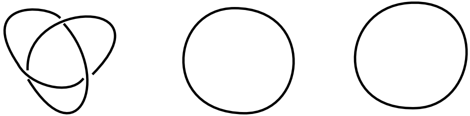

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- **Remark**: The entanglement entropies are all framing independent.

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- For $G = U(1)$, we can give a completely general formula for the entropy of a bi-partition of a general n-link $\mathcal{L}^n$:

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$$S_{EE} = \ln\left(\frac{k^m}{|\ker \boldsymbol{G}_{A,\bar{A}}|}\right)$$

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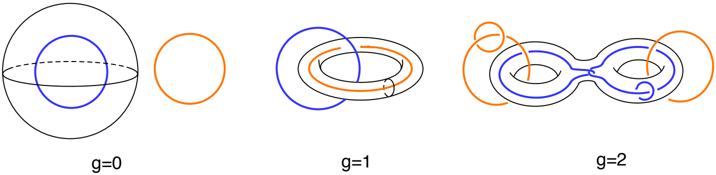

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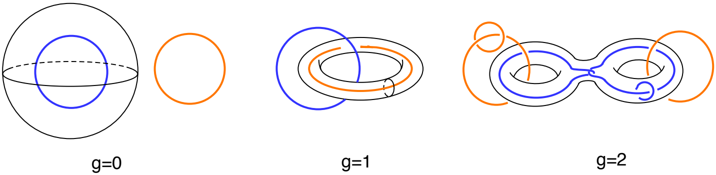

- The separating surface is not unique, but there is a unique such surface of *minimal-genus*.

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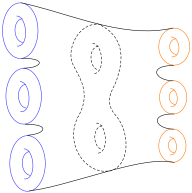

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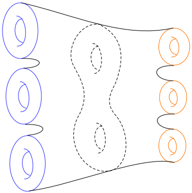

- This is reminscent of the area-law bounds in tensor network descriptions of critical states [Nozaki et al '12, Pastawski et al '15].

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$$|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right).$$

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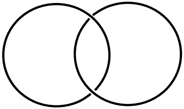
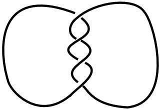
- In fact, all non-split, alternating, prime links are either torus or hyperbolic [Menasco '84].

Torus links

- Torus links are links which can be drawn on the surface of a torus without self-intersections.



2_1^2



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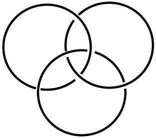

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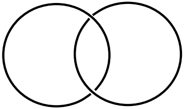

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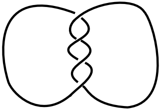

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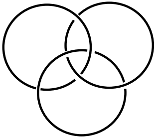

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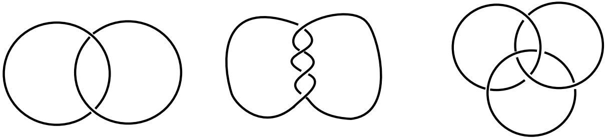

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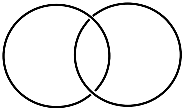

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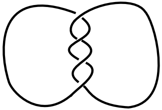

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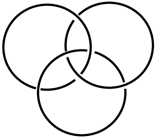

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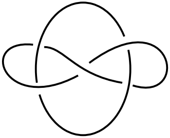
All torus links (with three of more components) have a GHZ-like entanglement structure.

- This can be proved by using the special structure of the colored link invariants of torus links [Labadista et al' 00].

Hyperbolic Links

- Hyperbolic links are links whose link-complements admit a hyperbolic structure.



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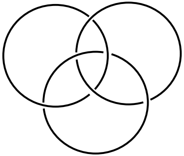

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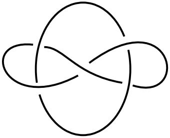

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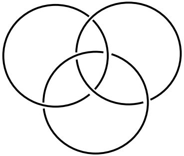

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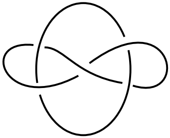

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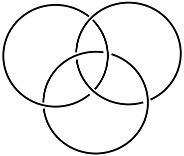

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Conjecture

Hyperbolic links (with three of more components) have a W-like entanglement structure.

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$$\langle j_1, j_2|\rho^\Gamma|\tilde{j}_1, \tilde{j}_2\rangle = \langle \tilde{j}_1, j_2|\rho|j_1, \tilde{j}_2\rangle.$$

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- Then the negativity is defined as

$$\mathcal{N} = \frac{||\rho^{\Gamma}|| - 1}{2},$$

where $||A|| = \mathrm{Tr}\left(\sqrt{A^{\dagger}A}\right)$ is the trace norm.

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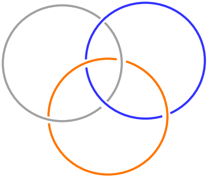
- A non-zero value of $\mathcal{N}$ is a sufficient (but not necessary) condition for the reduced density matrix to be non-separable.
- We numerically computed the entanglement negativities for 20 3-component hyperbolic links.



Link	Negativity at $k = 3$	Hyp. volume
L6a4	0.18547	7.32772
L6a5	0.11423	5.33349
L7a7	0.05008	7.70691
L8a16	0.097683	9.802
L8a18	0.189744	6.55174
L8a19	0.158937	10.667
L8n4	0.11423	5.33349
L8n5	0.18547	7.32772
L10a138	0.097683	10.4486
L10a140	0.0758142	12.2763
L10a145	0.11423	6.92738
L10a148	0.119345	11.8852
L10a156	0.0911946	15.8637
L10a161	0.0354207	7.94058
L10a162	0.0913699	13.464
L10a163	0.0150735	15.5509
L10n78	0.189744	6.55174
L10n79	0.097683	9.802
L10n81	0.15947	10.667
L10n92	0.11423	6.35459

We found in all the cases that the links had W-like entanglement. This provides some evidence that hyperbolic links generically have W-like entanglement.

Part 2: Machine Learning

The Volume conjecture

- For a knot K, let $J_{K,N}(q)$ be the colored Jones polynomial, where $N = 2j$ is the color and

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- In this limit, the *colored* Jones polynomial knows about the hyperbolic volume.

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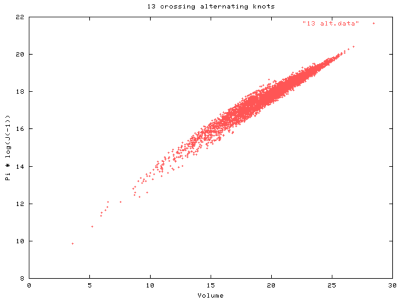

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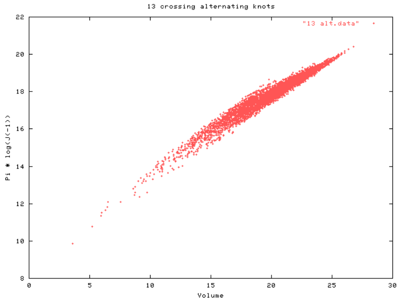

- But this only seems to work for alternating knots, and fails badly for non-alternating knots.

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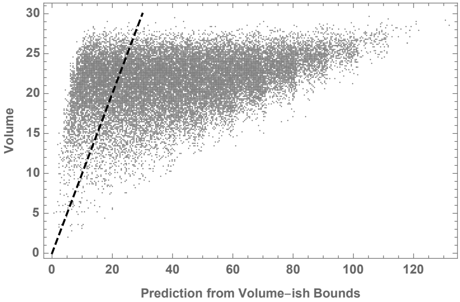
- But this bound is not very tight:



Further, the bounds are only proven for alternating knots.

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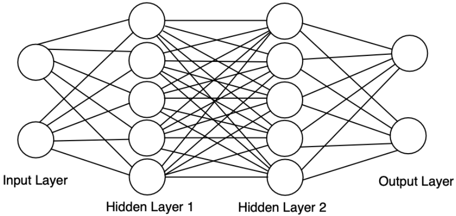
- In our case, the J_i are the Jones polynomials of knots, and the v_i are the volumes of those knots.
- A neural network f_θ is a function (with an *a priori* chosen architecture) which is designed to approximate the associations A efficiently.

Neural Net architecture

- The architecture of the neural net looks as follows:



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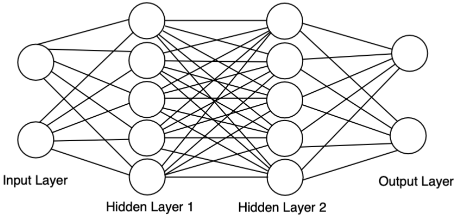

- We encode the Jones polynomial in a vector $\vec{J}_K = (a_n, \cdots, a_m)$, and feed it to the network:.

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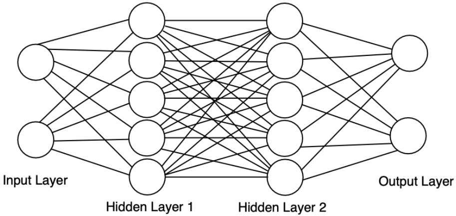

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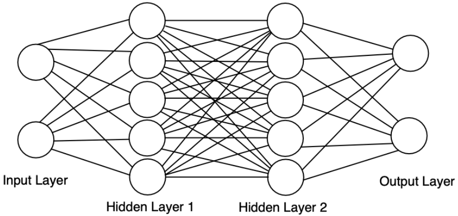

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- The non-linear function is the logistic sigmoid: $\sigma(x) = \frac{1}{1+e^{-x}}$.

```
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- The neural net is taught the associations on the training set by tuning the internal parameters θ to approximate A as closely as possible on T, by minimizing a suitable loss function:

$$h(\theta) = \sum_{i \in T} ||f_\theta(J_i) - v_i||^2.$$

Comparing with the true volumes

- Finally, we assess the performance of the trained network by applying it to the unseen inputs $J'_i \in T^c$ and comparing $f_\theta(J'_i)$ to the true answers $v'_i = A(J'_i)$.

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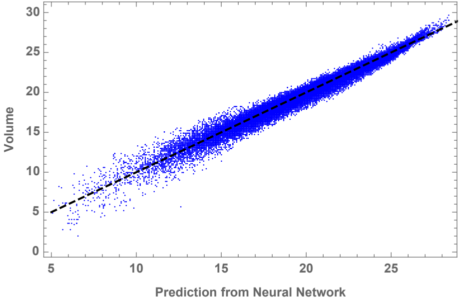

- By training on as little as 10% of data, the network can predict the volume with an accuracy of 97.5%, for both alternating and non-alternating knots.

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- The robustness of the network suggests that there might be a generalized volume conjecture which relates the hyperbolic volume to the Jones polynomial, i.e., the weak-backreaction but possibly strong-coupling regime.
- Neural networks might provide a novel and useful technique to search for mathematical relationships between topological invariants.