Generalized Mirror Models – Beyond Algebraic Toric Spaces

Tristan Hübsch

@ V-Tech, Blacksburg; 2023.04.24

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w/Per Berglund

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Beyond Algebraic Toric Spaces

Playbill

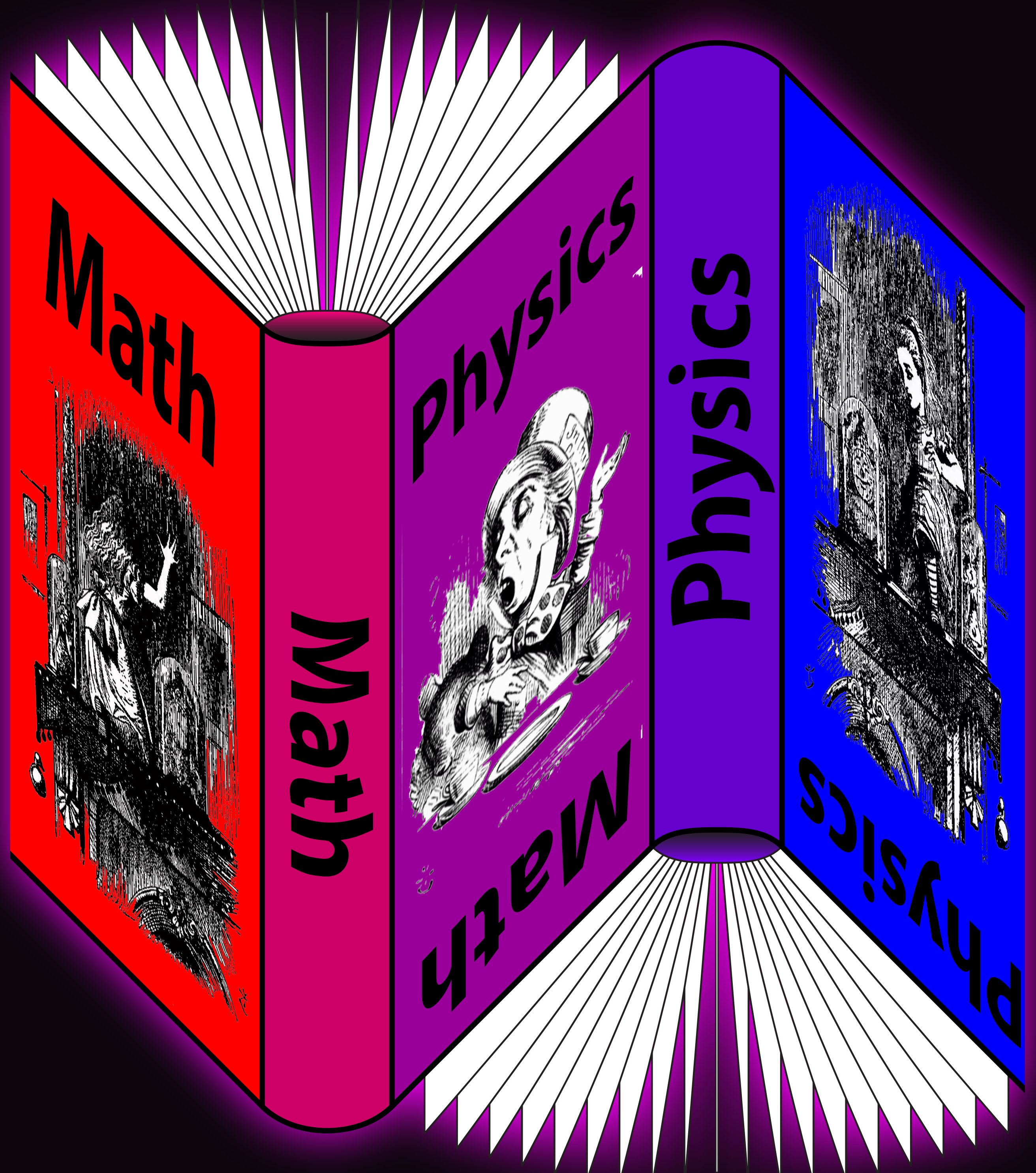
The Story so Far…

Laurent Largo

Meromorphic March

Laurent-Toric Fugue

New? Toric Spaces

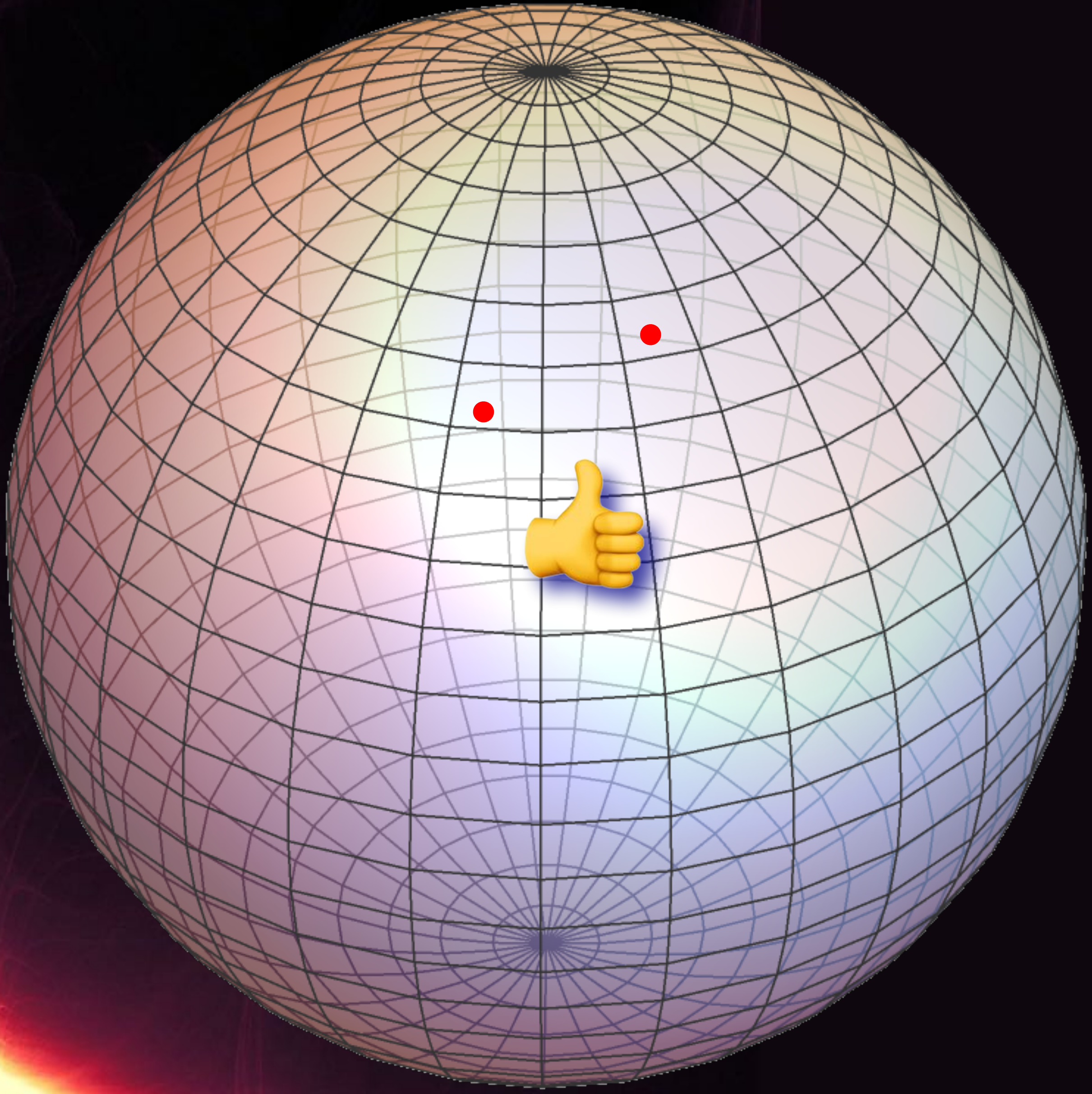


* "It doesn't matter what it's called,
…as long as it has substance."
— S.-T. Yau

How Hard Can it Be?

Constructing CY ⊂ Some "Nice" Ambient Space

- Reduce to 0 dimensions: $\mathbb{P}^4[5] \to \mathbb{P}^3[4] \to \mathbb{P}^2[3] \to \mathbb{P}^1[2]$



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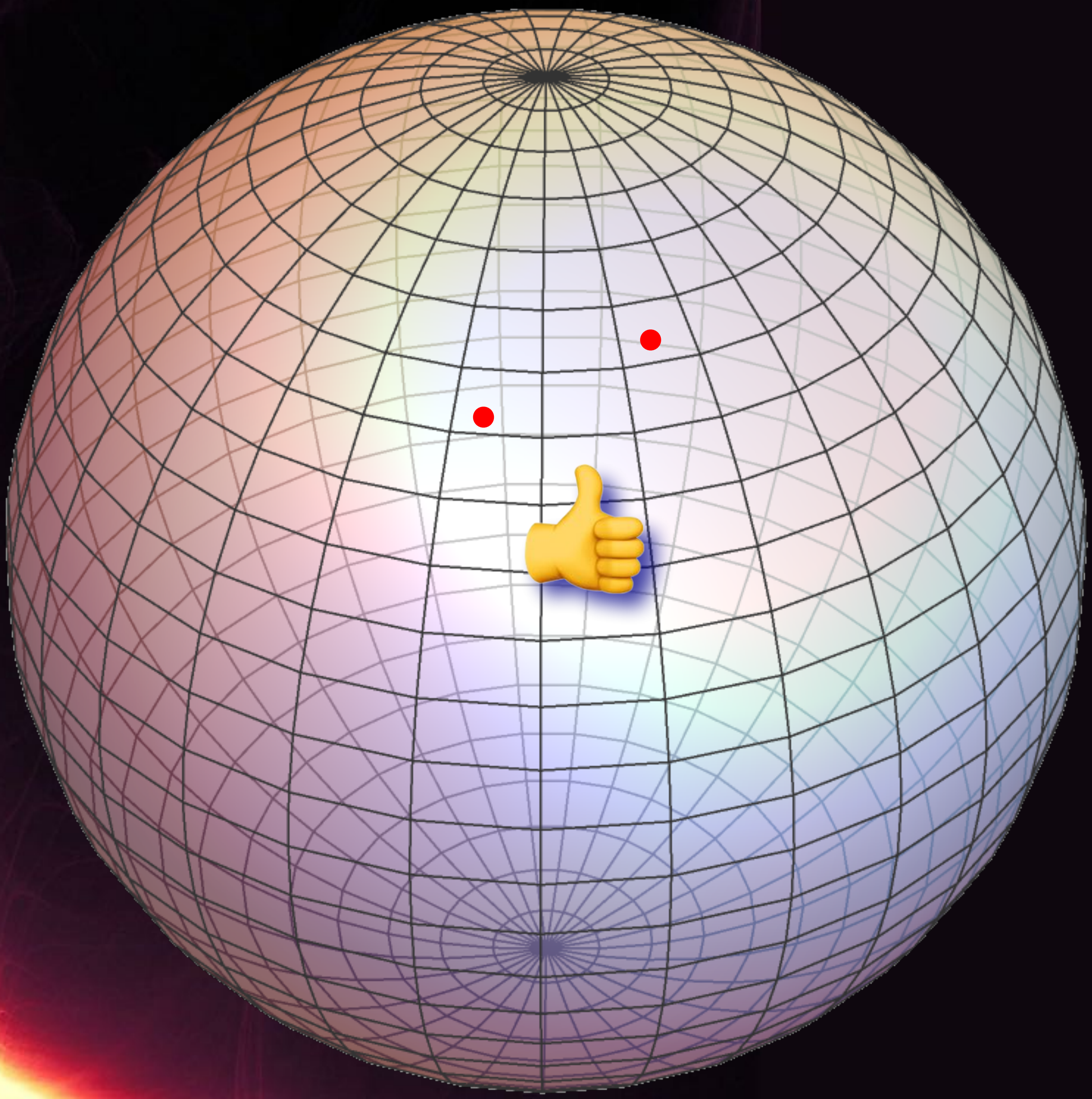

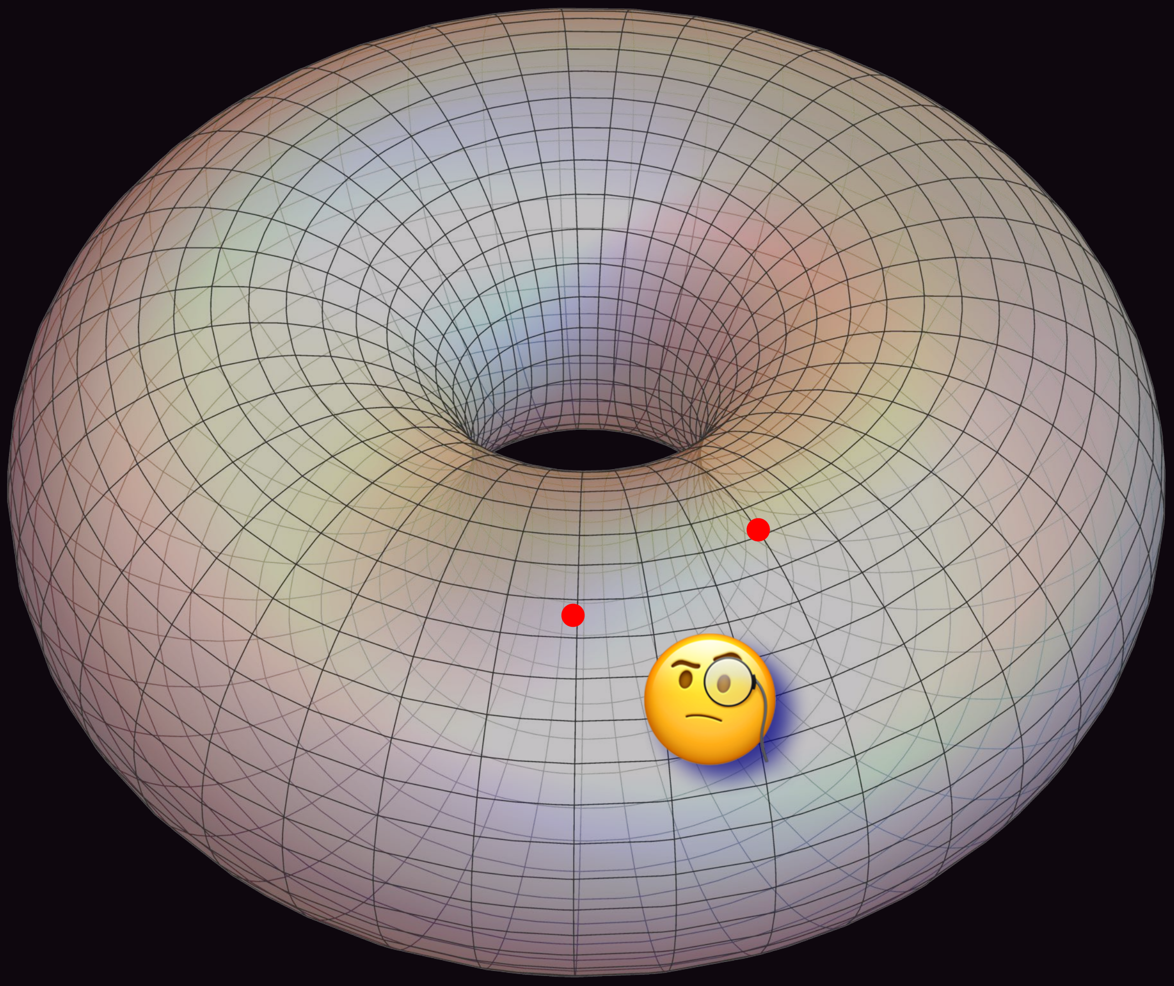

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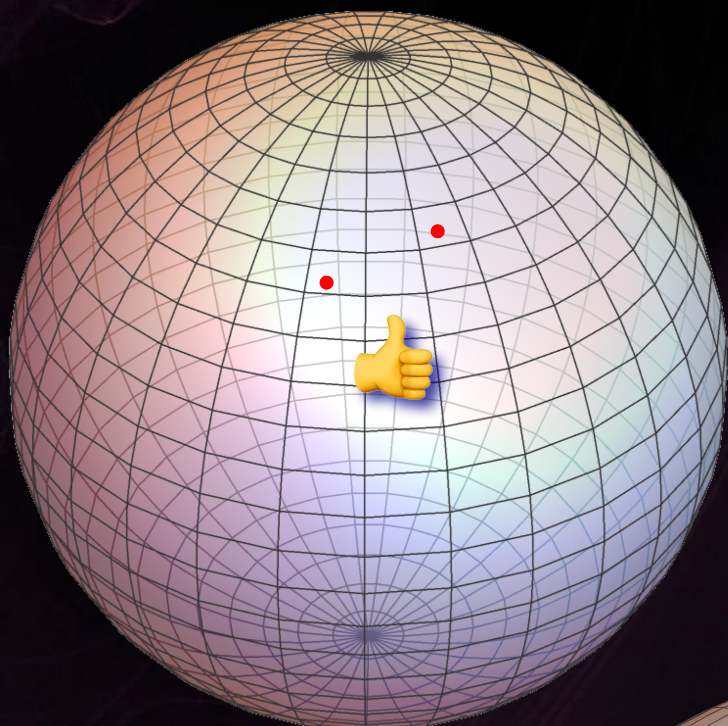
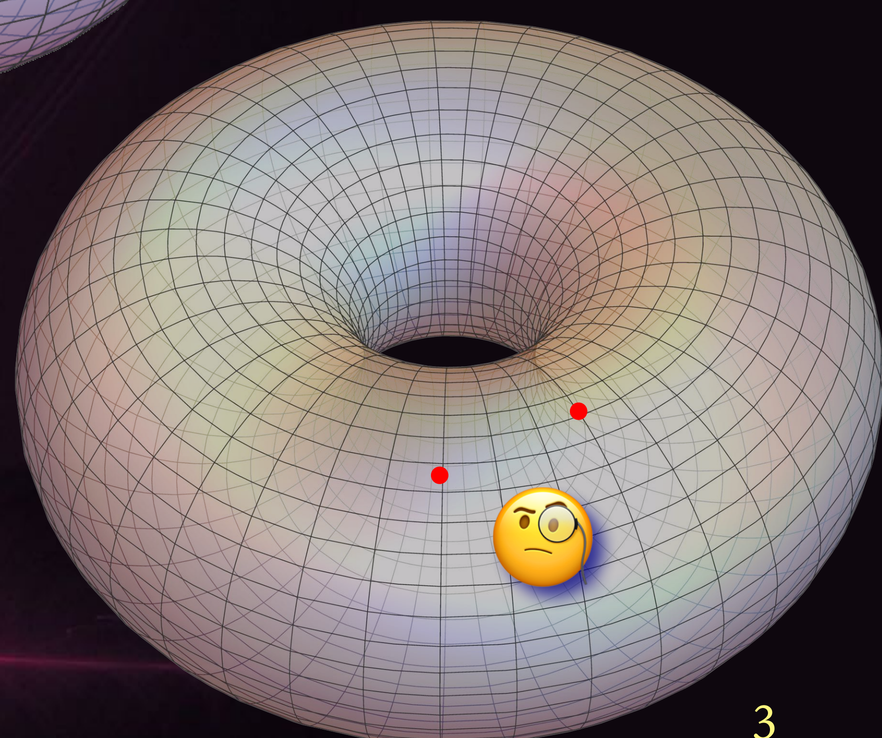
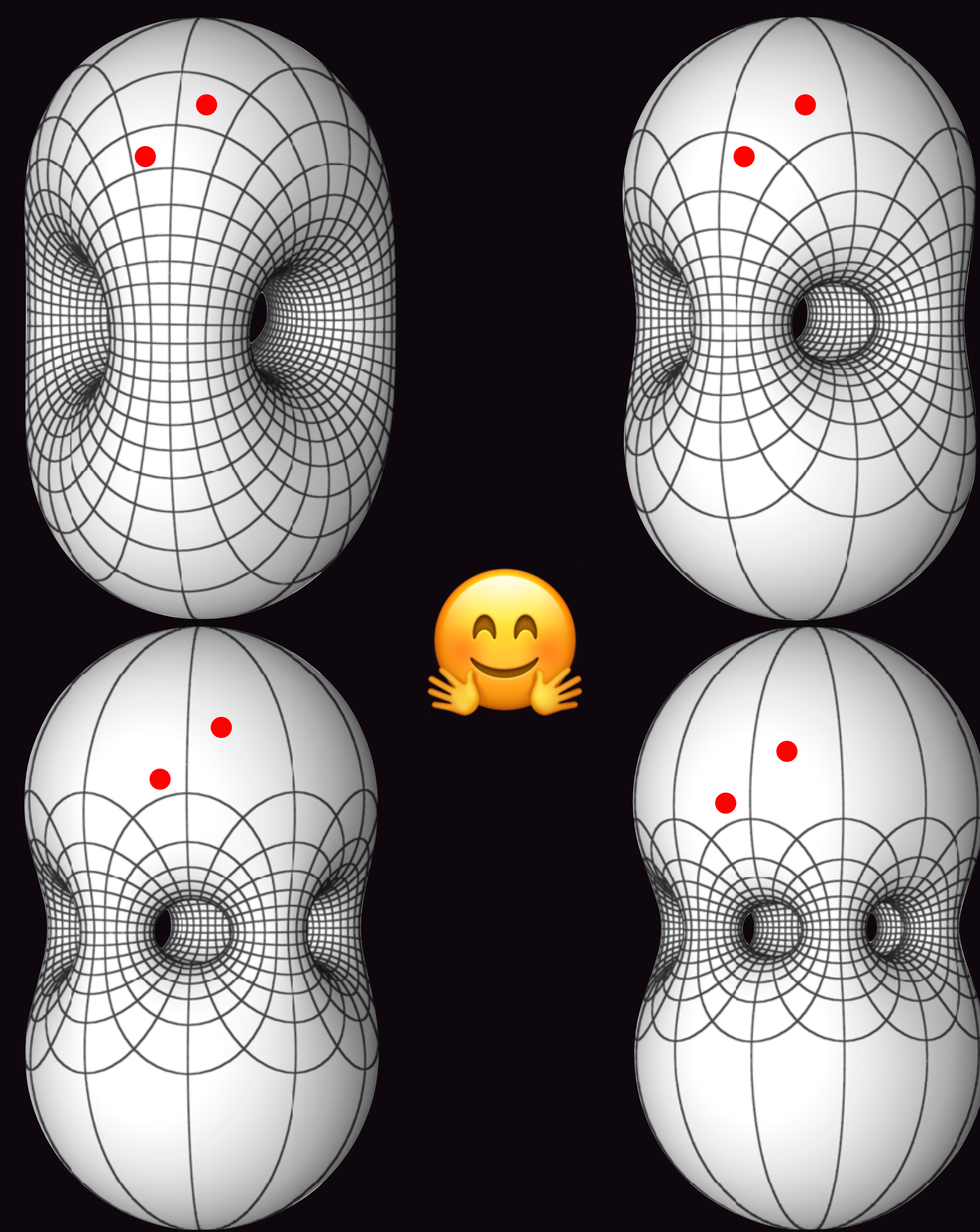
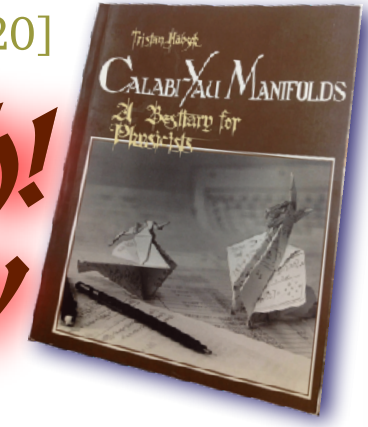






[arXiv:1606.07420]

31th! B-day



The Story so Far…

Classical Constructions

smooth$_{\mathbb{R}}$ models

special? symplectic

- E.g: $X_m \in \left[\begin{array}{c||c||c} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & m & 2-m \end{array}\right]^{(2,86)}_{-168}$

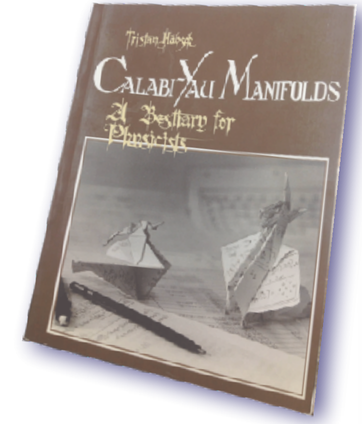
$b_2 = 2 = h^{1,1}$ dim. space of Kähler classes

$\frac{1}{2}b_3 - 1 = 86 = h^{2,1}$ dim. space of complex structures

$-168 = \chi = 2(h^{1,1} - h^{2,1})$ the Euler #

- Zero-set of $p(x,y)=0$, $\deg[p] = \binom{1}{m}$, & $q(x,y)=0$, $\deg[q] = \binom{4}{2-m}$
- Generic $\{q=0\} \cap \{p=0\}$ smooth; $\deg_{\mathbb{P}^n}[p] + \deg_{\mathbb{P}^n}[q] = n+1 \Rightarrow c_1 = 0$
- Sequentially: $X_m \xrightarrow{q=0} \left(F_m \xrightarrow{p=0} \mathbb{P}^4 \times \mathbb{P}^1\right)$ $\quad q(x,y) \overset{?}{\sim} \frac{q_0(x)}{y_0} + \frac{q_1(x)}{y_1}$
- Chern: $c = \frac{(1+J_1)^5(1+J_2)^2}{(1+J_1+mJ_2)(1+4J_1+(2-m)J_2)} = 1 + [6J_1^2 + (8-3m)J_1J_2] - [20J_1^3 - (32+15mJ_1^2J_2)]$.
- C.T.C. Wall: $(aJ_1+bJ_2)^3 = [2a + 3(\underline{4b+ma})]a^2$ $\quad C_{4-k}[(aJ_1+bJ_2)^k] = g(\underline{4b+ma})$
- $p_1[aJ_1+bJ_2] = -88a - 12(\underline{4b+ma})$… the same "$4b+ma$"
- So, $F_m \approx_{\mathbb{R}} F_{m\,(\mathrm{mod}\,4)}$ & $X_m \approx_{\mathbb{R}} X_{m\,(\mathrm{mod}\,4)}$: 4 <u>diffeomorphism types</u>
- …but, $m = 0, 1, 2, 3 \Rightarrow \deg[q] = \binom{4}{-1}$?!

The Story so Far…



Why Haven't We Thought of This Before?

- $\deg[q] = \binom{4}{-1}$ holomorphic sections?!
  - Not *everywhere* on $\mathbb{P}^4 \times \mathbb{P}^1$ — (simple poles)
  - but ***yes*** on $F_3^{(4)} \subset \mathbb{P}^4 \times \mathbb{P}^1$ — $\geqslant 105$ of 'em!

[AAGGL:1507.03235 + BH:1606.07420]
[+ GvG:1708.00517]

$$X_m \in \left[\begin{array}{c\|c\|c} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & m & 2-m \end{array}\right]^{(2,86)}_{-168}$$

for $m = 3$

- How? On $F_3^{(4)}$, $q(x,y) \simeq q(x,y) + \lambda \cdot p(x,y)$ ← *equivalence class!*
  - [Hirzebruch, 1951]⇒ $p = x_0 y_0{}^3 + x_1 y_1{}^3$ & $q = c(x)\left(\frac{x_0 y_0}{y_1{}^2} - \frac{x_1 y_1}{y_0{}^2}\right)$ $\deg[c] = \binom{3}{0}$
  - So, $q_\lambda = q(x,y) + \frac{\lambda\, c(x)}{(y_0 y_1)^2} p(x,y) \xlongequal{\lambda \to -1} c(x)\left(-2\frac{x_1 y_1}{y_0{}^2}\right)$ where $y_0 \neq 0$
    = Wu-Yang monopole!
  - & $q_\lambda = q(x,y) + \frac{\lambda\, c(x)}{(y_0 y_1)^2} p(x,y) \xlongequal{\lambda \to 1} c(x)\left(2\frac{x_0 y_0}{y_1{}^2}\right)$ where $y_1 \neq 0$
  - & $q_1(x,y) - q_0(x,y) = 2\frac{c(x)}{(y_0 y_1)^2}\, p(x,y) = 0$, on $F_3 := \{p(x,y) = 0\}$
  - [GvG, 1708.00517] scheme-theor. "generalized complete intersections"

Reverse-engineered: Mayer-Vietoris sequence & "patching" of the two charts

Laurent Largo

…in well-tempered counterpoint

[BH:1606.07420, 1611.10300 & 2205.12827] +more

- For $\{\underbrace{x_0 y_0{}^m + x_1 y_1{}^m}_{:=p(x,y;0)} = -\sum_{a,\ell} \epsilon_{a\ell}\, x_a\, y_0{}^{m-\ell} y_1{}^{\ell}\} = F^{(n)}_{m;\epsilon} \in \left[\begin{matrix}\mathbb{P}^n\\ \mathbb{P}^1\end{matrix}\middle\|\begin{matrix}1\\ m\end{matrix}\right]$

  even $p(x,y;0)$ is transverse, so $p^{-1}(0)$ is smooth

  - The central ($\epsilon = 0$) member of the family is a Hirzebruch scroll F_m :
  - Directrix: $S := \{\mathfrak{s}(x,y) = 0\}$, $[S] = [H_1] - m[H_2]$ & $[S]^n = -(n-1)m$;
    - where $\mathfrak{s}(x,y) := \left(\frac{x_0}{y_1{}^m} - \frac{x_1}{y_0{}^m}\right) + \frac{\lambda}{(y_0 y_1)^m}[x_0 y_0{}^m + x_1 y_1{}^m]$ degree $\left(\begin{smallmatrix}1\\ -m\end{smallmatrix}\right)$
  - & $h^0(K^*) = 3\binom{2n-1}{n} + \delta_{\epsilon,0}\vartheta_3^m\binom{2n-2}{2}(m-3)$, $h^0(T) = n^2 + 2 + \delta_{\epsilon,0}\vartheta_1^m(n-1)(m-1)$
  - & $h^1(K^*) = \delta_{\epsilon,0}\vartheta_3^m\binom{2n-2}{2}(m-3)$, $h^1(T) = \delta_{\epsilon,0}\vartheta_1^m(n-1)(m-1)$
  - All "*exceptionals*" cancel (*incrementally*) from H^* for $(\epsilon_\alpha \neq 0)$ deformations resulting in *discrete deformations* $F^{(n)}_m \to F^{(n)}_{(m_1,m_2,\cdots)}$ & ⋯ & $\approx_{\mathbb{R}} F^{(n)}_{[m\,(\mathrm{mod}\,n)]}$

- These $F^{(n)}_{(m_1,m_2,\cdots)}$'s are distinct toric varieties… w/$\{\mathfrak{s}_r, r \leqslant m_i\}$

Laurent Largo

…in well-tempered counterpoint

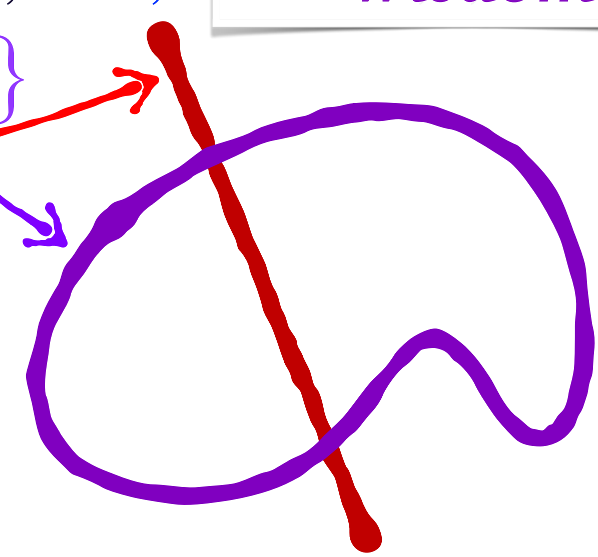
[BH:1606.07420, 1611.10300 & 2205.12827]
+more

- On $F^{(n)}_m$: $p(x,y;0) = x_0 y_0{}^m + x_1 y_1{}^m = 0 \Rightarrow x_0 = -x_1(y_1/y_0)^m$ & $x_1 \to X_1 = \mathfrak{s}$
  - & $(X_i,\ i=2,\cdots,n+2) = (x_2,\cdots,x_n; y_0, y_1)$
  - $\mathbb{P}^4 \times \mathbb{P}^1$ bi-degree → toric $(\mathbb{C}^\times)^2$-action:
  - BTW, $\det\left[\dfrac{\partial(p(x,y), \mathfrak{s}(x,y), x_2, \cdots; y_0, y_1)}{\partial(x_0, x_1, x_2, \cdots; y_0, y_1)}\right] = \text{const.}$
  - Need $\deg[f(X)] = \binom{4}{2-m}$, with $\deg[X_1 X_{5,6}^m] = \binom{1}{0} = \deg[X_{2,3,4}]$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	←$\mathbb{P}^4$
$-m$	0	0	0	1	1	←$\mathbb{P}^1$

- $f(X) = X_1{}^4 X_{5,6}^{2+3m} \oplus X_1{}^3 X_{2,3,4} X_{5,6}^{2+2m} \cdots \oplus X_1 X_{2,3,4}^3 X_{5,6}^2$ $\oplus X_{2,3,4}^4 X_{5,6}^{2-m}$

  standard wisdom
  - $m > 2$, $\{f(X)=0\} = \{X_1 = 0\} \cup \{\oplus_k X_1{}^k X_{2,3,4}^2 X_{5,6}^{2+km} = 0\}$



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…in well-tempered counterpoint

[BH:1606.07420, 1611.10300 & 2205.12827] +more

- On $F^{(n)}_m$: $p(x,y;0) = x_0 {y_0}^m + x_1 {y_1}^m = 0 \Rightarrow x_0 = -x_1 (y_1/y_0)^m$ & $x_1 \to X_1 = \mathfrak{s}$
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  - $\{f(X) = 0\}^\sharp = \{X_1 = 0\} \cap \{\oplus_k {X_1}^k X_{2,3,4}^2 X_{5,6}^{2+km} = 0\}$

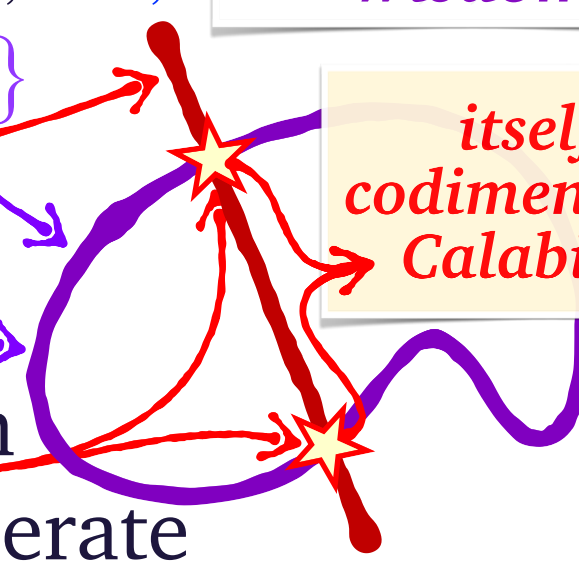
standard wisdom

itself a codimension-2 Calabi-Yau

$$\left[\begin{array}{c\|c\|c} \mathbb{P}^n & 1 \quad n-1 & 1 \\ \mathbb{P}^1 & m \quad 2 & -m \end{array}\right] = \left[\begin{array}{c\|c\|c\|c} \mathbb{P}^n & 1 & 1 & n-1 \\ \mathbb{P}^1 & m & -m & 2 \end{array}\right] \xrightarrow{\simeq} \left[\begin{array}{c\|c} \mathbb{P}^{n-2} & n-1 \\ \mathbb{P}^1 & 2 \end{array}\right]$$

$p = 0 = \mathfrak{s} \Leftrightarrow x_0 = 0 = x_1$

Tyurin degenerate



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…in well-tempered counterpoint

[BH:1606.07420, 1611.10300 & 2205.12827] +more

- On $F^{(n)}_m$: $p(x,y;0)=x_0y_0{}^m+x_1y_1{}^m=0 \Rightarrow x_0=-x_1(y_1/y_0)^m$ & $x_1 \to X_1=\mathfrak{s}$
  - & $(X_i,\ i=2,\cdots,n+2)=(x_2,\cdots,x_n;y_0,y_1)$
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  - BTW, $\det\left[\dfrac{\partial(p(x,y),\mathfrak{s}(x,y),x_2,\cdots;y_0,y_1)}{\partial(x_0,x_1,x_2,\cdots;y_0,y_1)}\right]=\text{const.}$
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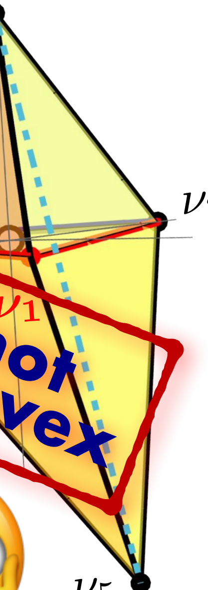
REM*



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standard wisdom

  - $m>2$, $\{f(X)=0\}=\{X_1=0\}\cup\{\oplus_k X_1{}^kX_{2,3,4}^2X_{5,6}^{2+km}=0\}$
  - $\{f(X)=0\}^\sharp=\{X_1=0\}\cap\{\oplus_k X_1{}^kX_{2,3,4}^2X_{5,6}^{2+km}=0\}$

$$\left[\begin{smallmatrix}\mathbb{P}^n\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}1 & n-1\\ m & 2\end{smallmatrix}\middle|\begin{smallmatrix}1\\ -m\end{smallmatrix}\right]=\left[\begin{smallmatrix}\mathbb{P}^n\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}1\\ m\end{smallmatrix}\middle|\begin{smallmatrix}1\\ -m\end{smallmatrix}\,\begin{smallmatrix}n-1\\ 2\end{smallmatrix}\right]\overset{\simeq}{\to}\left[\begin{smallmatrix}\mathbb{P}^{n-2}\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}n-1\\ 2\end{smallmatrix}\right]$$

$p=0=\mathfrak{s}\ \Leftrightarrow\ x_0=0=x_1$

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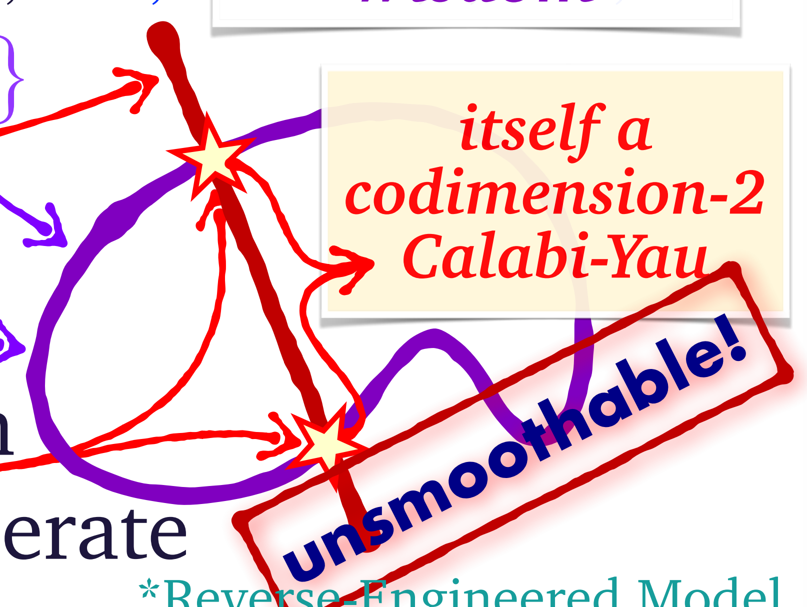
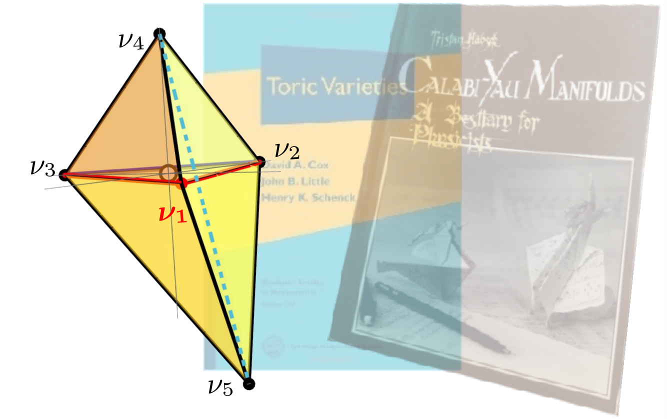


*Reverse-Engineered Model

Laurent Largo

…with a meandering melody



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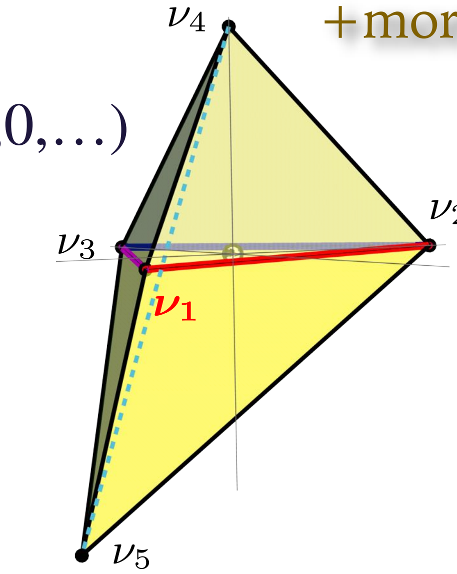
+more



- Deform: $p_1(x,y) = x_0 y_0{}^5 + x_1 y_1{}^5 + x_2 y_0 y_1{}^4$ toric $F^{(n)}_{(4,1,0,\dots)}$
  - Find: $\mathfrak{s}_{1,1}(x,y) = \frac{x_0 y_0}{y_1{}^5} + \frac{x_2}{y_1{}^4} - \frac{x_1}{y_1{}^4}$ & $\mathfrak{s}_{1,2}(x,y) = \frac{x_0}{y_1} - \frac{x_2}{y_0} - \frac{x_1 y_1{}^4}{y_0{}^5}$
  - & $\det\left[\frac{\partial(p_1, \mathfrak{s}_{1,1}, \mathfrak{s}_{1,2}, x_3, \cdots; y_0, y_1)}{\partial(x_0, x_1, x_2, x_3, \cdots; y_0, y_1)}\right] = \text{const.}$

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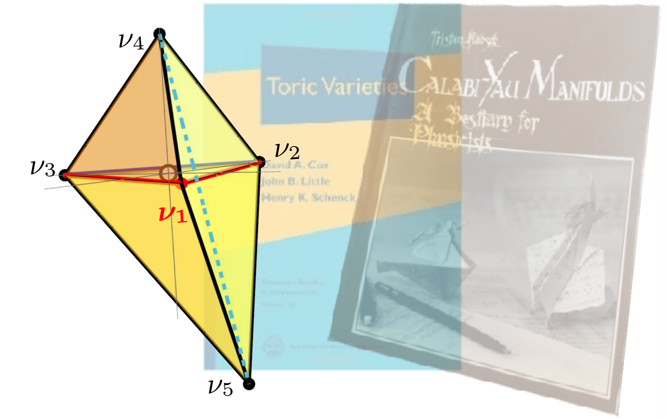

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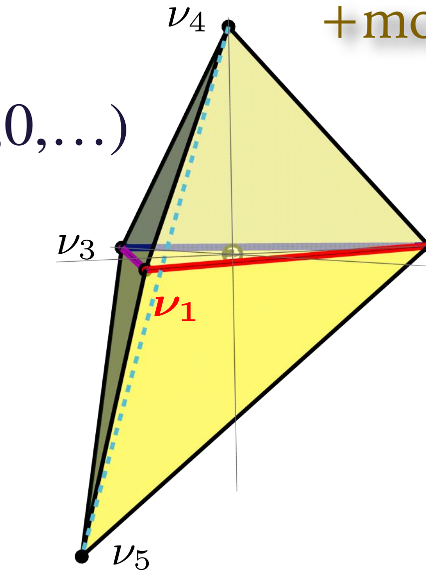

- Deform: $p_2(x,y) = x_0 y_0{}^5 + x_1 y_1{}^5 + x_2 y_0{}^2 y_1{}^3$ toric $F^{(n)}_{(3,2,0,\dots)}$
  - Find: $\mathfrak{s}_{2,1}(x,y) = \frac{x_0 y_0{}^2}{y_1{}^5} + \frac{x_2}{y_1{}^3} - \frac{x_1}{y_1{}^3}$ & $\mathfrak{s}_{2,2}(x,y) = \frac{x_0}{y_1{}^2} - \frac{x_2}{y_0{}^2} - \frac{x_1 y_1{}^3}{y_0{}^5}$
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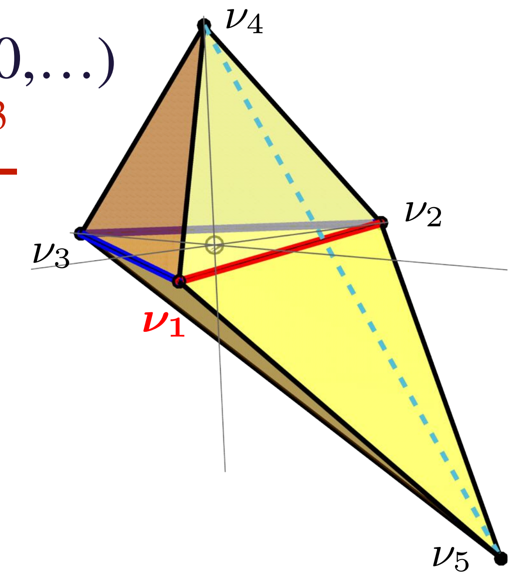

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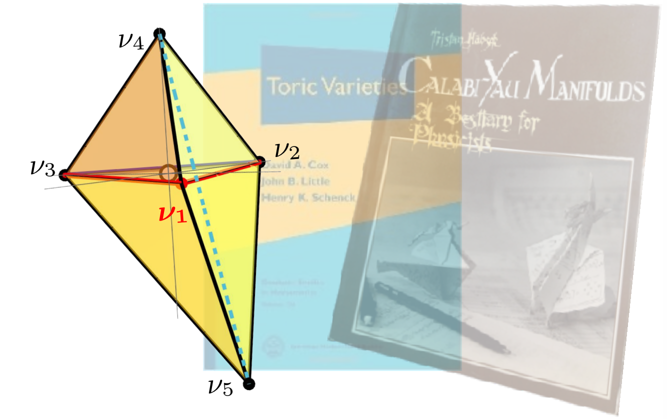
…with a meandering melody



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+more

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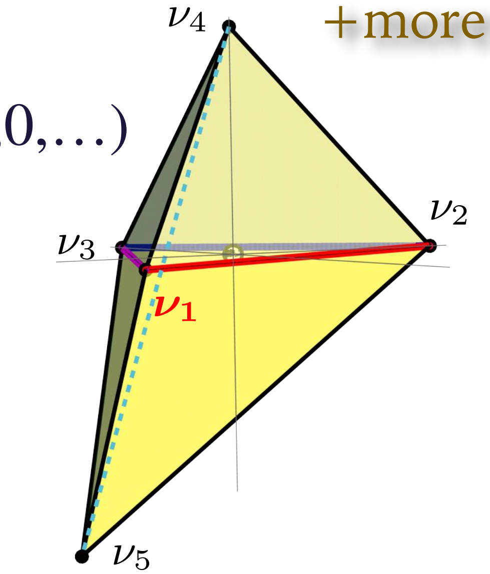
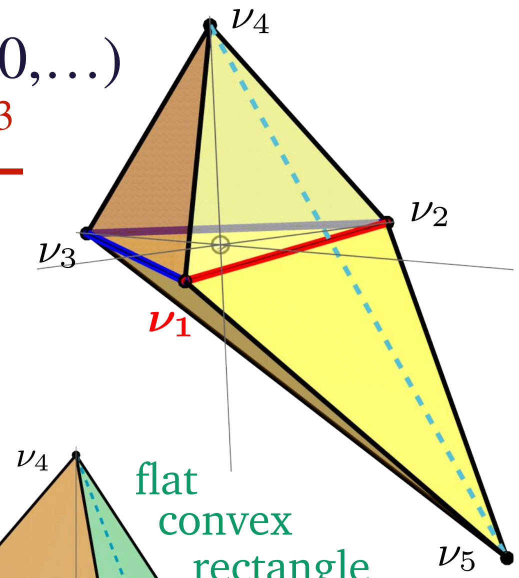
  - & $\det\left[\frac{\partial(p_1, \mathfrak{s}_{1,1}, \mathfrak{s}_{1,2}, x_3, \cdots; y_0, y_1)}{\partial(x_0, x_1, x_2, x_3, \cdots; y_0, y_1)}\right] = \text{const.}$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	←$\mathbb{P}^4$
−4	−1	0	0	1	1	←$\mathbb{P}^1$

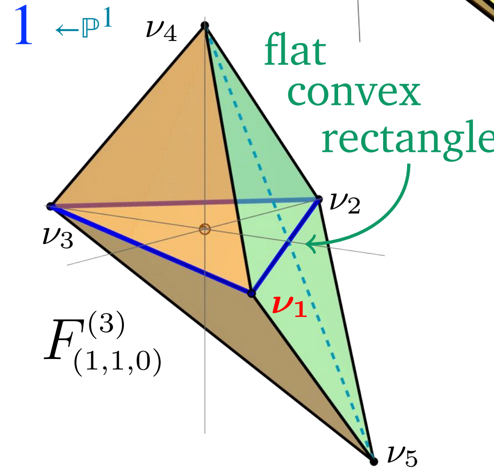


- Deform: $p_2(x,y) = x_0 y_0^5 + x_1 y_1^5 + x_2 y_0^2 y_1^3$ toric $F^{(n)}_{(3,2,0,\dots)}$
  - Find: $\mathfrak{s}_{2,1}(x,y) = \frac{x_0 y_0^2}{y_1^5} + \frac{x_2}{y_1^3} - \frac{x_1}{y_1^3}$ & $\mathfrak{s}_{2,2}(x,y) = \frac{x_0}{y_1^2} - \frac{x_2}{y_0^2} - \frac{x_1 y_1^3}{y_0^5}$
  - & $\det\left[\frac{\partial(p_2, \mathfrak{s}_{2,1}, \mathfrak{s}_{2,2}, x_3, \cdots; y_0, y_1)}{\partial(x_0, x_1, x_2, x_3, \cdots; y_0, y_1)}\right] = \text{const.}$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	←$\mathbb{P}^4$
−3	−2	0	0	1	1	←$\mathbb{P}^1$

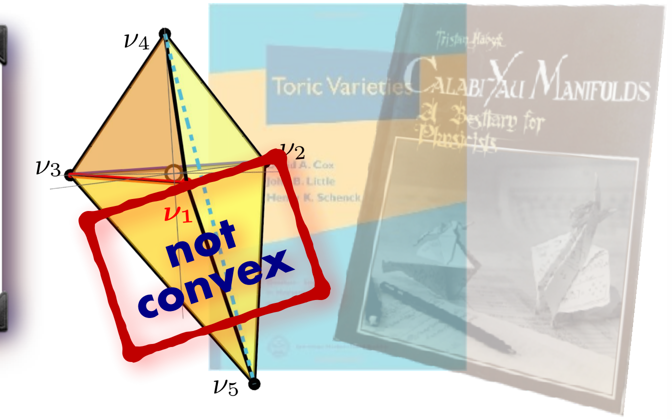


- … and $p_3(x,y) = x_0 y_0^5 + x_1 y_1^5 + x_2 y_0^2 y_1^3 + x_3 y_0^3 y_1^2$
  - → toric $F^{(n)}_{(2,2,1,\cdots)}$ for $n=3$, $F^{(3)}_{(2,2,1)} \approx F^{(3)}_{(1,1,0)}$



Laurent Largo

$$F^{(n)}_{m;\epsilon} \in \left[\begin{matrix}\mathbb{P}^n \\ \mathbb{P}^1\end{matrix}\middle\|\begin{matrix}1 \\ m\end{matrix}\right]$$

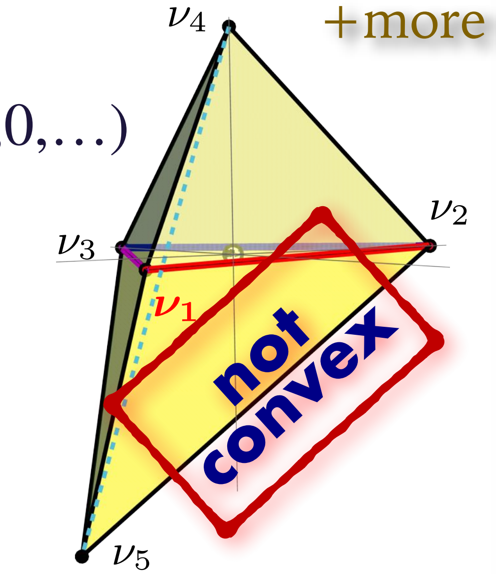


…with a meandering melody

[BH:1606.07420, 1611.10300 & 2205.12827]
+more

- Deform: $p_1(x,y) = x_0 y_0^5 + x_1 y_1^5 + x_2 y_0 y_1^4$ toric $F^{(n)}_{(4,1,0,\ldots)}$
  - Find: $\mathfrak{s}_{1,1}(x,y) = \frac{x_0 y_0}{y_1^5} + \frac{x_2}{y_1^4} - \frac{x_1}{y_1^4}$ & $\mathfrak{s}_{1,2}(x,y) = \frac{x_0}{y_1} - \frac{x_2}{y_0} - \frac{x_1 y_1^4}{y_0^5}$
  - & $\det\left[\frac{\partial(p_1, \mathfrak{s}_{1,1}, \mathfrak{s}_{1,2}, x_3, \cdots; y_0, y_1)}{\partial(x_0, x_1, x_2, x_3, \cdots; y_0, y_1)}\right] = \text{const.}$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow\mathbb{P}^4$
−4	−1	0	0	1	1	$\leftarrow\mathbb{P}^1$



- Deform: $p_2(x,y) = x_0 y_0^5 + x_1 y_1^5 + x_2 y_0^2 y_1^3$ toric $F^{(n)}_{(3,2,0,\ldots)}$
  - Find: $\mathfrak{s}_{2,1}(x,y) = \frac{x_0 y_0^2}{y_1^5} + \frac{x_2}{y_1^3} - \frac{x_1}{y_1^3}$ & $\mathfrak{s}_{2,2}(x,y) = \frac{x_0}{y_1^2} - \frac{x_2}{y_0^2} - \frac{x_1 y_1^3}{y_0^5}$
  - & $\det\left[\frac{\partial(p_2, \mathfrak{s}_{2,1}, \mathfrak{s}_{2,2}, x_3, \cdots; y_0, y_1)}{\partial(x_0, x_1, x_2, x_3, \cdots; y_0, y_1)}\right] = \text{const.}$

X_1	X_2	X_3	X_4	X_5	X_6	
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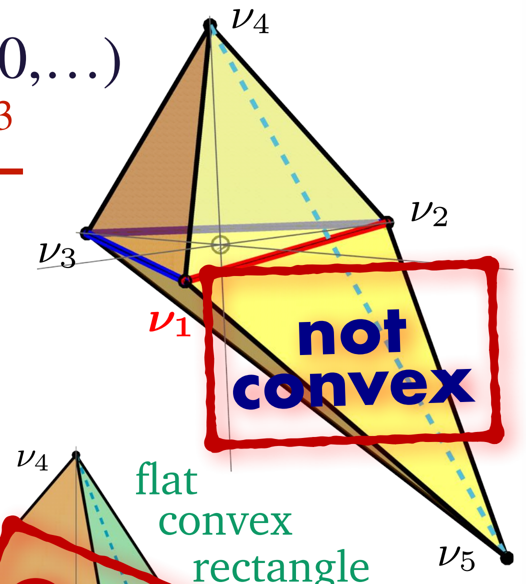
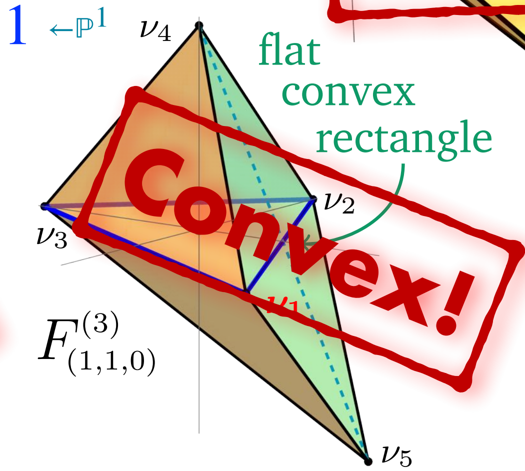


- … and $p_3(x,y) = x_0 y_0^5 + x_1 y_1^5 + x_2 y_0^2 y_1^3 + x_3 y_0^3 y_1^2$
  - → toric $F^{(n)}_{(2,2,1,\cdots)}$ for $n=3$, $F^{(3)}_{(2,2,1)} \approx F^{(3)}_{(1,1,0)}$ Fano!



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…with a meandering melody

[BH:1606.07420, 1611.10300 & 2205.12827] +more

- Algorithm:

Construction 2.1 *Given a degree-$\binom{1}{m}$ hypersurface $\{p_{\vec\epsilon}(x,y)\,0\} \subset \mathbb{P}^n\times\mathbb{P}^1$ as in (2.2), construct*

$$\deg = \binom{1}{m-r_0-r_1}: \quad \mathfrak{s}_{\vec\epsilon}(x,y;\lambda) := \underset{y_0}{\mathrm{Flip}}\Big[\frac{1}{{y_0}^{r_0}\,{y_1}^{r_1}}\,p_{\vec\epsilon}(x,y)\Big] \pmod{p_{\vec\epsilon}(x,y)}, \qquad \left[\begin{matrix}\mathbb{P}^n\\ \mathbb{P}^1\end{matrix}\middle\|\begin{matrix}1\\ m\end{matrix}\right]$$

progressively decreasing $r_0{+}r_1 = 2m, 2m{-}1, \cdots$, and keeping only Laurent polynomials containing both y_0- and y_1-denominators but no y_0, y_1-mixed ones. The “Flip_{y_i}” operator changes the relative sign of the rational monomials with y_i-denominators. For algebraically independent such sections, restrict to a subset with maximally negative degrees that are not overall (y_0,y_1)-multiples of each other.

- E.g.: $m=2$

```
p0 = x₀ y₀² + x₁ y₁²; ep[α_] := Table[1/(y₀^(α-i) y₁^i), {i, 0, α}]; Expand /@ (p0 {ep[5], ep[4], ep[3]})
```

$$\Big\{\Big\{\frac{x_0}{y_0^3}+\frac{x_1 y_1^2}{y_0^5},\ \frac{x_0}{y_0^2 y_1}+\frac{x_1 y_1}{y_0^4},\ \frac{x_1}{y_0^3}+\frac{x_0}{y_0 y_1^2},\ \frac{x_0}{y_1^3}+\frac{x_1}{y_0^2 y_1},\ \frac{x_0 y_0}{y_1^4}+\frac{x_1}{y_0 y_1^2},\ \frac{x_0 y_0^2}{y_1^5}+\frac{x_1}{y_1^3}\Big\},$$

$$\Big\{\frac{x_0}{y_0^2}+\frac{x_1 y_1^2}{y_0^4},\ \frac{x_0}{y_0 y_1}+\frac{x_1 y_1}{y_0^3},\ \frac{x_1}{y_0^2}+\frac{x_0}{y_1^2},\ \frac{x_0 y_0}{y_1^3}+\frac{x_1}{y_0 y_1},\ \frac{x_0 y_0^2}{y_1^4}+\frac{x_1}{y_1^2}\Big\},\ \Big\{\frac{x_0}{y_0}+\frac{x_1 y_1^2}{y_0^3},\ \frac{x_0}{y_1}+\frac{x_1 y_1}{y_0^2},\ \frac{x_1}{y_0}+\frac{x_0 y_0}{y_1^2},\ \frac{x_0 y_0^2}{y_1^3}+\frac{x_1}{y_1}\Big\}\Big\}$$

$\cdot y_1, \quad \cdot y_0$

- finds $\mathfrak{s}(x,y) = \Big(\frac{x_0}{y_1{}^2} - \frac{x_1}{y_0{}^2}\Big) \bmod (x_0 y_0{}^2 + x_1 y_1{}^2)$; $\deg = \binom{1}{-2}$, $[\mathfrak{s}^{-1}(0)] = [J_1] - 2[J_2]$.

& [AAGGL:1507.03235]

THE **exceptional curve $[S]^2 = -1$ in $F_2^{(2)}$**

Meromorphic March

…back to the median motif

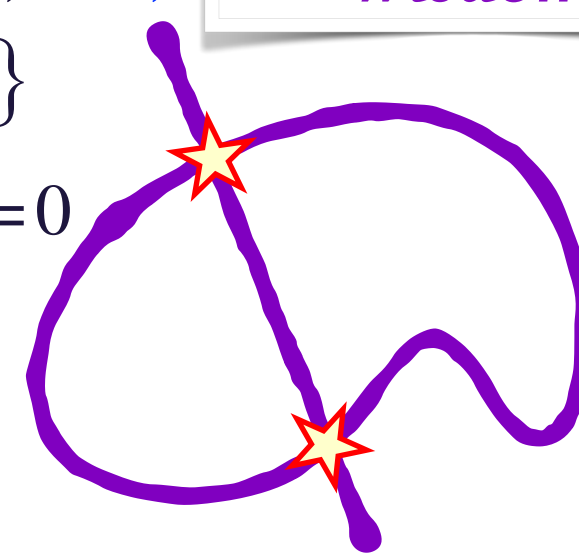
- On $F_m^{(n)}$: $x_0 {y_0}^m + x_1 {y_1}^m = 0 \;\Rightarrow\; x_0 = -x_1 (y_1/y_0)^m \;\;\&\;\; x_1 \to X_1 = \mathfrak{s}$
  - & $(X_i,\ i=2,\cdots,n+2) = (x_2,\cdots,x_n; y_0, y_1)$
  - $\mathbb{P}^4 \times \mathbb{P}^1$ bi-degree $\to$ toric $(\mathbb{C}^\times)^2$-action:
  - BTW, $\det\left[\frac{\partial(p(x,y),\,\mathfrak{s}(x,y),\,x_2,\,\cdots;\,y_0,\,y_1)}{\partial(x_0,\,x_1,\,x_2,\,\cdots;\,y_0,\,y_1)}\right] = \text{const.}$
  - Need $[f(X)] = \binom{4}{2-m}$, with $\deg[X_1 X_{5,6}^m] = \binom{1}{0} = \deg[X_{2,3,4}]$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow \mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow \mathbb{P}^1$

- $f(X) = {X_1}^4 X_{5,6}^{2+3m} \oplus {X_1}^3 X_{2,3,4} X_{5,6}^{2+2m} \cdots \oplus X_1 X_{2,3,4}^3 X_{5,6}^2$

standard wisdom

  - $m > 2,\ \{f(X)=0\} = \{X_1 = 0\} \cup \{\oplus_k {X_1}^k X_{2,3,4}^2 X_{5,6}^{2+km} = 0\}$
  - $\{f(X)=0\}^\sharp = \{X_1 = 0\} \cap \{\oplus_k {X_1}^k X_{2,3,4}^2 X_{5,6}^{2+km} = 0\}$: $R_{\mu\nu} = 0$



Meromorphic March

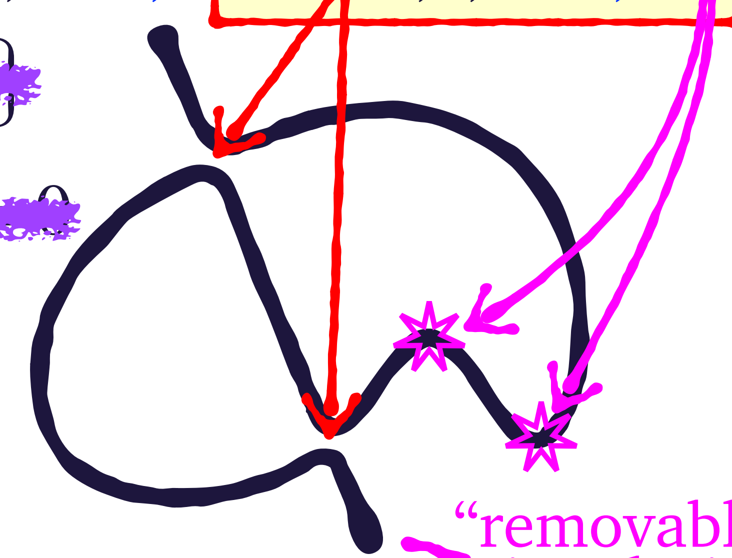
1611.10300 & 2205.12827
+much more

…back to the median motif

- On $F_m^{(n)}$: $x_0 y_0{}^m + x_1 y_1{}^m = 0 \;\Rightarrow\; x_0 = -x_1 (y_1/y_0)^m$ & $x_1 \to X_1 = \mathfrak{s}$
  - & $(X_i,\ i=2,\cdots,n+2) = (x_2,\cdots,x_n;\, y_0, y_1)$
  - $\mathbb{P}^4 \times \mathbb{P}^1$ bi-degree → toric $(\mathbb{C}^\times)^2$-action:
  - BTW, $\det\left[\frac{\partial(p(x,y),\,\mathfrak{s}(x,y),\,x_2,\,\cdots;\,y_0,\,y_1)}{\partial(x_0,\,x_1,\,x_2,\,\cdots;\,y_0,\,y_1)}\right] = \text{const.}$
  - Need $[f(X)] = \binom{4}{2-m}$, with $\deg[X_1 X_{5,6}^m] = \binom{1}{0} = \deg[X_{2,3,4}]$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	←$\mathbb{P}^4$
$-m$	0	0	0	1	1	←$\mathbb{P}^1$

- $f(X) = X_1{}^4 X_{5,6}^{2+3m} \oplus X_1{}^3 X_{2,3,4} X_{5,6}^{2+2m} \cdots \oplus X_1 X_{2,3,4}^3 X_{5,6}^2 \oplus X_{2,3,4}^4 X_{5,6}^{2-m}$
  - $m > 2$, $\{f(X)=0\}$ ~~$= \{X_1 = 0\} \cup \{\oplus_k X_1{}^k X_{2,3,4}^2 X_{5,6}^{2+km} = 0\}$~~
  - ~~$\{f(X)=0\}^\sharp = \{X_1 = 0\} \cup \{\oplus_k X_1{}^k X_{2,3,4}^2 X_{5,6}^{2+km} = 0\} \cup P_{\mu\nu} = 0$~~
  - Embrace the Laurent terms = transverse
  - "Intrinsic limit" (*L'Hôpital-"repaired"*) → smooth (*pre?*)complex spaces

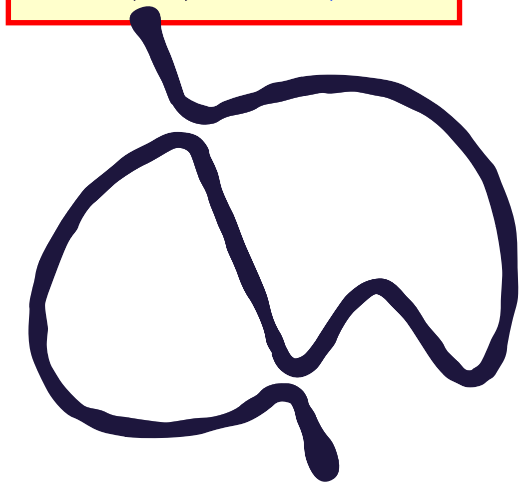


Meromorphic March

1611.10300 & 2205.12827
+much more

…back to the median motif

- $f(X) = X_1{}^4 X_{5,6}^{2+3m} \oplus X_1{}^3 X_{2,3,4} X_{5,6}^{2+2m} \cdots \oplus X_1 X_{2,3,4}^3 X_{5,6}^2 \oplus X_{2,3,4}^4 X_{5,6}^{2-m}$
  - $m > 2$, Laurent terms & "intrinsic limit" 🤔



- Virtual varieties [F. Severi] [🙏 A. Gholampour], i.e., Weil divisors
  - E.g., $\mathbb{P}^2_{(3:1:1)}[5]$: $0 = x_3{}^5 + x_4{}^5 + \frac{x_2{}^2}{x_4} = \frac{x_3{}^5 x_4 + x_4{}^6 + x_2{}^2}{x_4}$
  - Denominator contributions tend to subtract from those of the numerator
- Change variables [David Cox] [🙏 H. Schenck]: $(x_2, x_3, x_4) \mapsto (z_3\sqrt{z_2}, z_1{}^2, z_2)$
  - $x_3{}^5 + x_4{}^5 + \frac{x_2{}^2}{x_4} \mapsto z_1{}^{10} + z_2{}^5 + z_3{}^2$ in $\mathbb{P}^2_{(1:2:5)}[10]$
  - **Generalized to all** $F_m^{(n)}[c_1]$ ✅ — *not a fluke*
  - A <u>*desingularized*</u> <u>*finite quotient*</u> of a <u>*branched multiple cover*</u>
  - …and a variety of "general type" ($c_1 < 0$ or even $c_1 \gtrless 0$)

…there's ∞ of those, just as of VEX polytopes!

Meromorphic March

1611.10300 & 2205.12827 +much more

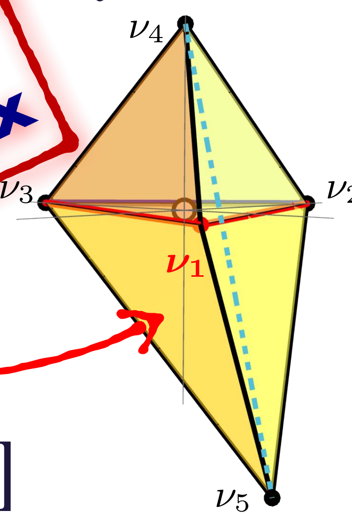
…back to the median motif

- On $F^{(n)}_{m}$: $x_0{y_0}^m + x_1{y_1}^m = 0$; $\det\left[\frac{\partial(p(x,y),\mathfrak{s}(x,y),x_2,\cdots;y_0,y_1)}{\partial(x_0,x_1,x_2,\cdots;y_0,y_1)}\right] = \text{const.}$ & $p(x,y)=0$.
  - $\mathbb{P}^n\times\mathbb{P}^1$-degrees → Mori vectors
  - central in family $F^{(n)}_{m;\epsilon} \in \left[\begin{smallmatrix}\mathbb{P}^n \\ \mathbb{P}^1\end{smallmatrix}\middle\| \begin{smallmatrix}1 \\ m\end{smallmatrix}\right]$
  - deformations $p(x,y;\epsilon) := p(x,y;0) + \sum_{a\ell} \epsilon_{a\ell}\,\delta p_{a\ell}$
  - have less non-convex sp. polytopes & less singular $\Gamma[\mathscr{K}^*(F^{(n)}_{\vec{m}})]$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	←$\mathbb{P}^4$
$-m$	0	0	0	1	1	←$\mathbb{P}^1$

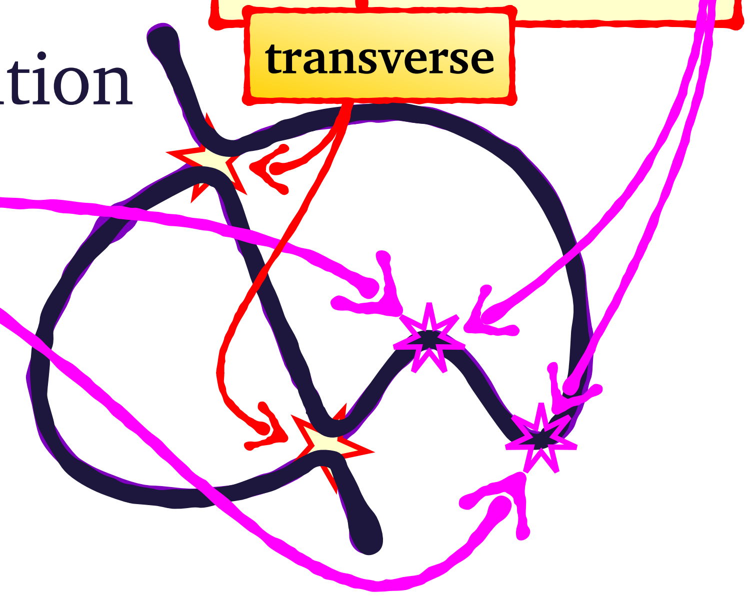
not convex

REM*



- $f(X) = {X_1}^4 X_{5,6}^{2+3m} \oplus {X_1}^3 X_{2,3,4} X_{5,6}^{2+2m} \cdots \oplus X_1 X_{2,3,4}^3 X_{5,6}^2 \oplus X_{2,3,4}^4 X_{5,6}^{2-m}$
  - $m>2$, regular = "unsmoothable" Turin degeneration
  - Laurent smoothing (w/L'Hôpital repair)
  - CY = Weyl divisors in non-Fano
  - desingularized finite quotient of branched multiple covers ↔ general type var's

transverse




IDEA

Laurent-Toric Fugue

(a *not-so-new* Toric Geometry)

A Generalized Construction of
Calabi-Yau Mirror Models
arXiv:1611.10300 + 2205.12827
+ lots more…

Laurent-Toric Fugue

1611.10300 & 2205.12827 +much more

& Non-Convex Mirrors $m=3$ —2D *Proof-of-Concept*—

- $X_1{}^2 X_2{}^0 (X_3 \oplus X_4)^{2+1m}$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow \mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow \mathbb{P}^1$

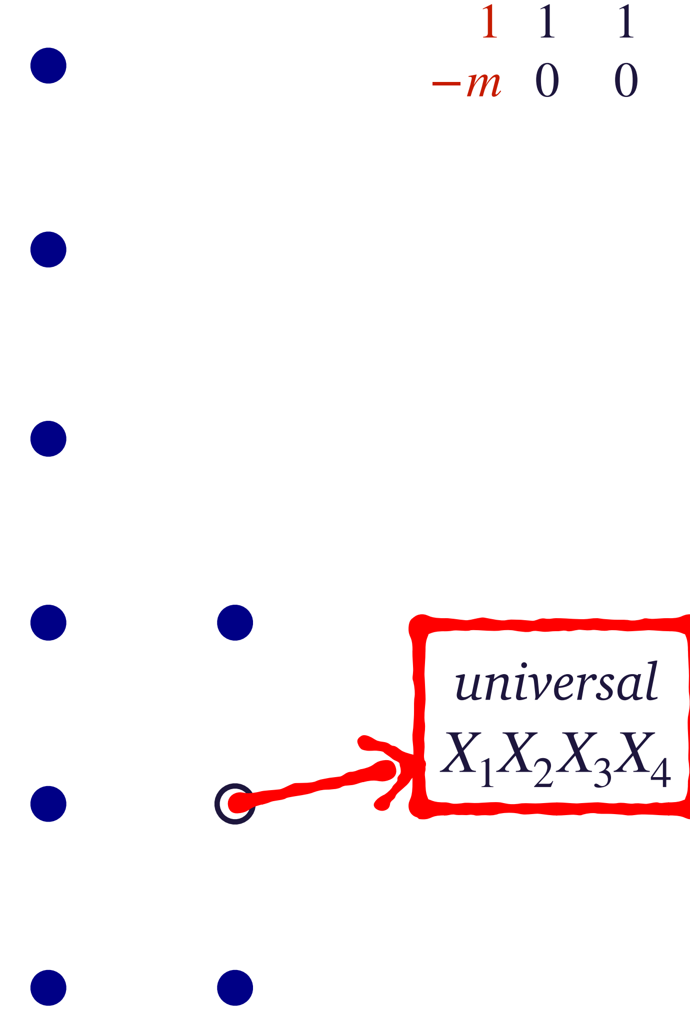
Laurent-Toric Fugue

1611.10300 & 2205.12827
+much more

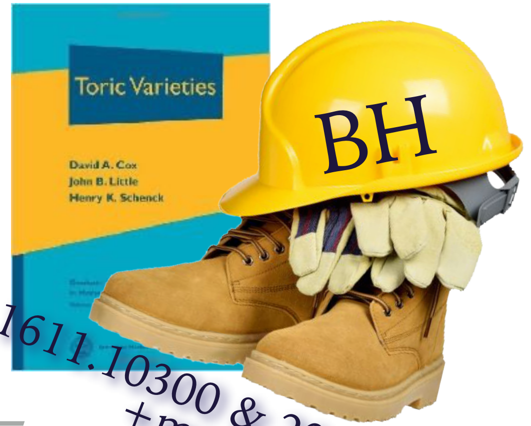
& Non-Convex Mirrors $_{m=3}$ —2D *Proof-of-Concept*—

- $X_1{}^2 X_2{}^0 (X_3 \oplus X_4)^{2+1m} \oplus X_1{}^1 X_2{}^1 (X_3 \oplus X_4)^{2+0m}$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow \mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow \mathbb{P}^1$



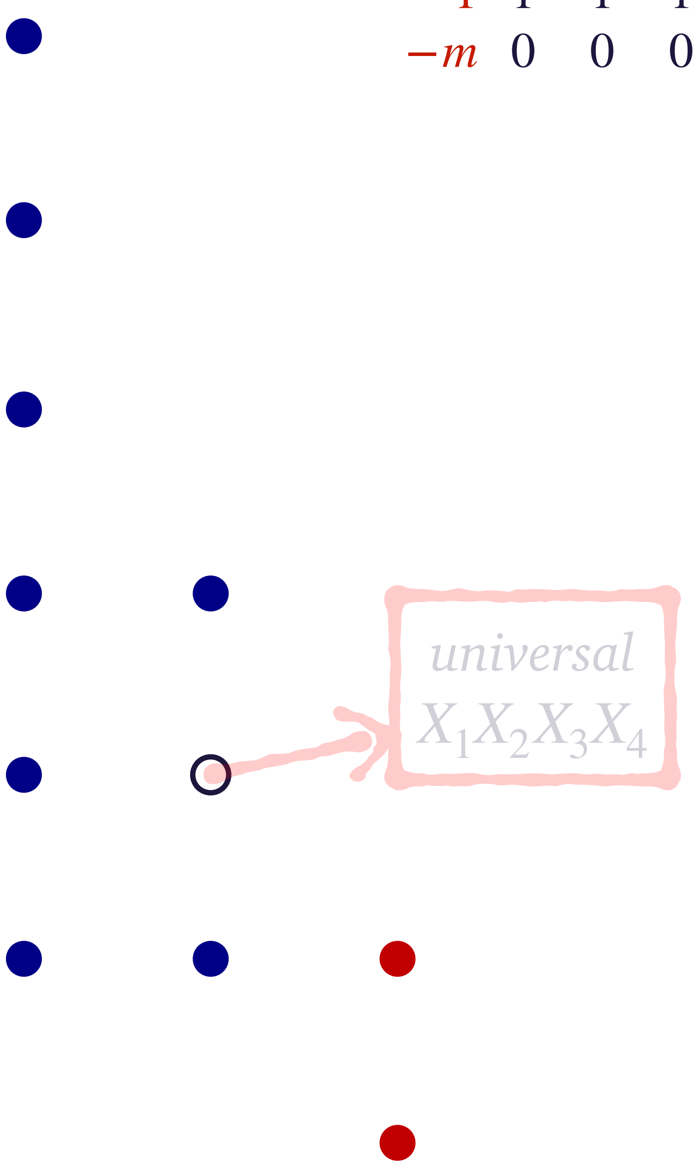
Laurent-Toric Fugue



& Non-Convex Mirrors $m=3$ —2D *Proof-of-Concept*—

- $X_1^{\,2}X_2^{\,0}(X_3\oplus X_4)^{2+1m}\oplus X_1^{\,1}X_2^{\,1}(X_3\oplus X_4)^{2+0m}\oplus X_1^{\,0}X_2^{\,2}(X_3\oplus X_4)^{2-1m}$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow\mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow\mathbb{P}^1$



Laurent-Toric Fugue

1611.10300 & 2205.12827
+much more

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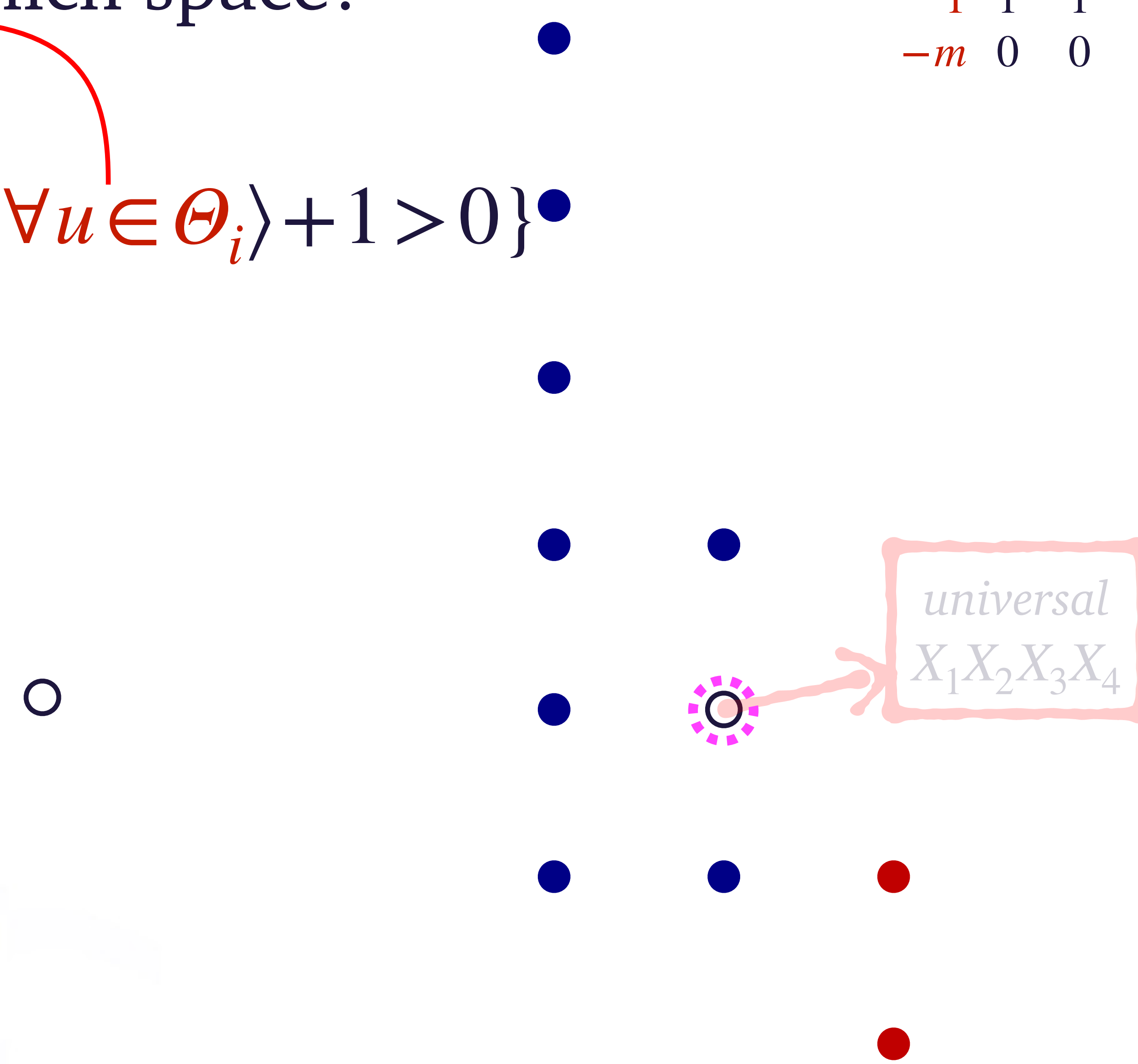
- $X_1{}^2 X_2{}^0 (X_3 \oplus X_4)^{2+1m} \oplus X_1{}^1 X_2{}^1 (X_3 \oplus X_4)^{2+0m} \oplus X_1{}^0 X_2{}^2 (X_3 \oplus X_4)^{2-1m}$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow \mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow \mathbb{P}^1$

- *Trans*polar: functions on which space?
  - $\Delta \to \bigcup_i (\text{convex } \Theta_i)$;
  - Compute $\Theta_i \to \Theta_i^\circ := \{v\colon \langle v | \forall u \in \Theta_i \rangle + 1 > 0\}$



Laurent-Toric Fugue

1611.10300 & 2205.12827 +much more

& Non-Convex Mirrors $m=3$ —2D *Proof-of-Concept*—

- $X_1{}^2 X_2{}^0 (X_3 \oplus X_4)^{2+1m} \oplus X_1{}^1 X_2{}^1 (X_3 \oplus X_4)^{2+0m} \oplus X_1{}^0 X_2{}^2 (X_3 \oplus X_4)^{2-1m}$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow \mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow \mathbb{P}^1$

- *Trans*polar: functions on which space?
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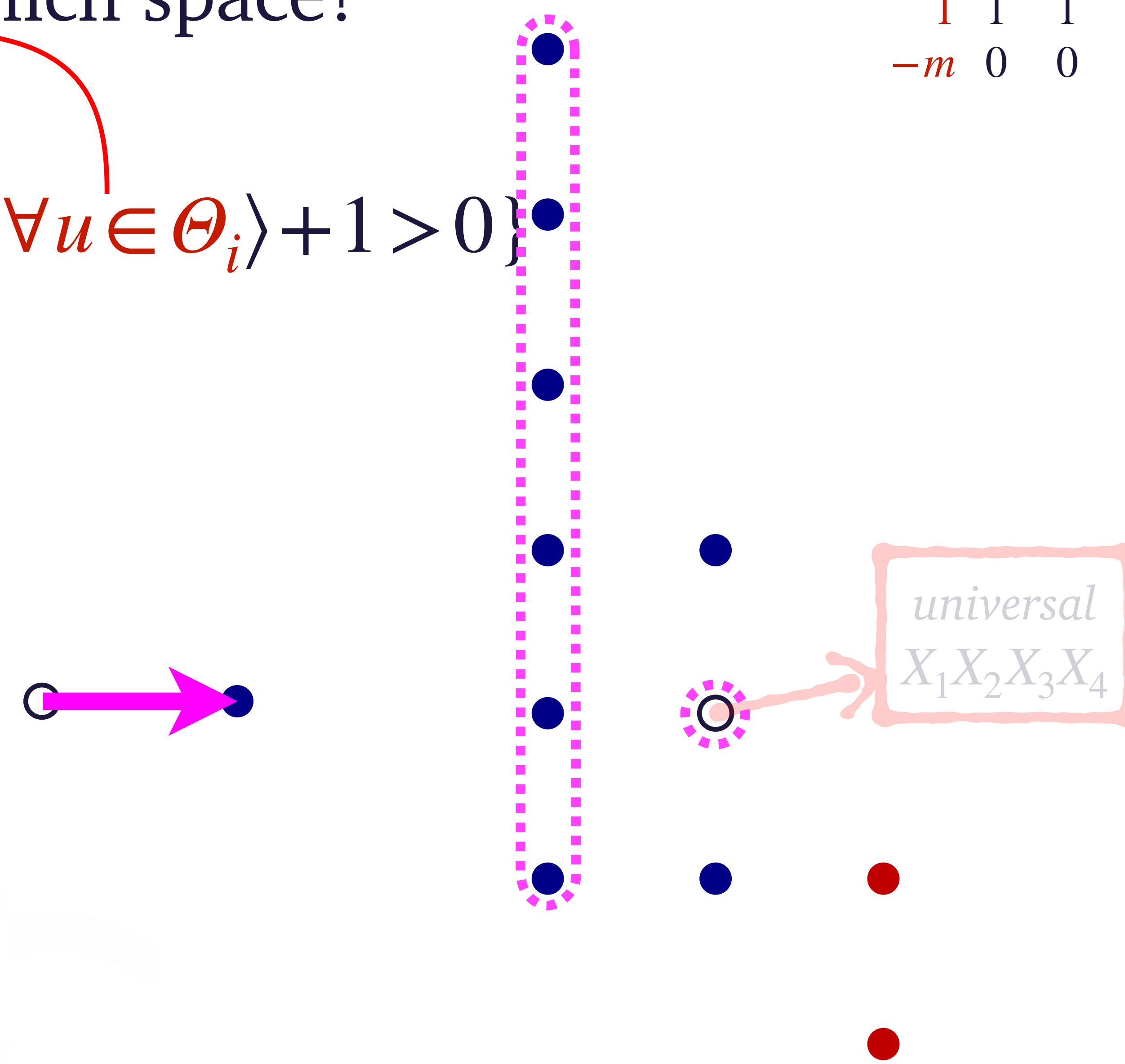
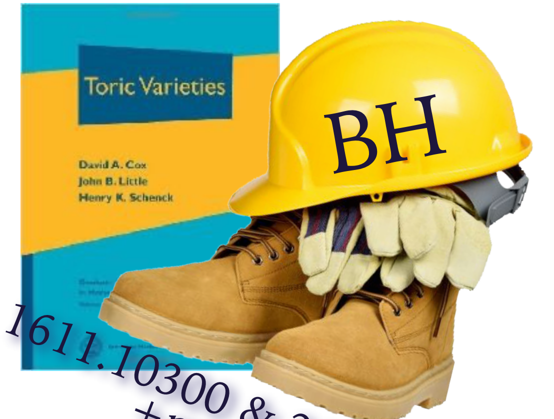




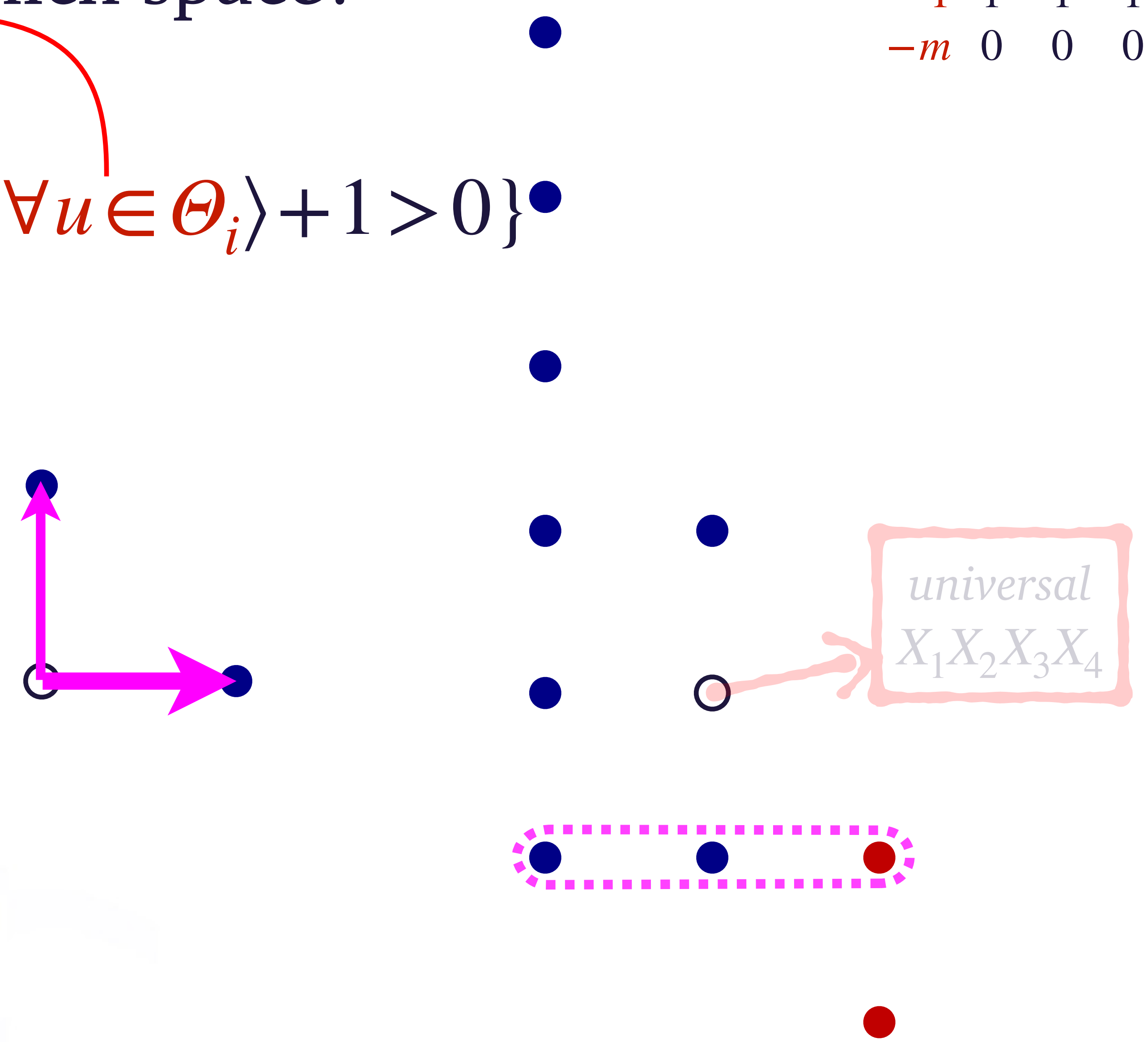
Laurent-Toric Fugue

& Non-Convex Mirrors $m=3$ —2D *Proof-of-Concept*—

1611.10300 & 2205.12827 +much more

- $X_1{}^2 X_2{}^0 (X_3 \oplus X_4)^{2+1m} \oplus X_1{}^1 X_2{}^1 (X_3 \oplus X_4)^{2+0m} \oplus X_1{}^0 X_2{}^2 (X_3 \oplus X_4)^{2-1m}$
- *Trans*polar: functions on which space?
  - $\Delta \to \bigcup_i (\text{convex } \Theta_i)$;
  - Compute $\Theta_i \to \Theta_i^\circ := \{v\!: \langle v | \forall u \in \Theta_i \rangle + 1 > 0\}$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow \mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow \mathbb{P}^1$



Laurent-Toric Fugue

& Non-Convex Mirrors $m=3$ —2D Proof-of-Concept—

1611.10300 & 2205.12827 +much more

- $X_1{}^2 X_2{}^0 (X_3 \oplus X_4)^{2+1m} \oplus X_1{}^1 X_2{}^1 (X_3 \oplus X_4)^{2+0m} \oplus X_1{}^0 X_2{}^2 (X_3 \oplus X_4)^{2-1m}$

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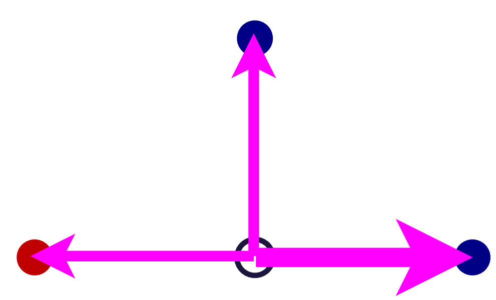
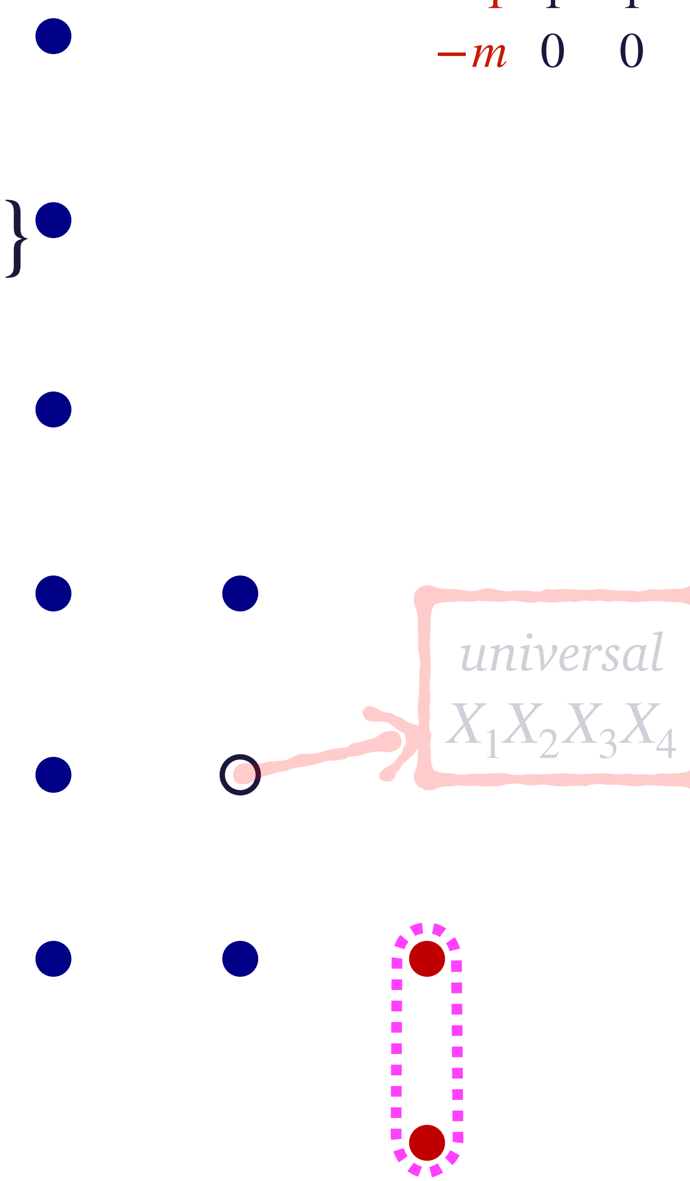
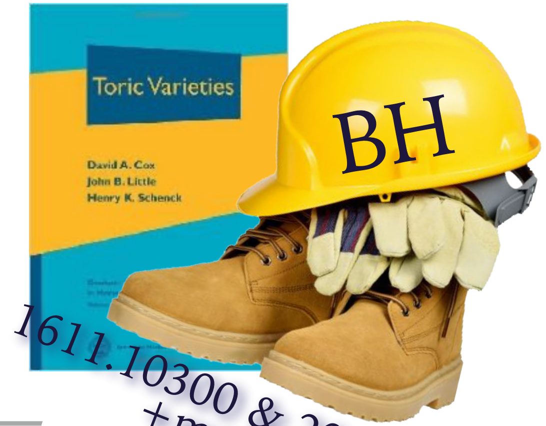




Laurent-Toric Fugue

& Non-Convex Mirrors $\quad m=3$ —2D *Proof-of-Concept*—



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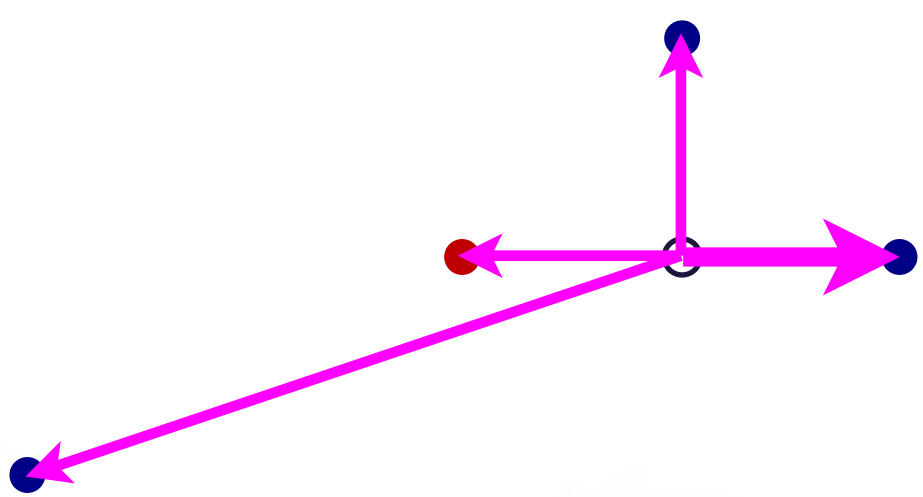
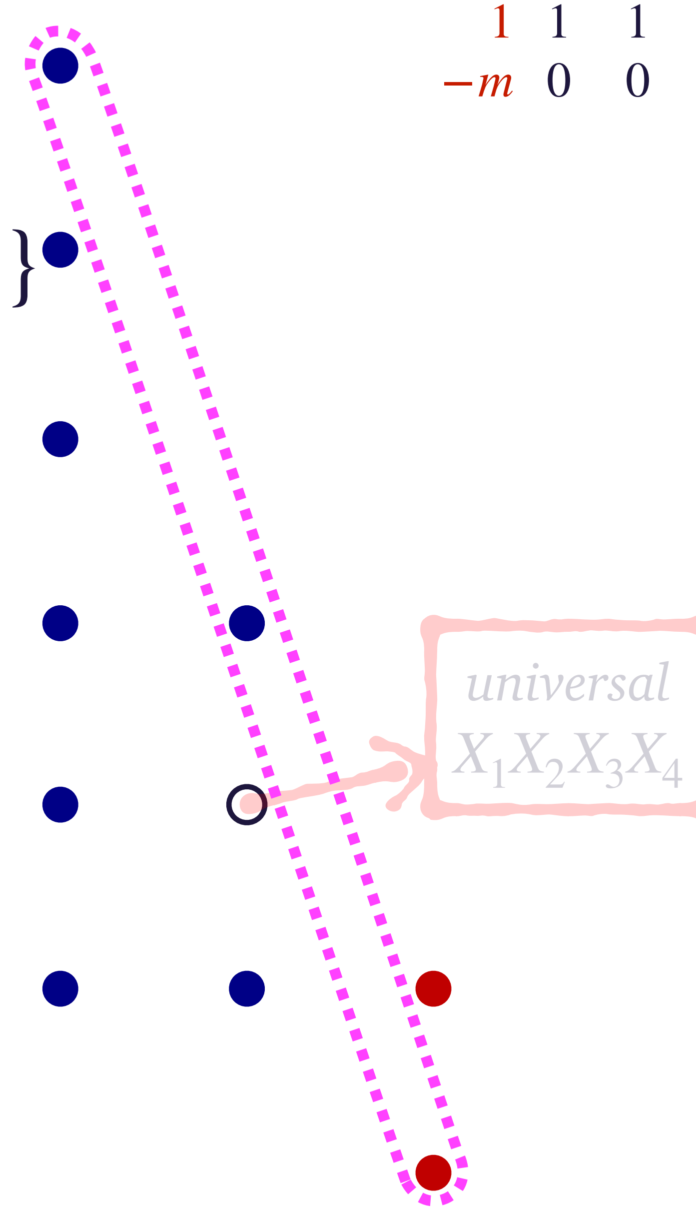
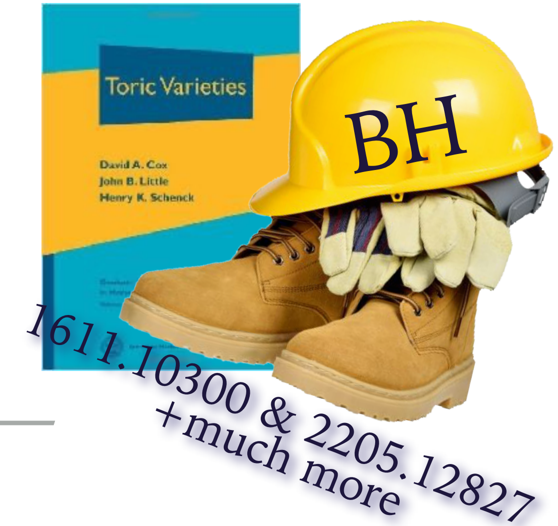




Laurent-Toric Fugue



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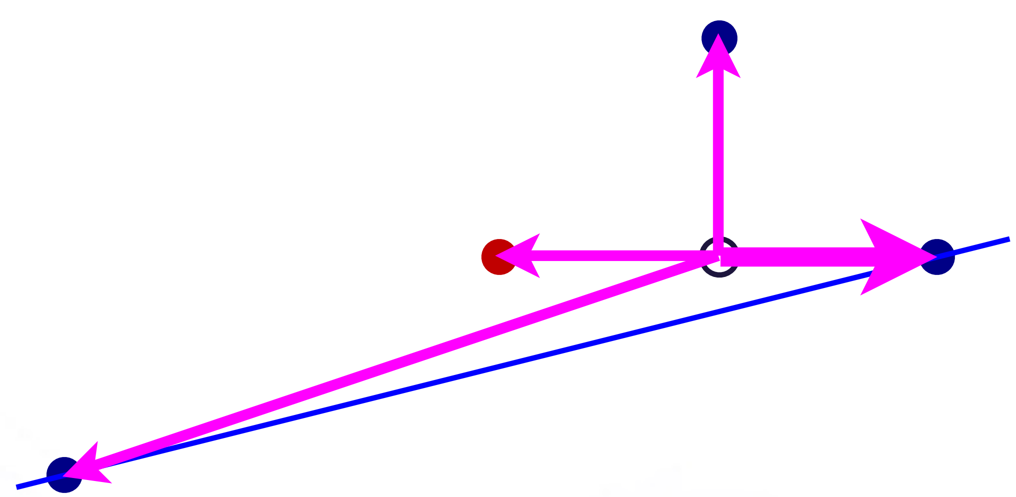

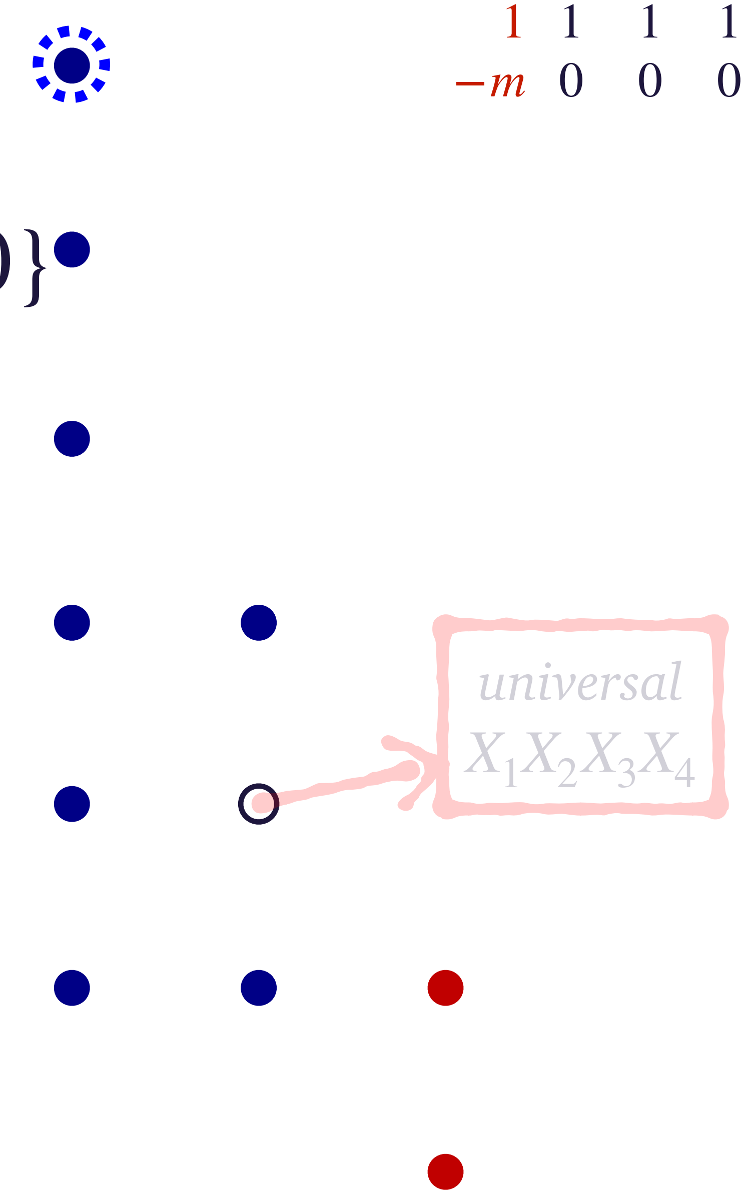

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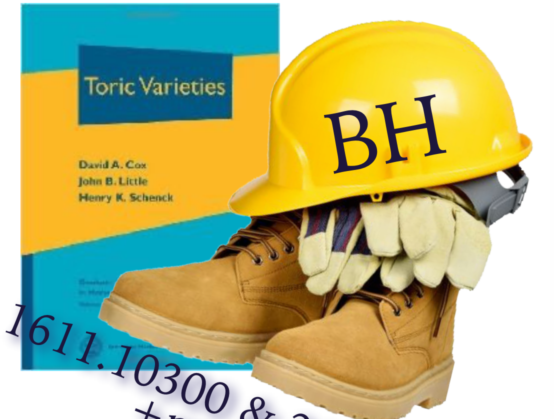

1611.10300 & 2205.12827 +much more

& Non-Convex Mirrors

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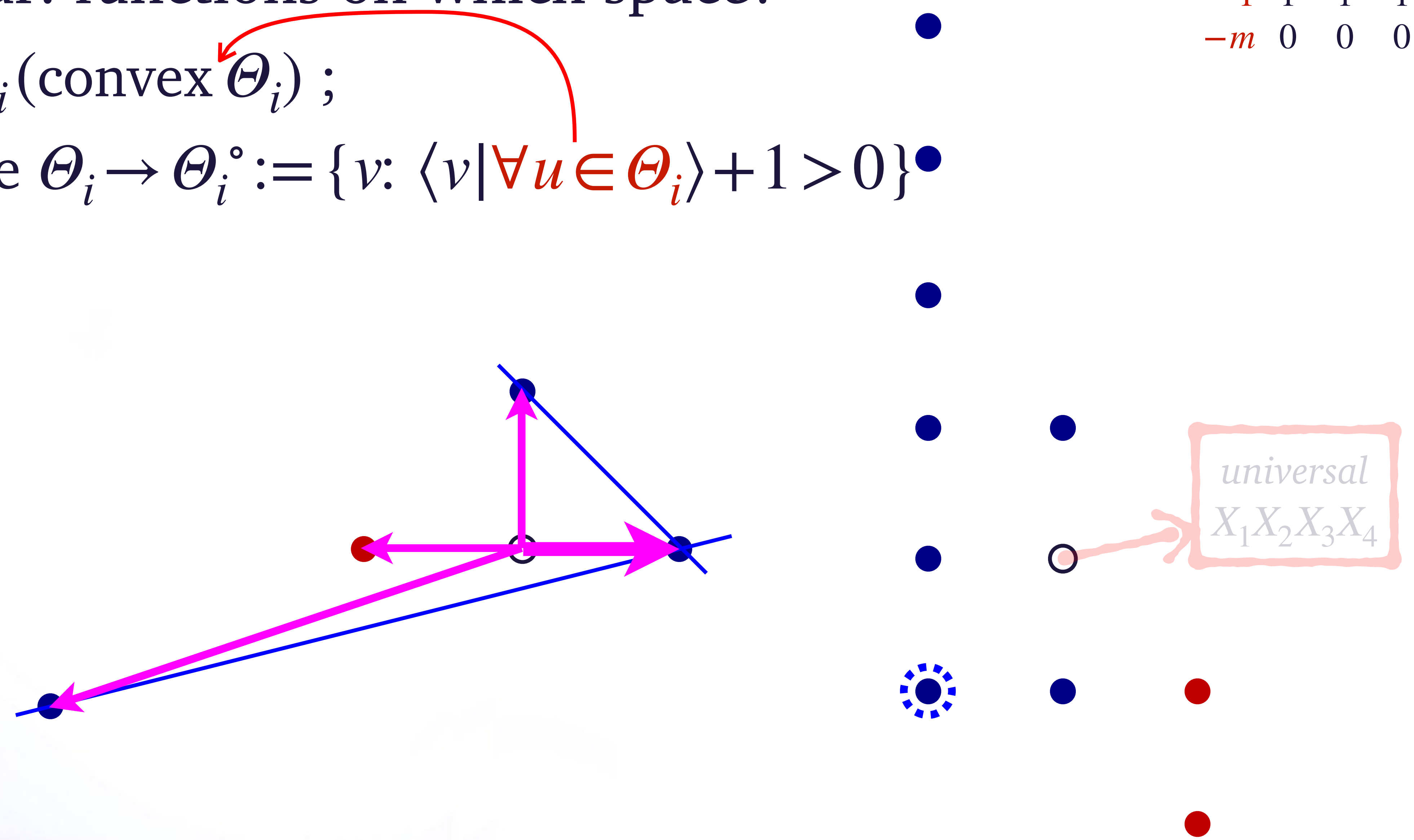

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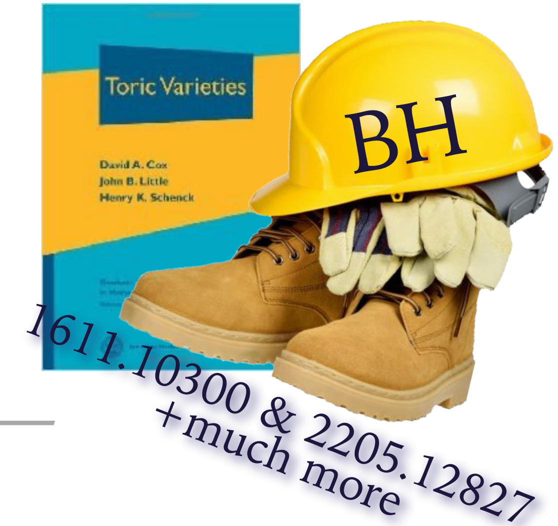

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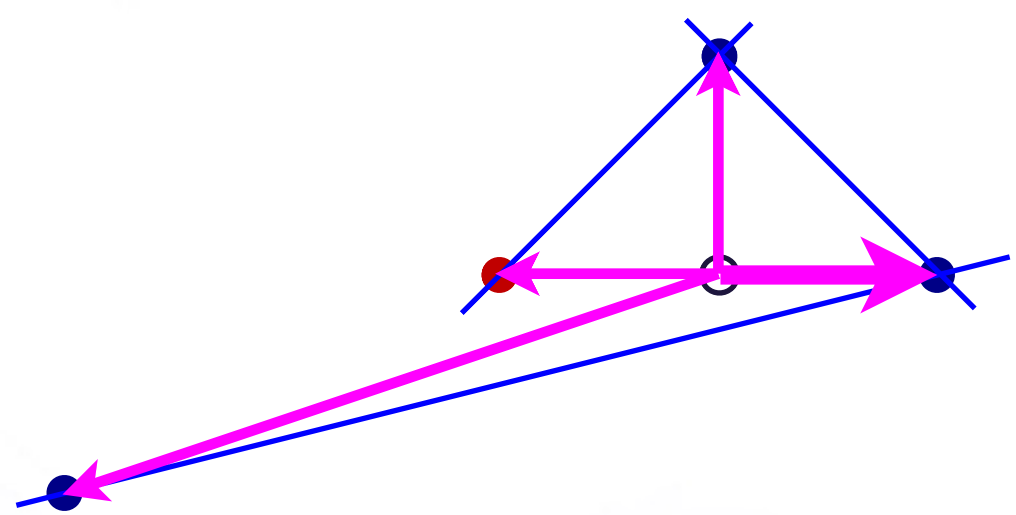

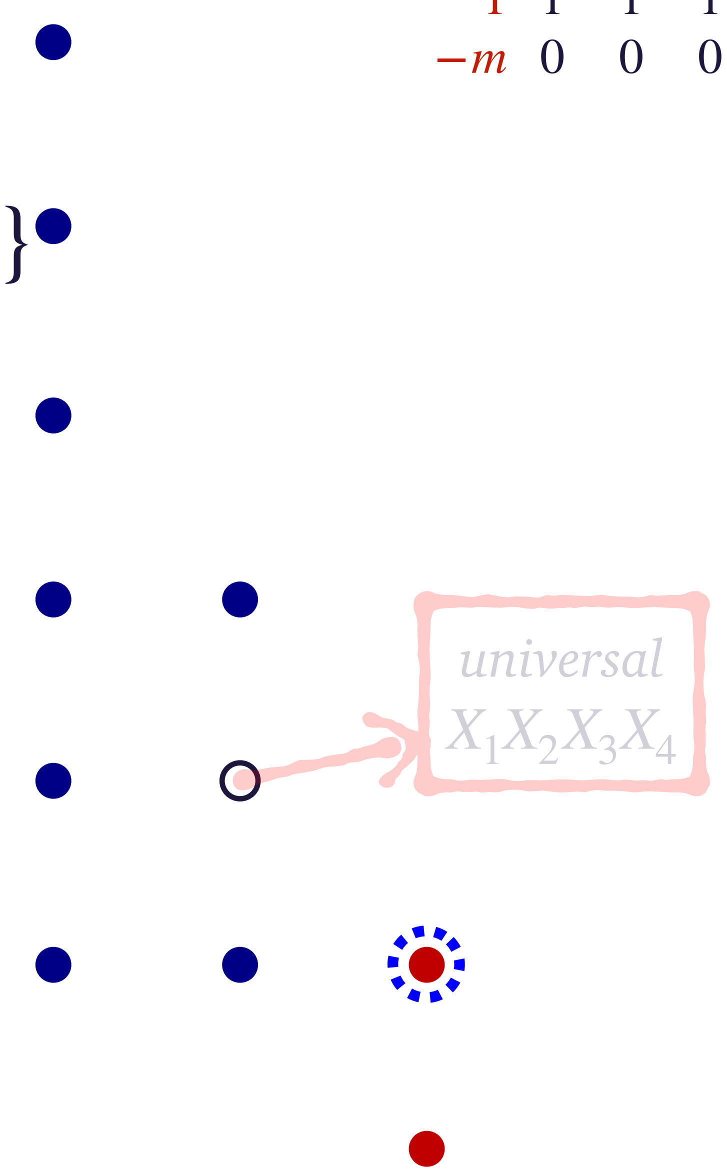

Laurent-Toric Fugue

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1611.10300 & 2205.12827 +much more

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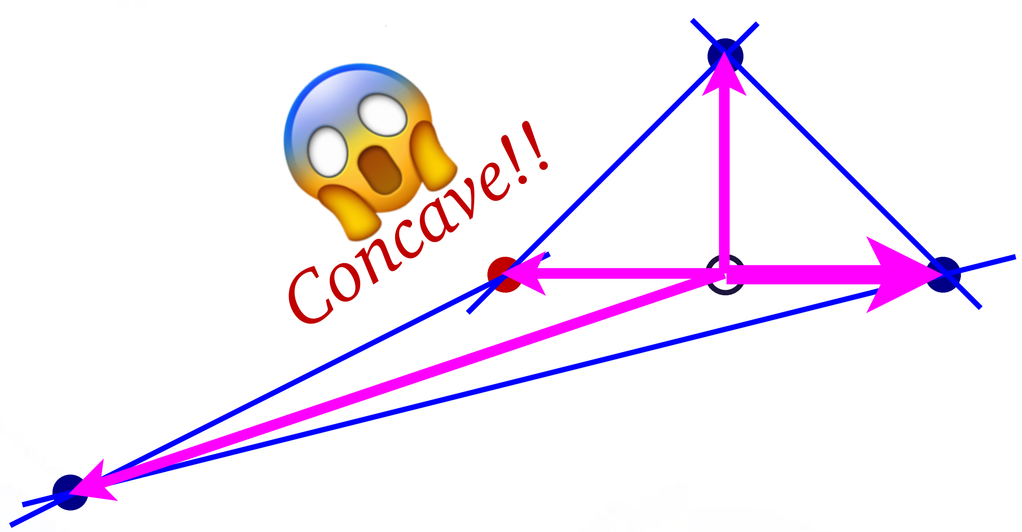

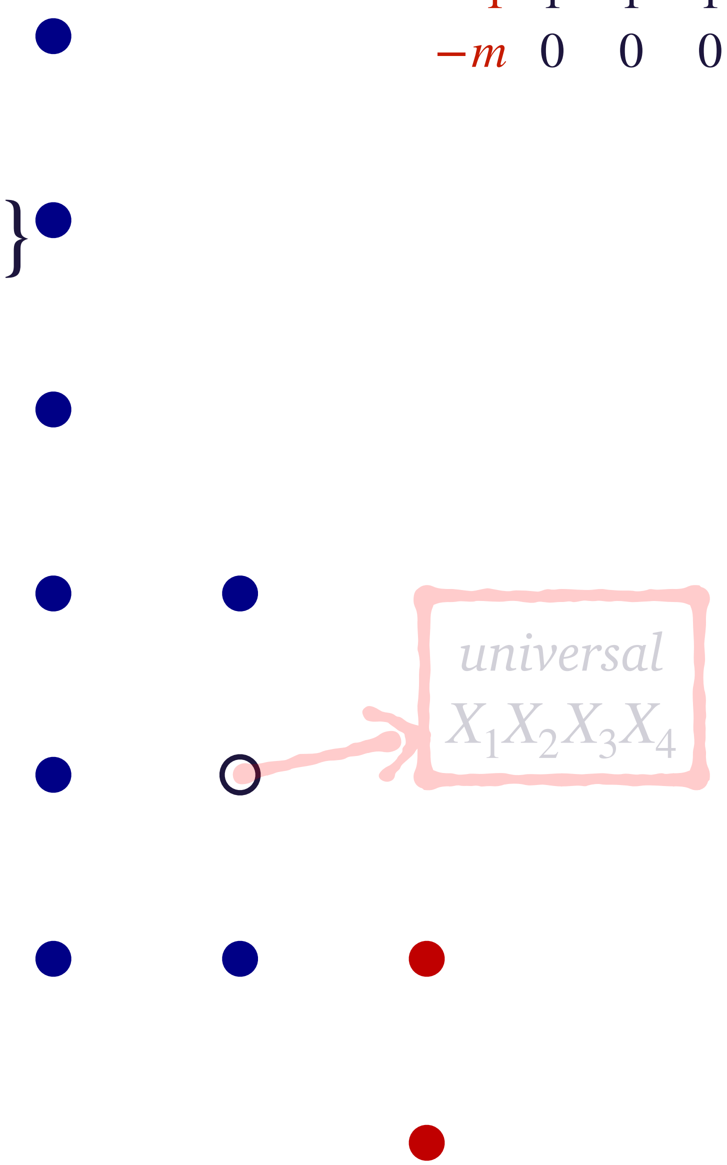

Laurent-Toric Fugue

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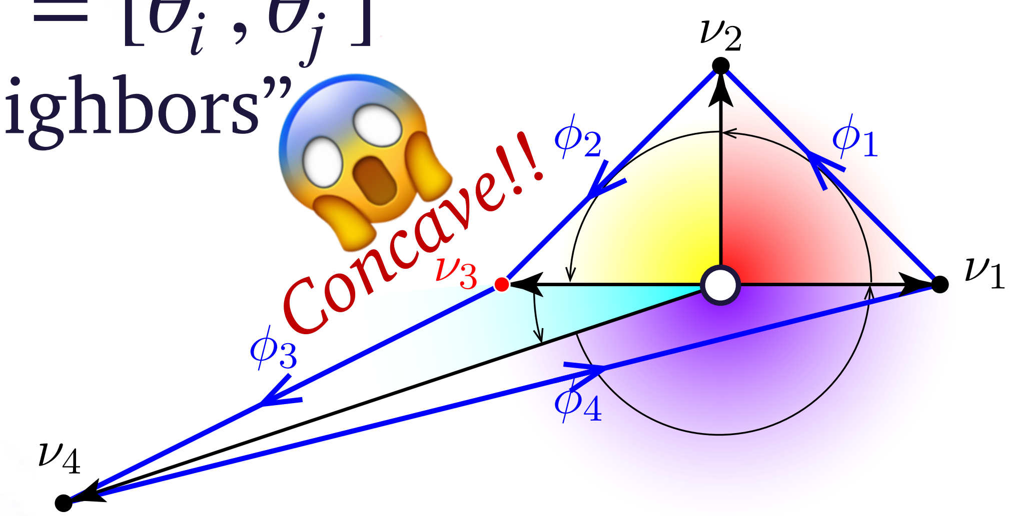

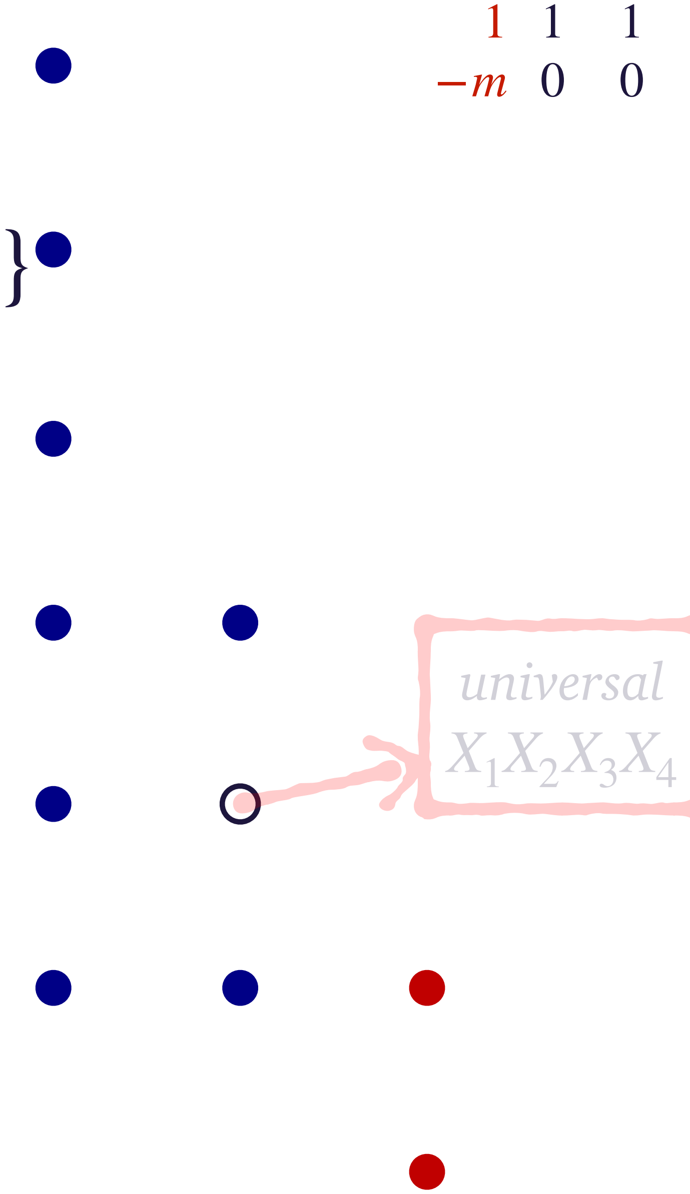

Laurent-Toric Fugue

1611.10300 & 2205.12827
+much more

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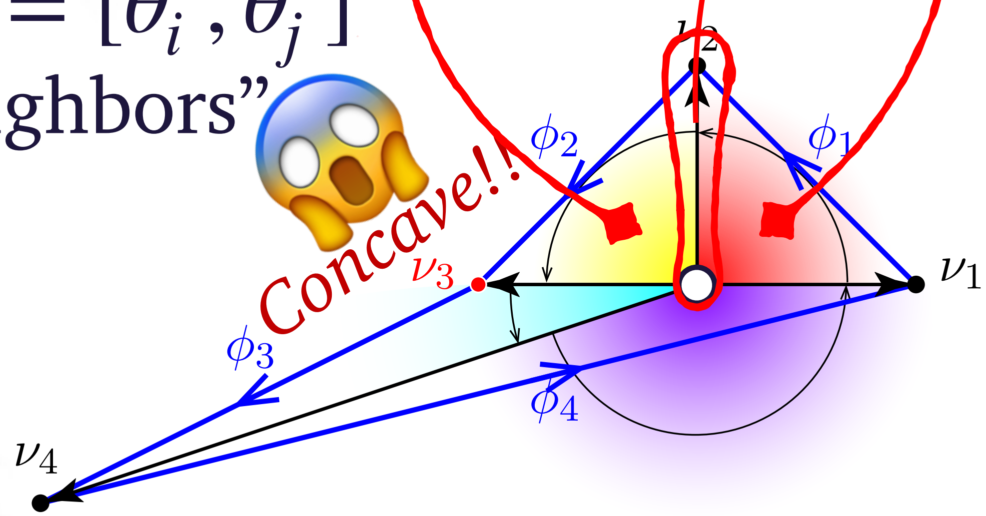

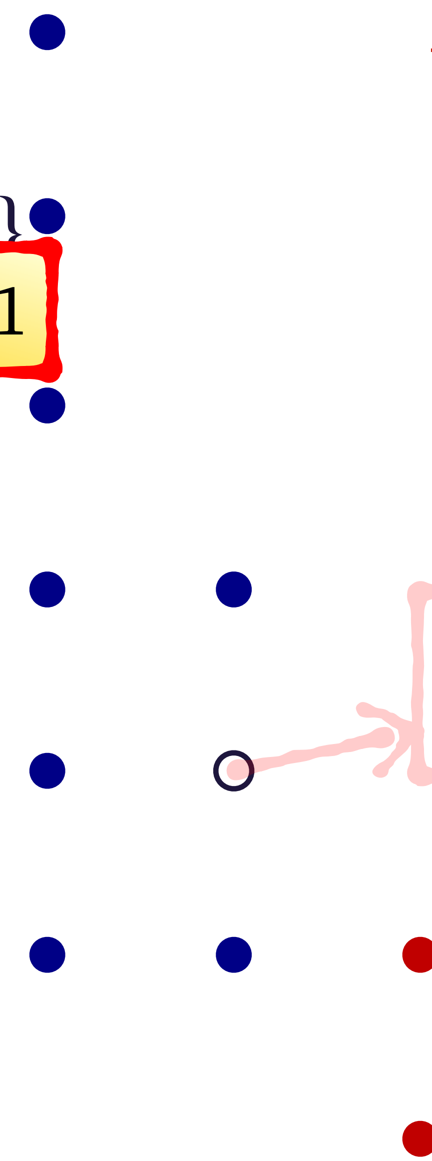

universal $X_1X_2X_3X_4$

Laurent-Toric Fugue

& Non-Convex Mirrors

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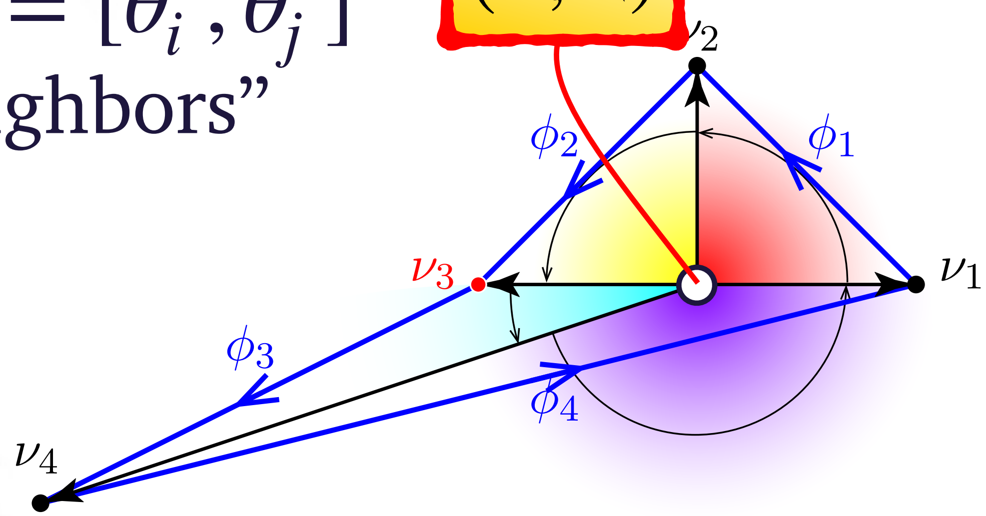
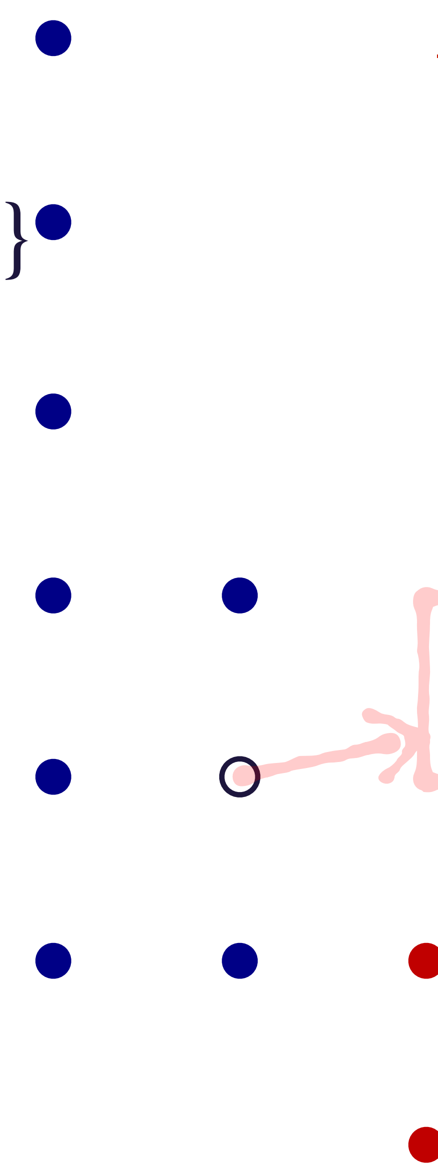
[Fulton] F_3

$(\Sigma, \prec)$





universal
$X_1X_2X_3X_4$

Laurent-Toric Fugue

1611.10300 & 2205.12827
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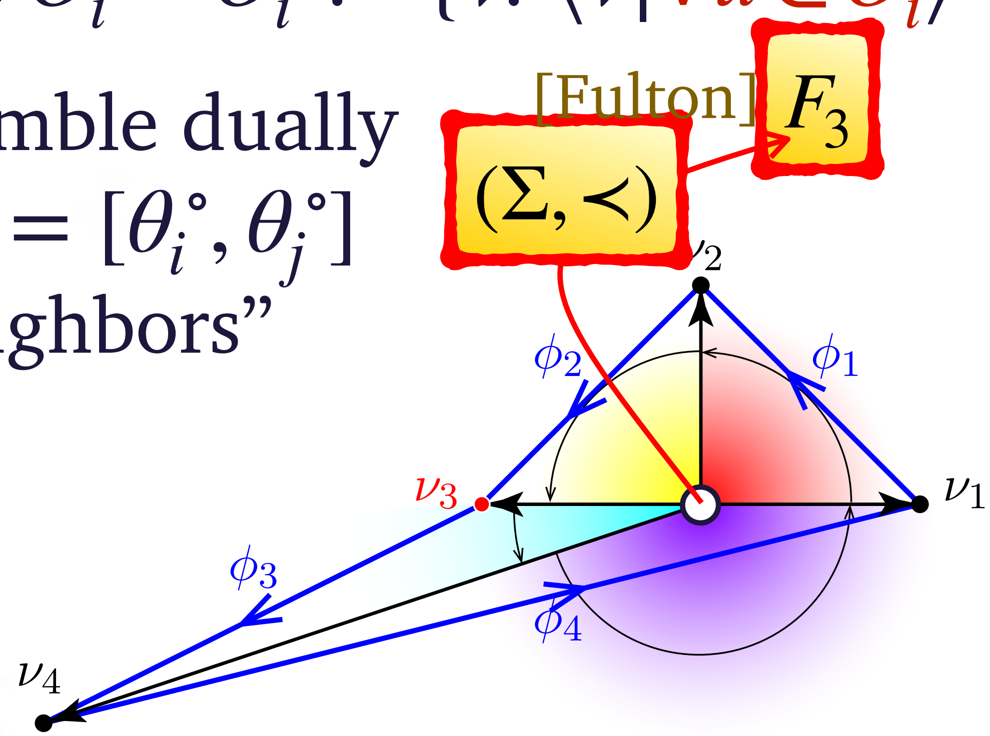

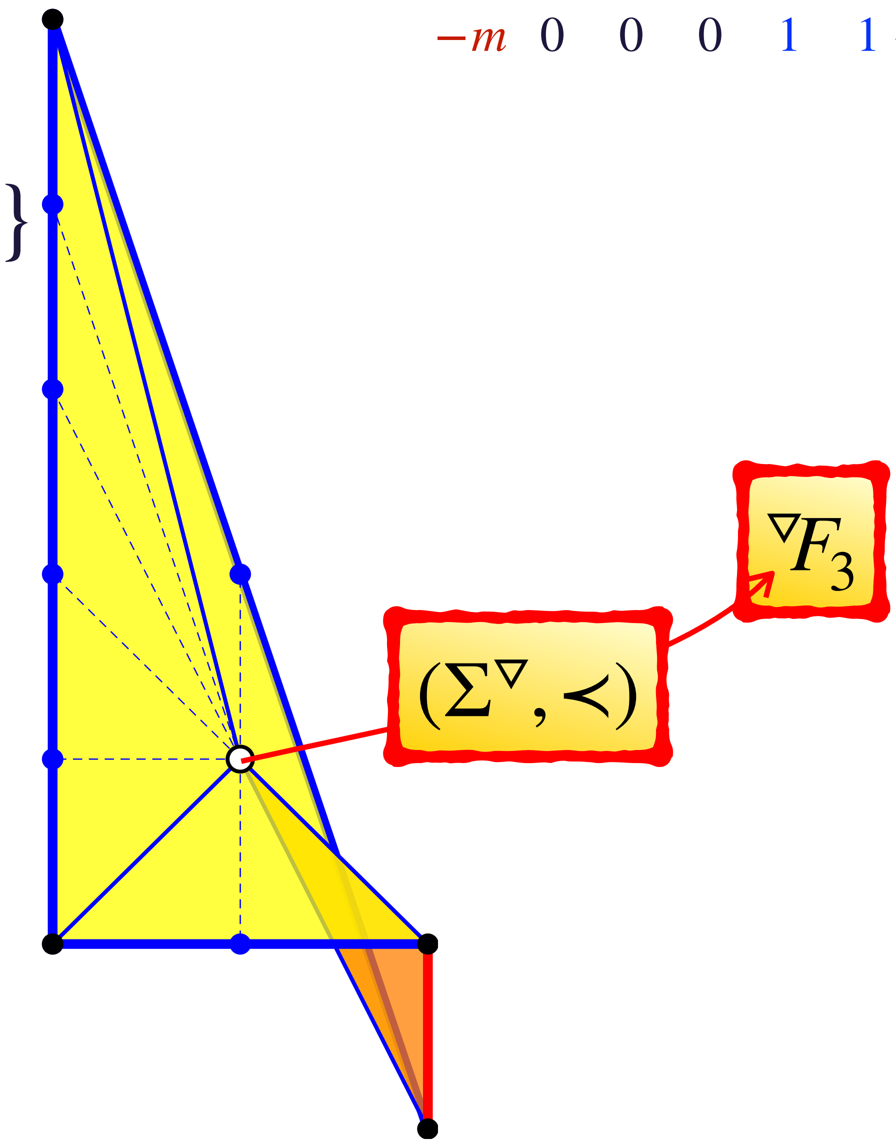

Laurent-Toric Fugue

& Non-Convex Mirrors

$m=3$ —2D *Proof-of-Concept*—

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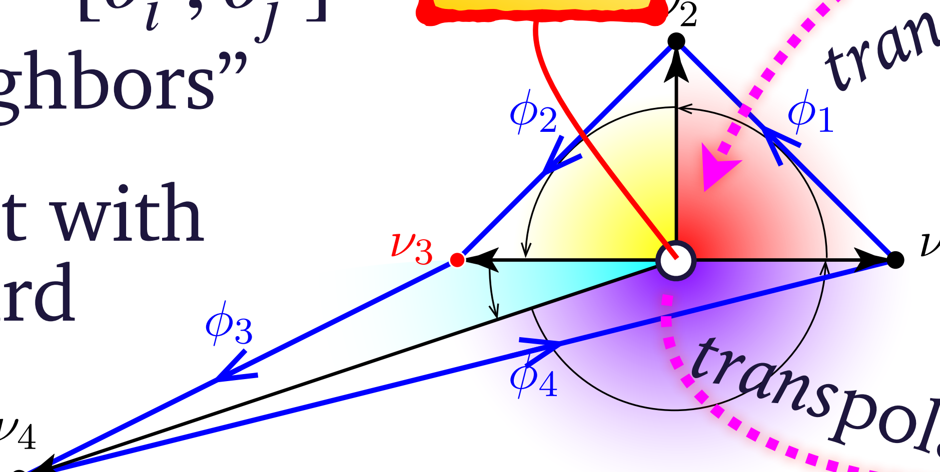
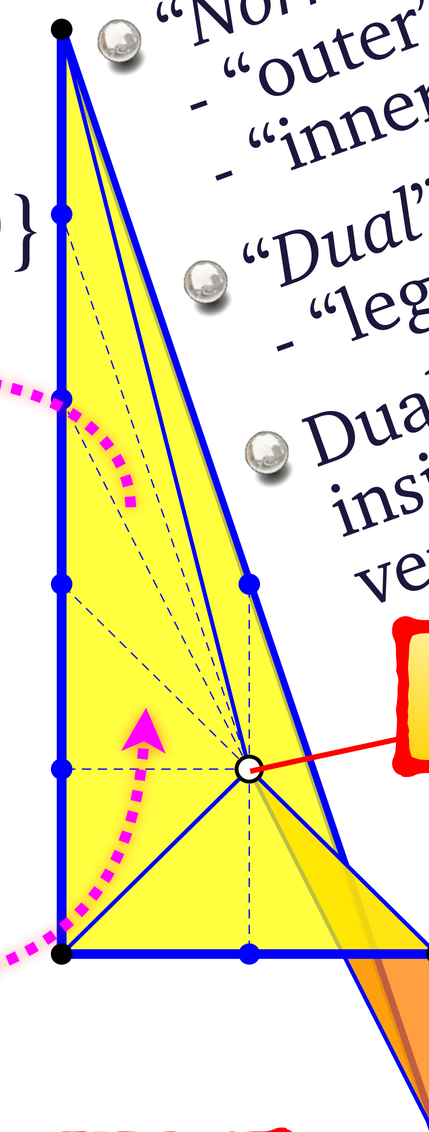
  - (Re)assemble dually $(\theta_i\cap\theta_j)^\circ = [\theta_i^\circ, \theta_j^\circ]$ with "neighbors"
  - Consistent with all standard methods

[Fulton] F_3

$(\Sigma,\prec)$





- "Normal fan"
  - "outer" [GE]
  - "inner/local" [C,L&S]
- "Dual"
  - "legal loops" [P&RV]
- Dual cones ↦ inside opening vertex-cones [?]

$^\nabla F_3$

$(\Sigma^\nabla,\prec)$

'92: Khovanskii +Pukhlikov
'93: Karshon +Tolman
'99: Hattori +Masuda
+lots of

(pre)complex *algebraic geometry*

poset $(\Sigma,\prec)$ ↔ **dual poset** $(\Sigma^\nabla,\prec)$

$F_3[c_1] \overset{\text{MM}}{\longleftrightarrow} {}^\nabla F_3[c_1]$ → *(pre)symplectic geometry*

Laurent-Toric Fugue

& Non-Convex Mirrors

—3D *Proof-of-Concept*—

combinatorially many choices…
= multiple mirrors

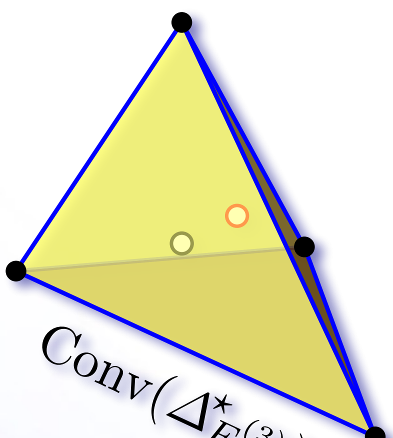
1611.10300 & 2205.12827
+much more

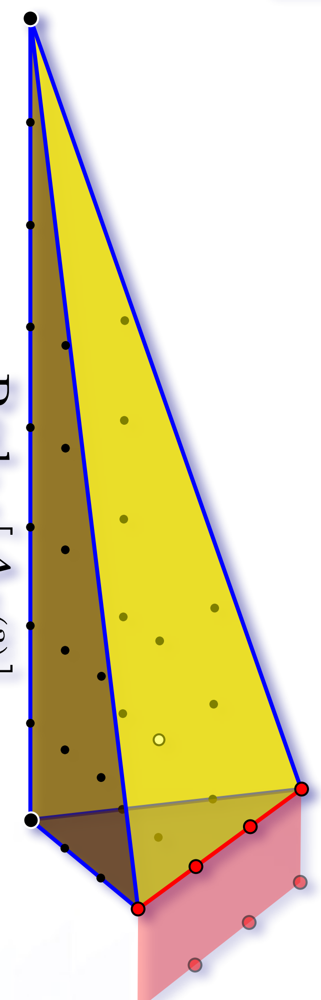
- (Toric) transposition:

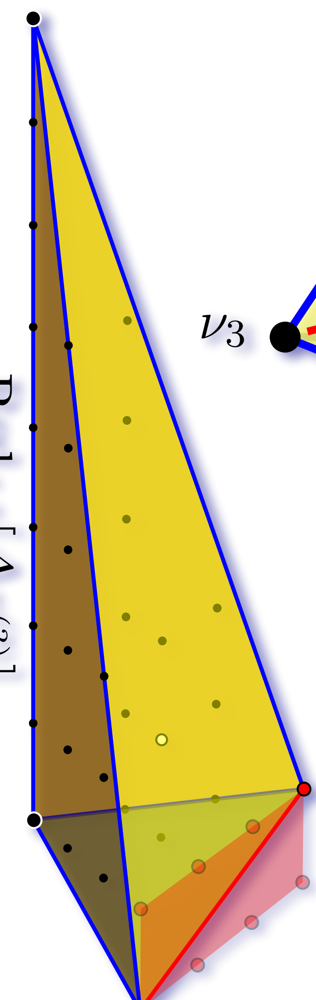
$$f(x;\Delta_{F_m^{(3)}}) = a_1\, x_1^{\,3}\, x_4^{\,2m+2} + a_2\, x_1^{\,3}\, x_5^{\,2m+2} + a_3\, \frac{x_2^{\,3}}{x_4^{\,m-2}} + a_4\, \frac{x_2^{\,3}}{x_5^{\,m-2}} + a_5\, \frac{x_3^{\,3}}{x_4^{\,m-2}} + a_6\, \frac{x_3^{\,3}}{x_5^{\,m-2}}$$

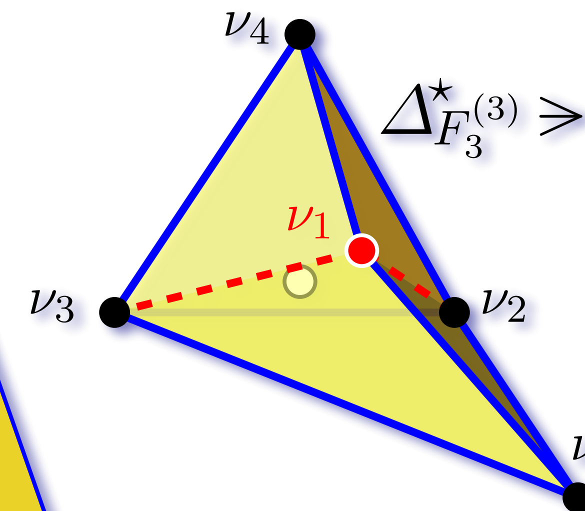
$$g(y;\Delta^\star_{F_m^{(3)}}) = \underbrace{b_1\, y_1^{\,3}\, y_2^{\,3}}_{\nu_1} + b_2\, y_3^{\,3}\, y_4^{\,3} + b_3\, y_5^{\,3}\, y_6^{\,3} + b_4\, \frac{y_1^{\,2m+2}}{(y_3\, y_5)^{m-2}} + b_5\, \frac{y_2^{\,2m+2}}{(y_4\, y_6)^{m-2}}$$

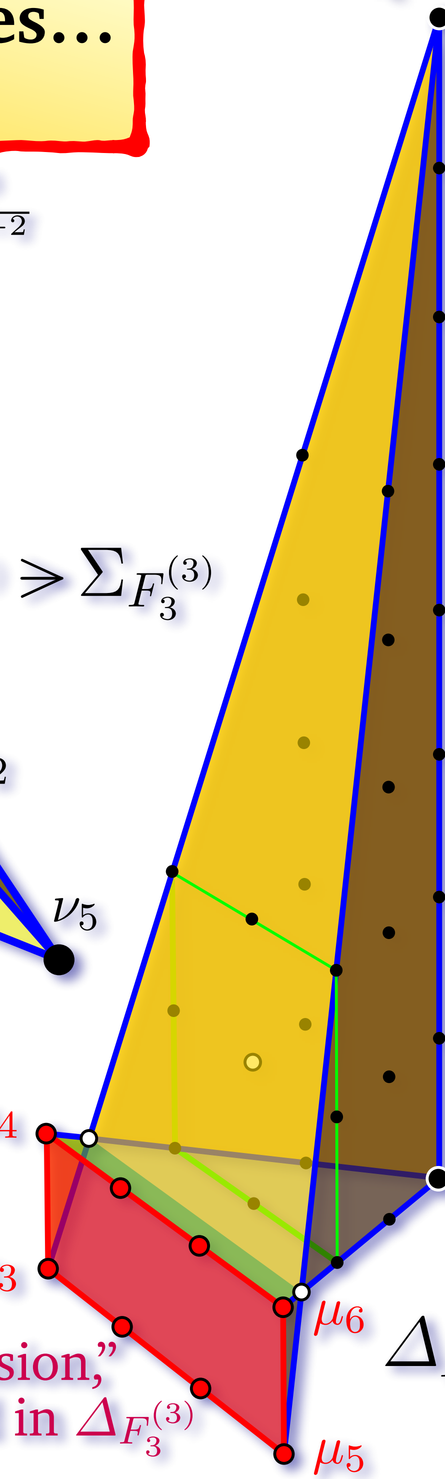
$$\mathbb{E} = \begin{bmatrix} 3 & 0 & 0 & 2m{+}2 & 0 \\ 3 & 0 & 0 & 0 & 2m{+}2 \\ 0 & 3 & 0 & 2{-}m & 0 \\ 0 & 3 & 0 & 0 & 2{-}m \\ 0 & 0 & 3 & 2{-}m & 0 \\ 0 & 0 & 3 & 0 & 2{-}m \end{bmatrix}$$











Laurent-Toric Fugue

& Non-Convex Mirrors

—3D *Proof-of-Concept*—

1611.10300 & 2205.12827 +much more

- (Toric) trans-position:

$$g(y)^{\mathsf{T}} = f(x) = a_1\, x_1^{\,3}\, x_4^{\,2m+2} + a_2\, x_1^{\,3}\, x_5^{\,2m+2} + \underline{a_3}\,\frac{x_2^{\,3}}{x_4^{\,m-2}} + a_4\,\frac{x_3^{\,3}}{x_4^{\,m-2}} + \underline{a_5}\,\frac{x_2^{\,3}}{x_5^{\,m-2}} + a_6\,\frac{x_3^{\,3}}{x_5^{\,m-2}}$$

5×6 matrix of exponents ↕ transpose

$$f(x)^{\mathsf{T}} = g(y) = b_1\, y_1^{\,3}\, y_2^{\,3} + b_2\, \underline{y_3}^{\,3}\, y_4^{\,3} + b_3\, \underline{y_5}^{\,3}\, y_6^{\,3} + b_4\,\frac{y_1^{\,2m+2}}{(\underline{y_3}\,\underline{y_5})^{m-2}} + b_5\,\frac{y_2^{\,2m+2}}{(y_4\, y_6)^{m-2}}$$

$x_1=1$, $\underline{a_3}, \underline{a_5}=0$ $\quad \mathbb{P}^3_{(3:3:1:1)}[8]$

$$a_1\, x_4^{\,8} + a_2\, x_5^{\,8} + a_4\,\frac{x_2^{\,3}}{x_5} + a_6\,\frac{x_3^{\,3}}{x_5} : \left\{\begin{array}{c} (\mathbb{Z}_3\!: \tfrac{1}{3}, \tfrac{2}{3}, 0, 0) \\ (\mathbb{Z}_{24}\!: \tfrac{1}{24}, \tfrac{1}{24}, 0, \tfrac{1}{8}) \\ \hline (\mathbb{Z}_8\!: \tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{8}, \tfrac{1}{8}) \end{array}\right. \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} : \left\{\begin{array}{l} \mathcal{G} = \mathbb{Z}_3 \times \mathbb{Z}_{24}, \\ \hline \mathcal{Q} = \mathbb{Z}_8. \end{array}\right.$$

$b_1=0$, $\underline{y_3}, \underline{y_5}=1$ $\quad \mathbb{P}^3_{(3:5:8:8)}[24]$

$$b_2\, y_4^{\,3} + b_3\, y_6^{\,3} + b_4\, y_1^{\,8} + b_5\,\frac{y_2^{\,8}}{y_4\, y_6} : \left\{\begin{array}{c} (\mathbb{Z}_8\!: \tfrac{1}{8}, 0, 0, 0) \\ (\mathbb{Z}_3\!: 0, 0, \tfrac{1}{3}, \tfrac{2}{3}) \\ \hline (\mathbb{Z}_8\!: \tfrac{5}{24}, \tfrac{3}{24}, \tfrac{1}{3}, \tfrac{1}{3}) \end{array}\right. \begin{bmatrix} y_1 \\ y_2 \\ y_4 \\ y_6 \end{bmatrix} : \left\{\begin{array}{l} \mathcal{G}^{\nabla} = \mathbb{Z}_8 \times \mathbb{Z}_3, \\ \hline \mathcal{Q}^{\nabla} = \mathbb{Z}_{24}. \end{array}\right.$$

$x_1=1$, $a_4, a_5=0$ $\quad \mathbb{P}^3_{(3:3:1:1)}[8]$

$$a_1\, x_4^{\,8} + a_2\, x_5^{\,8} + a_4\,\frac{x_2^{\,3}}{x_5} + a_5\,\frac{x_3^{\,3}}{x_4} : \left\{\begin{array}{c} (\mathbb{Z}_3\!: \tfrac{1}{3}, \tfrac{1}{3}, 0, 0) \\ (\mathbb{Z}_{24}\!: \tfrac{1}{24}, \tfrac{23}{24}, \tfrac{1}{8}, \tfrac{7}{8}) \\ \hline (\mathbb{Z}_8\!: \tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{8}, \tfrac{1}{8}) \end{array}\right. \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} : \left\{\begin{array}{l} \mathcal{G} = \mathbb{Z}_3 \times \mathbb{Z}_{24}, \\ \hline \mathcal{Q} = \mathbb{Z}_8. \end{array}\right.$$

$b_1=1$, $y_4, y_5=0$ $\quad \mathbb{P}^3_{(1:1:2:2)}[6]$

$$b_2\, y_4^{\,3} + b_3\, y_5^{\,3} + b_4\,\frac{y_1^{\,8}}{y_5} + b_5\,\frac{y_2^{\,8}}{y_4} : \left\{\begin{array}{c} (\mathbb{Z}_4\!: \tfrac{1}{4}, \tfrac{1}{4}, 0, 0) \\ (\mathbb{Z}_{24}\!: \tfrac{1}{24}, \tfrac{23}{24}, \tfrac{1}{3}, \tfrac{2}{3}) \\ \hline (\mathbb{Z}_6\!: \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{3}) \end{array}\right. \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_6 \end{bmatrix} : \left\{\begin{array}{l} \mathcal{G}^{\nabla} = \mathbb{Z}_4 \times \mathbb{Z}_{24}, \\ \hline \mathcal{Q}^{\nabla} = \mathbb{Z}_6. \end{array}\right.$$

Laurent-Toric Fugue

& Non-Convex Mirrors

—3D *Proof-of-Concept*—

1611.10300 & 2205.12827
+much more

- (Toric) trans-position: $g(y)^{\mathsf{T}} = f(x) = a_1\, x_1^{\,3}\, x_4^{\,2m+2} + a_2\, x_1^{\,3}\, x_5^{\,2m+2} + \underline{a_3}\, \frac{x_2^{\,3}}{x_4^{\,m-2}} + a_4\, \frac{x_3^{\,3}}{x_4^{\,m-2}} + \underline{a_5}\, \frac{x_2^{\,3}}{x_5^{\,m-2}} + a_6\, \frac{x_3^{\,3}}{x_5^{\,m-2}}$

  5×6 matrix of exponents ↕ transpose

  $f(x)^{\mathsf{T}} = g(y) = b_1\, y_1^{\,3}\, y_2^{\,3} + b_2\, \underline{y_3}^{\,3}\, y_4^{\,3} + b_3\, \underline{y_5}^{\,3}\, y_6^{\,3} + b_4\, \frac{y_1^{\,2m+2}}{(\underline{y_3}\, \underline{y_5})^{m-2}} + b_5\, \frac{y_2^{\,2m+2}}{(y_4\, y_6)^{m-2}}$

deformation

$x_1 = 1$, $\underline{a_3}, \underline{a_5} = 0$ $\quad \mathbb{P}^3_{(3:3:1:1)}[8]$

$$a_1\, x_4^{\,8} + a_2\, x_5^{\,8} + a_4\, \frac{x_2^{\,3}}{x_5} + a_6\, \frac{x_3^{\,3}}{x_5} : \left\{ \frac{\begin{matrix}(\mathbb{Z}_3 : \tfrac{1}{3}, \tfrac{2}{3}, 0, 0)\\ (\mathbb{Z}_{24} : \tfrac{1}{24}, \tfrac{1}{24}, 0, \tfrac{1}{8})\end{matrix}}{(\mathbb{Z}_8 : \tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{8}, \tfrac{1}{8})} \right. \begin{bmatrix} x_2\\ x_3\\ x_4\\ x_5 \end{bmatrix} : \left\{ \frac{\mathcal{G} = \mathbb{Z}_3 \times \mathbb{Z}_{24},}{\mathcal{Q} = \mathbb{Z}_8.} \right.$$

$b_1 = 0$, $\underline{y_3}, \underline{y_5} = 1$ $\quad \mathbb{P}^3_{(3:5:8:8)}[24]$

$$b_2\, y_4^{\,3} + b_3\, y_6^{\,3} + b_4\, y_1^{\,8} + b_5\, \frac{y_2^{\,8}}{y_4\, y_6} : \left\{ \frac{\begin{matrix}(\mathbb{Z}_8 : \tfrac{1}{8}, 0, 0, 0)\\ (\mathbb{Z}_3 : 0, 0, \tfrac{1}{3}, \tfrac{2}{3})\end{matrix}}{(\mathbb{Z}_8 : \tfrac{5}{24}, \tfrac{3}{24}, \tfrac{1}{3}, \tfrac{1}{3})} \right. \begin{bmatrix} y_1\\ y_2\\ y_4\\ y_6 \end{bmatrix} : \left\{ \frac{\mathcal{G}^{\nabla} = \mathbb{Z}_8 \times \mathbb{Z}_3,}{\mathcal{Q}^{\nabla} = \mathbb{Z}_{24}.} \right.$$

quotient ***either*** one of the two models by the $\mathbb{Z}_3$

$x_1 = 1$, $a_4, a_5 = 0$ $\quad \mathbb{P}^3_{(3:3:1:1)}[8]$

$$a_1\, x_4^{\,8} + a_2\, x_5^{\,8} + a_4\, \frac{x_2^{\,3}}{x_5} + a_5\, \frac{x_3^{\,3}}{x_4} : \left\{ \frac{\begin{matrix}(\mathbb{Z}_3 : \tfrac{1}{3}, \tfrac{1}{3}, 0, 0)\\ (\mathbb{Z}_{24} : \tfrac{1}{24}, \tfrac{23}{24}, \tfrac{1}{8}, \tfrac{7}{8})\end{matrix}}{(\mathbb{Z}_8 : \tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{8}, \tfrac{1}{8})} \right. \begin{bmatrix} x_2\\ x_3\\ x_4\\ x_5 \end{bmatrix} : \left\{ \frac{\mathcal{G} = \mathbb{Z}_3 \times \mathbb{Z}_6}{\mathcal{Q} = \mathbb{Z}_8 \times \mathbb{Z}_4} \right.$$

$/\mathbb{Z}_4$

for example

$b_1 = 1$, $y_4, y_5 = 0$ $\quad \mathbb{P}^3_{(1:1:2:2)}[6]$

$$b_2\, y_4^{\,3} + b_3\, y_5^{\,3} + b_4\, \frac{y_1^{\,8}}{y_5} + b_5\, \frac{y_2^{\,8}}{y_4} : \left\{ \frac{\begin{matrix}(\mathbb{Z}_4 : \tfrac{1}{4}, \tfrac{1}{4}, 0, 0)\\ (\mathbb{Z}_{24} : \tfrac{1}{24}, \tfrac{23}{24}, \tfrac{1}{3}, \tfrac{2}{3})\end{matrix}}{(\mathbb{Z}_6 : \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{3})} \right. \begin{bmatrix} y_1\\ y_2\\ y_3\\ y_6 \end{bmatrix} : \left\{ \frac{\mathcal{G}^{\nabla} = \mathbb{Z}_4 \times \mathbb{Z}_8}{\mathcal{Q}^{\nabla} = \mathbb{Z}_6 \times \mathbb{Z}_3} \right.$$

$/\mathbb{Z}_3$

Laurent Family Picture

arXiv: "real soon"

Summary

— …*threescore-six moons ago, today* —

- CY($n-1$)-folds in Hirzebruch n-folds
  - Euler characteristic ✅
  - Chern class, term-by-term ✅
  - Hodge numbers ✅ *(jump @ $^\sharp\mathscr{X}$)*
  - Cornerstone polynomials & mirror ✅
  - Phase-space regions & mirror ✅
  - Phase-space discriminant & mirror ✅
  - The "other way around" ✅ (*limited!*)
  - Yukawa couplings ✅
  - World-sheet instantons ✅
  - Gromov-Witten invariants SOON ?✅
    - *Will there be anything else? …being ML-datamined*

$d(\theta^{(k)}) := k!\,\mathrm{Vol}(\theta^{(k)})$ [BH: *signed* by orientation!]

- Oriented polytopes
- Trans-polar$^\triangledown$ constr.
- Newton $\Delta_X := (\Delta^\star_X)^\triangledown$
- VEX polytopes s.t.: $((\Delta)^\triangledown)^\triangledown = \Delta$
- Star-triangulable w/flip-folded faces
- Polytope *extension* ⇔ Laurent monomials

B^3H^2K mirrors

& GLSM
Toric textbooks to be *…extended*

Laurent Family Picture

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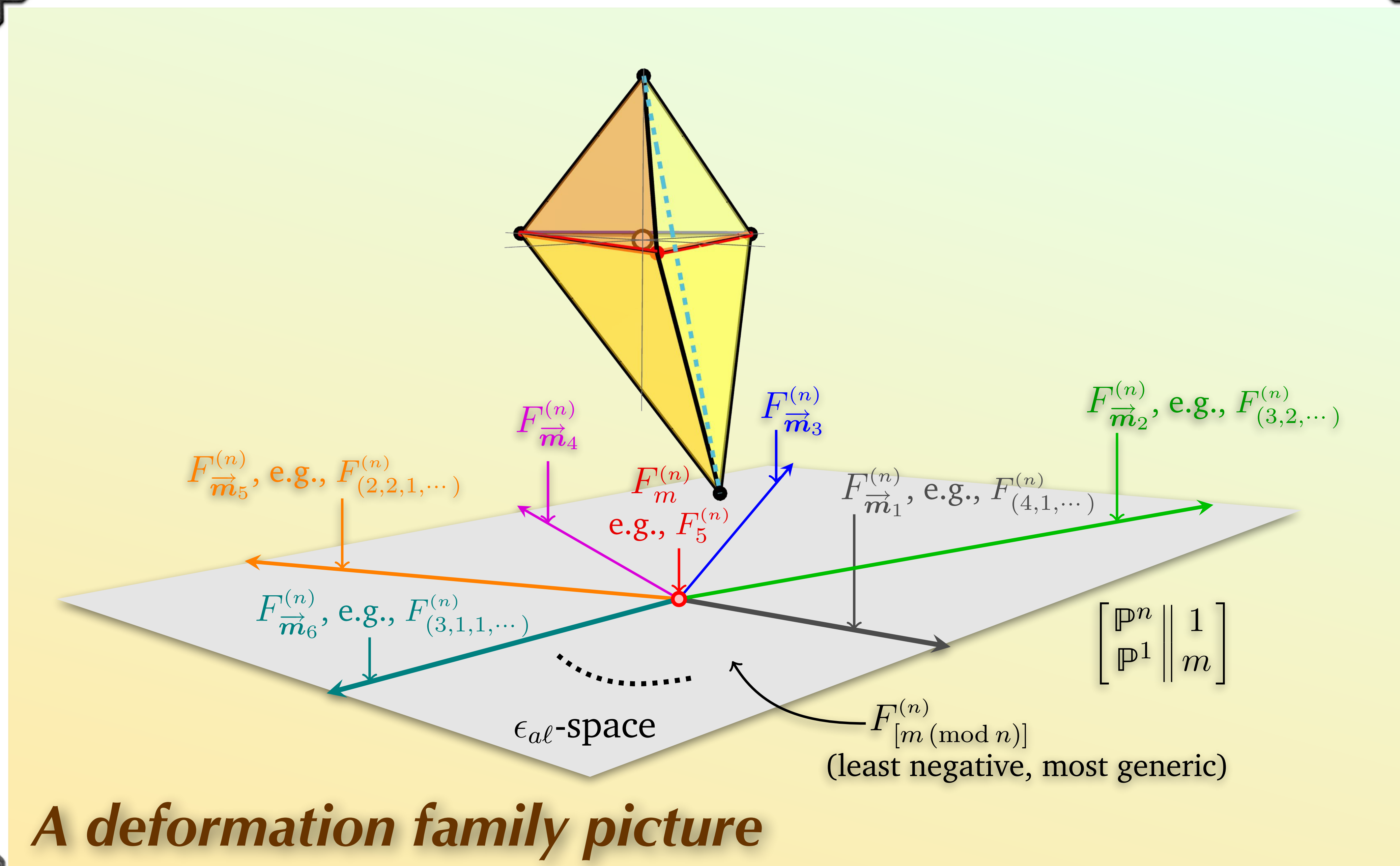
- Oriented polytopes



str.

$X)^{\triangledown}$

B³H²K mirrors

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Laurent Family Picture

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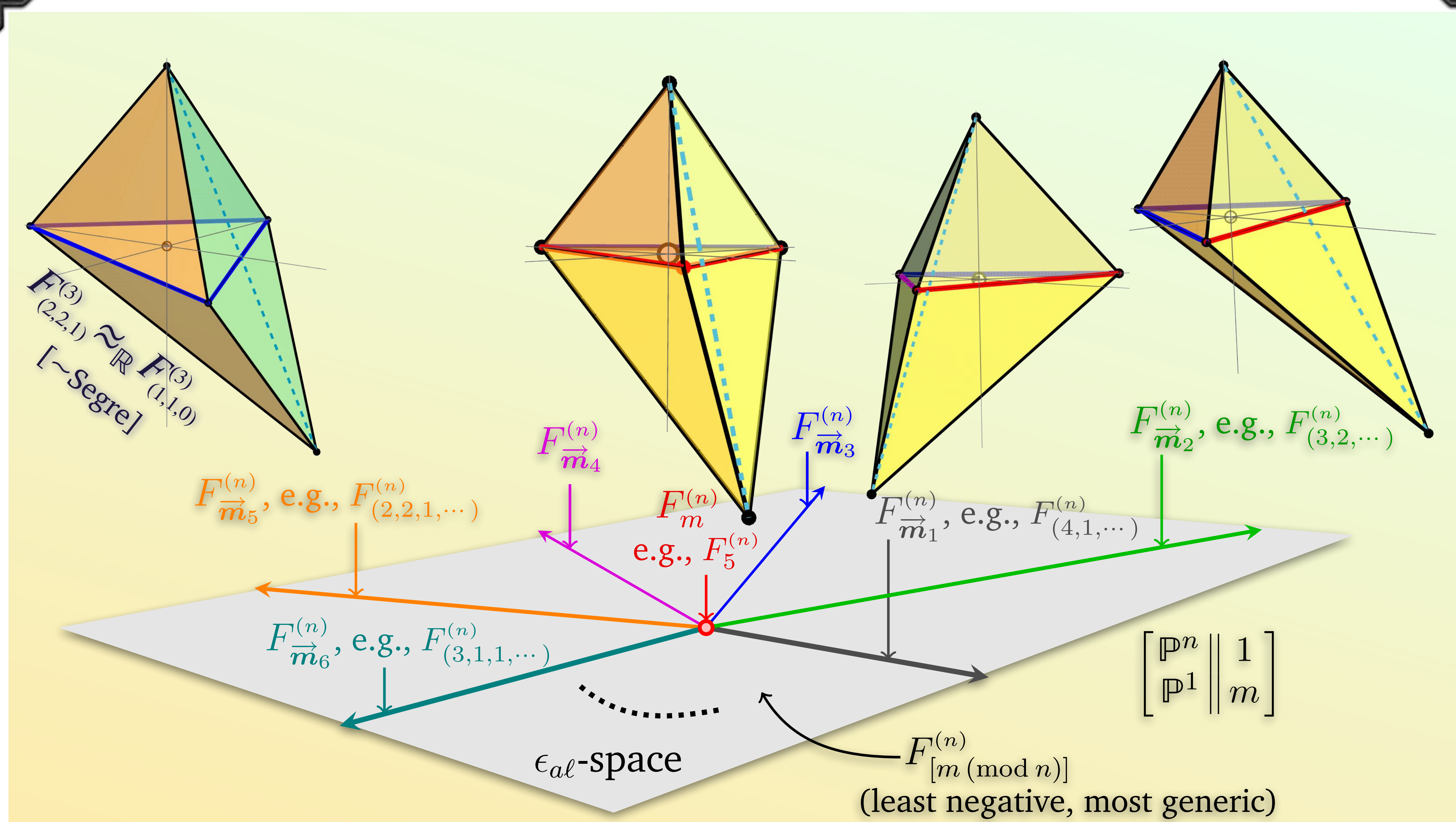

A deformation family picture

B³H²K mirrors

Laurent Family Picture

arXiv: "real soon"

Summary

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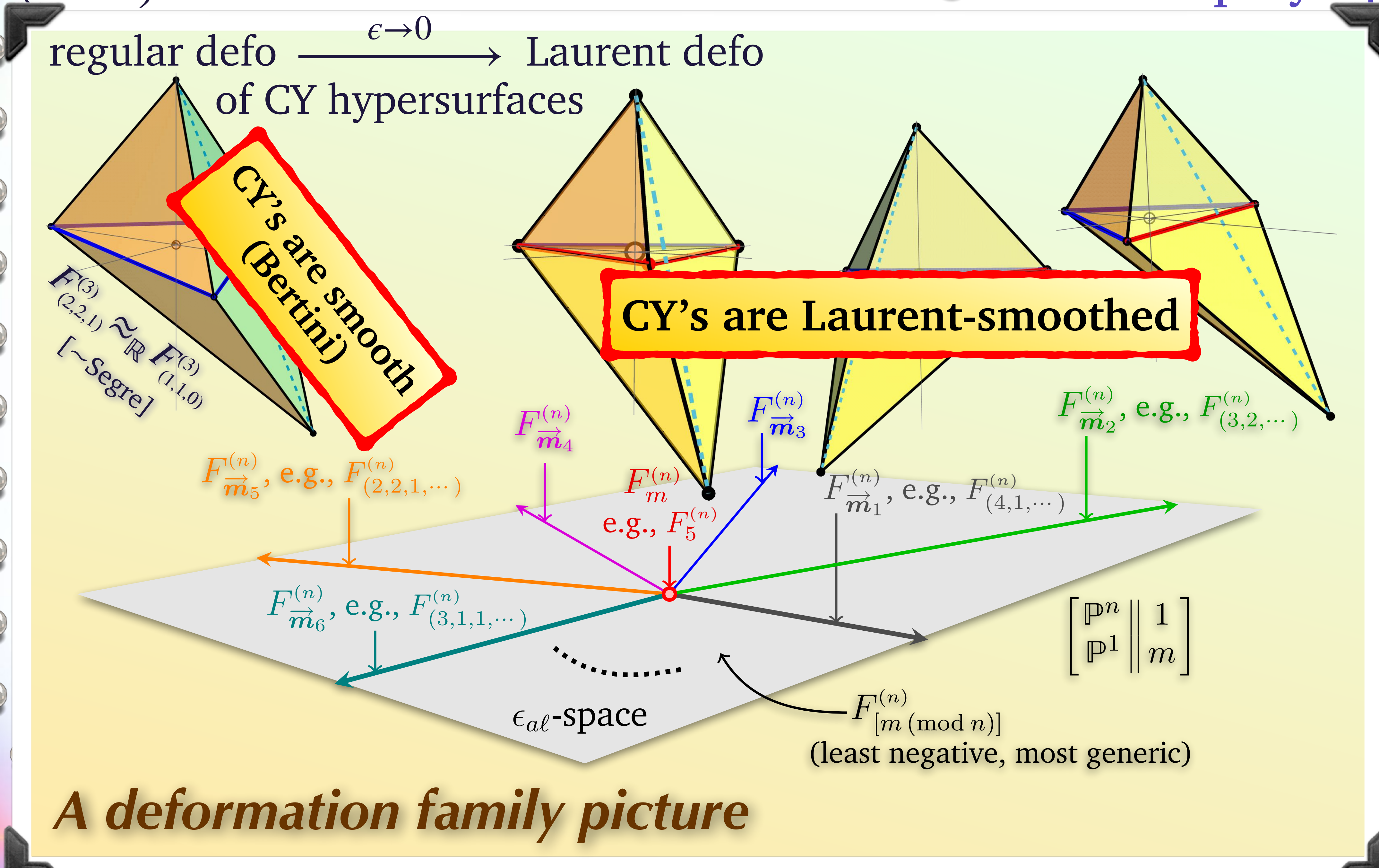

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$(\ldots_X^\star)^\nabla$

B^{3}H^{2}K mirrors

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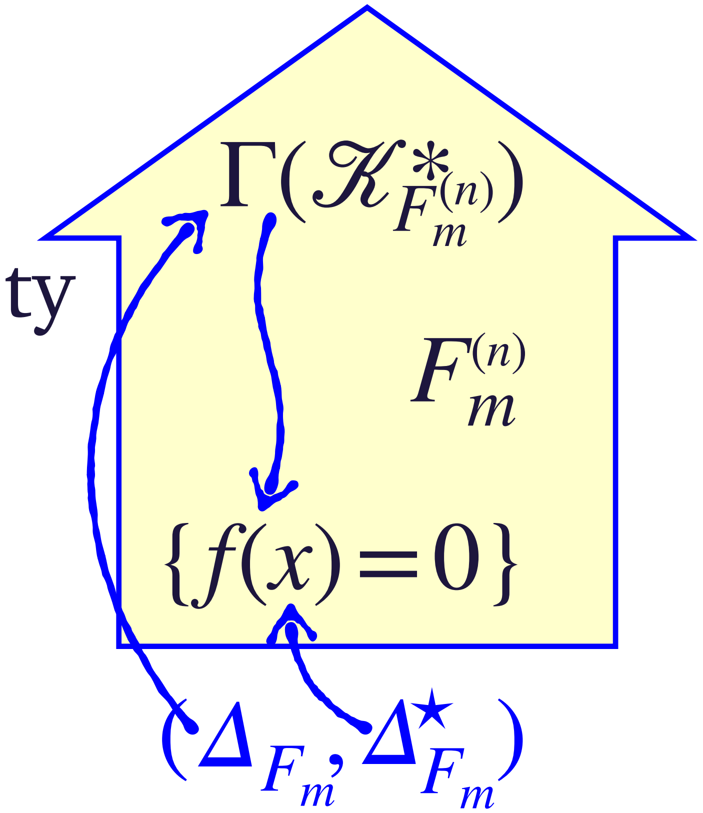
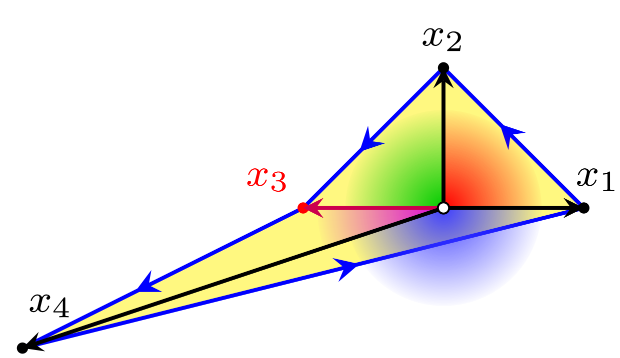
New? Toric Spaces

1611.10300 & 2205.12827
+much more

Sit Tight and Assess

- Step back for the “big picture”
- Toric (complex algebraic) variety
  - A deformation family of CY hypersurfaces: $F_m^{(n)}[c_1]$
  - In toric-speak (blueprint):





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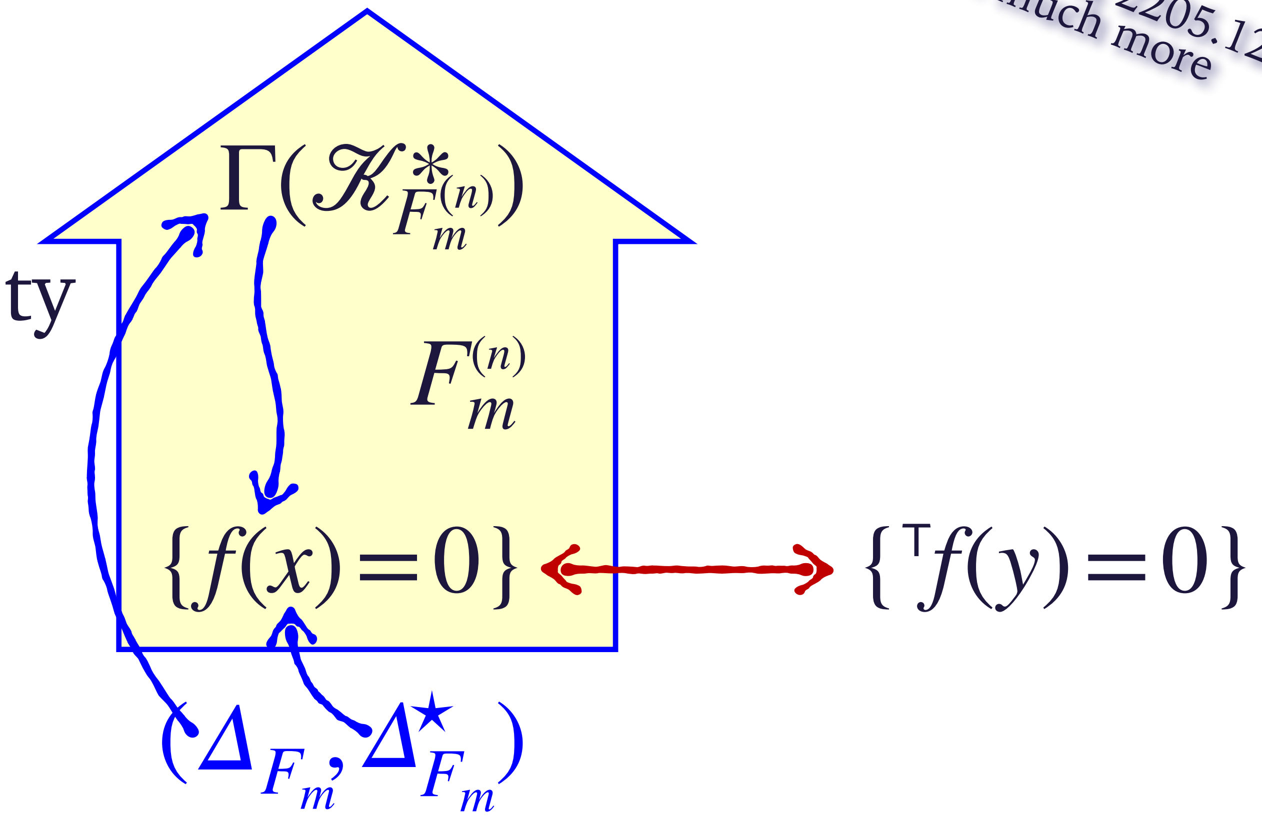

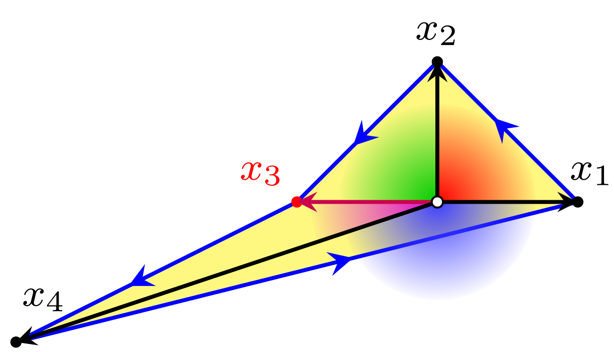

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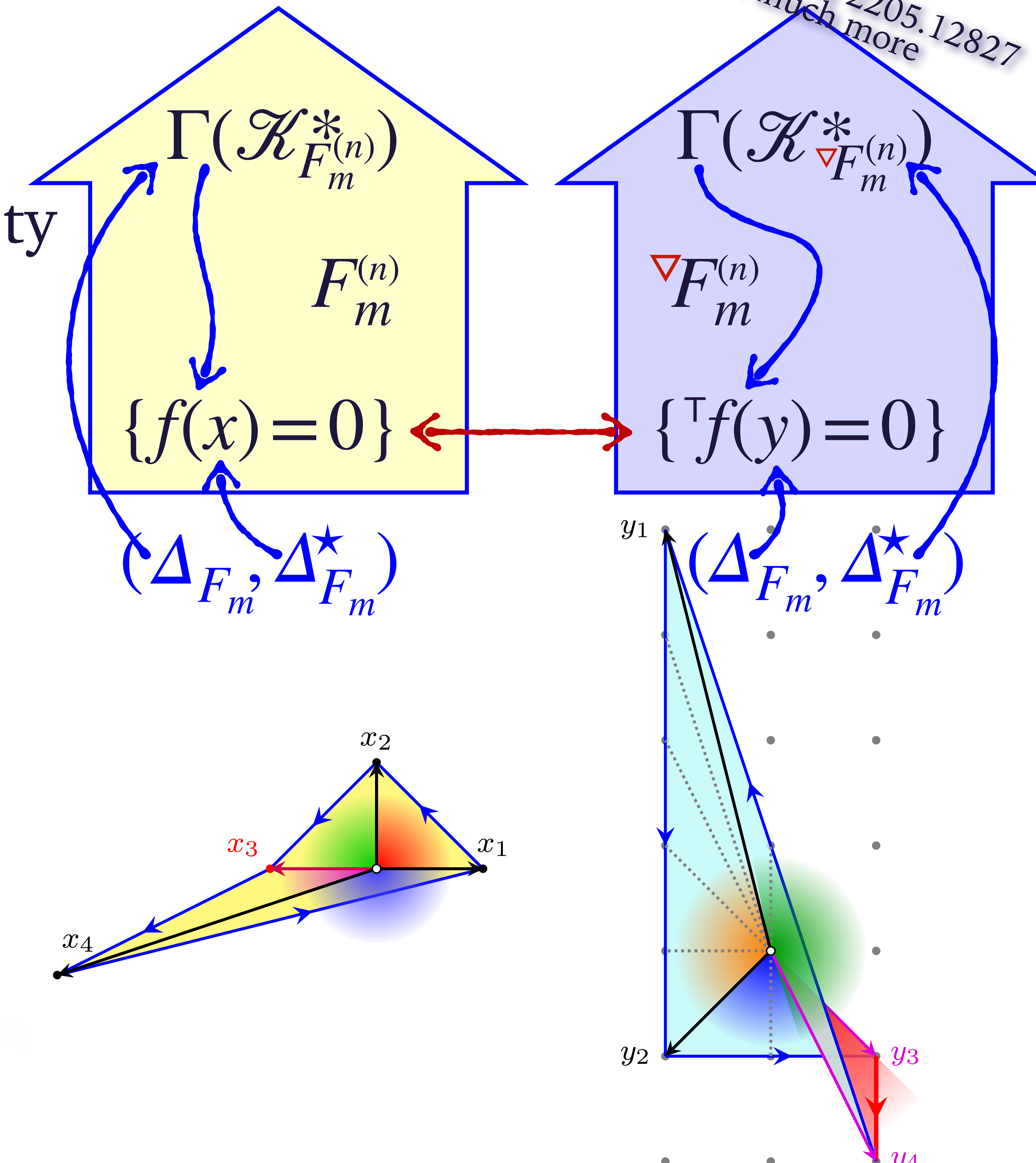

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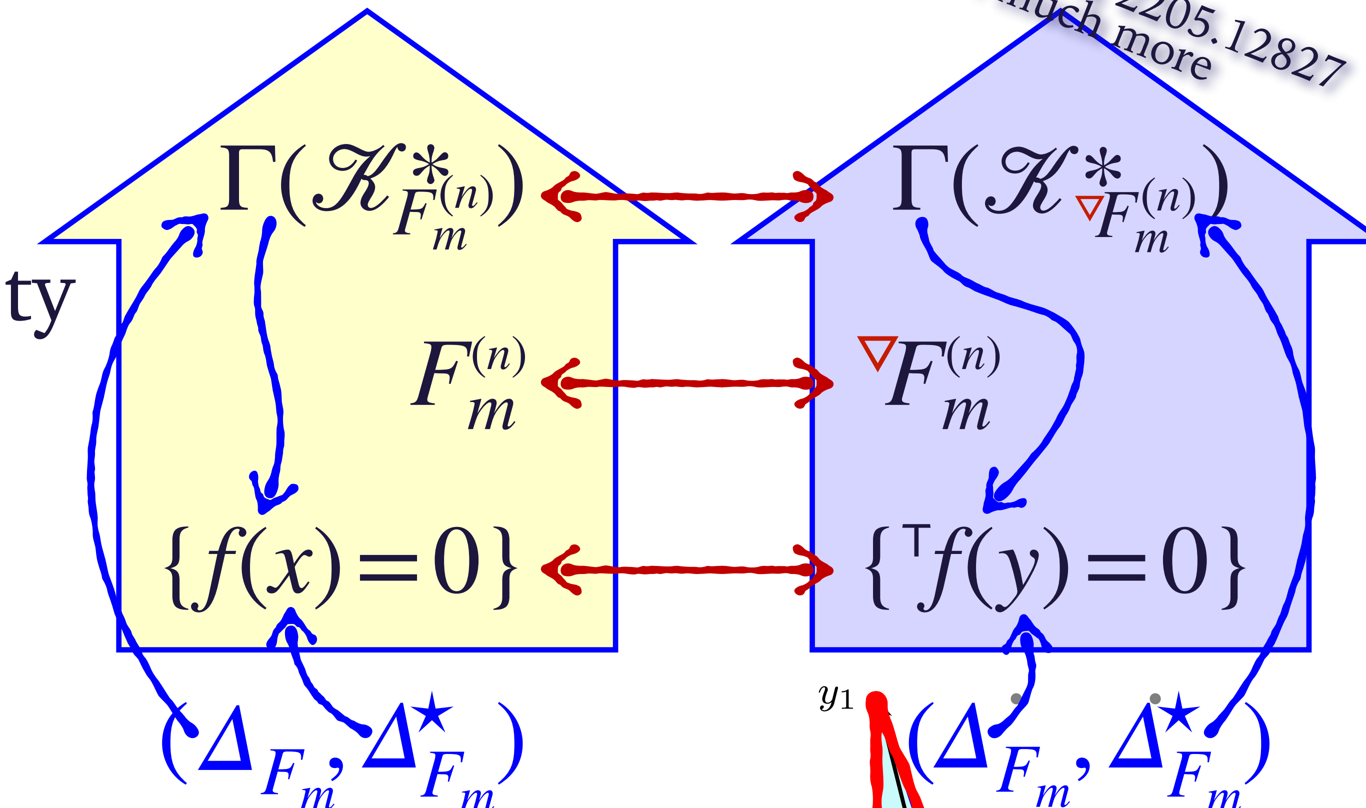
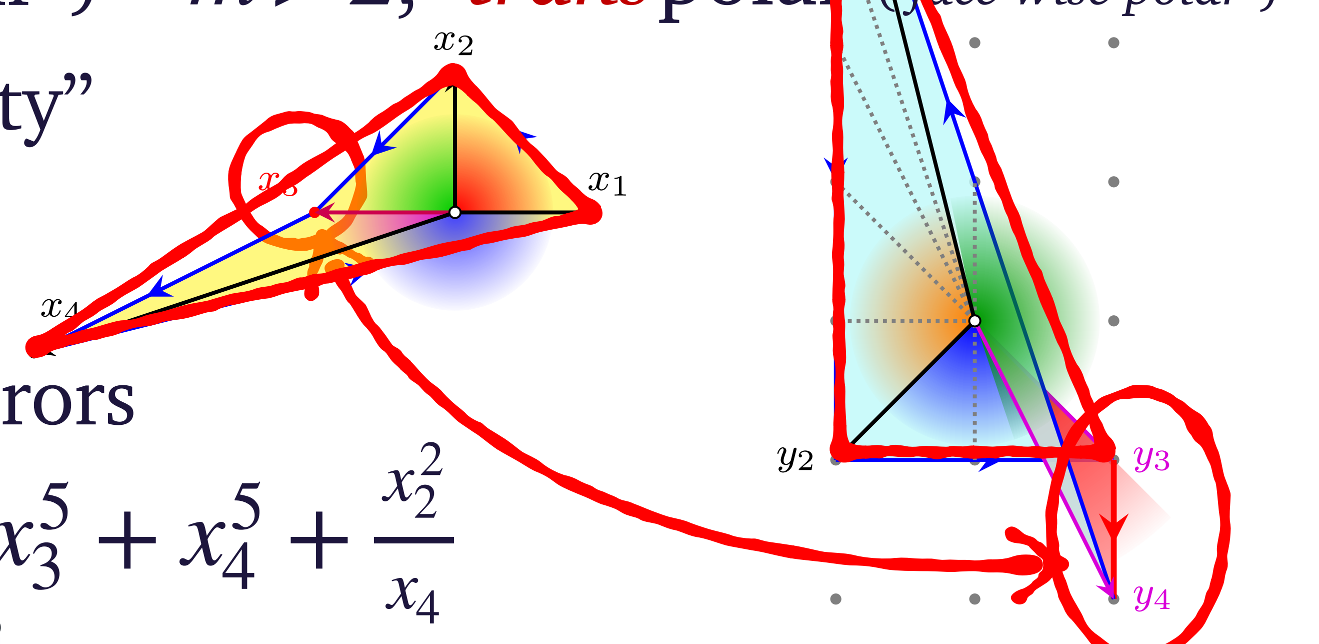
  - Fano ($m = 0,1,2$): "$\triangledown = \circ$" ("polar")
  - The "extension" ↔ "non-convexity" for all $m > 2$
  - Pick simplicial subsets for defining sections → multiple mirrors



$m > 2$, *trans*polar (*face-wise polar*)



$$x_3^5 + x_4^5 + \frac{x_2^2}{x_4}$$

New? Toric Spaces

1611.10300 & 2205.12827
+much more

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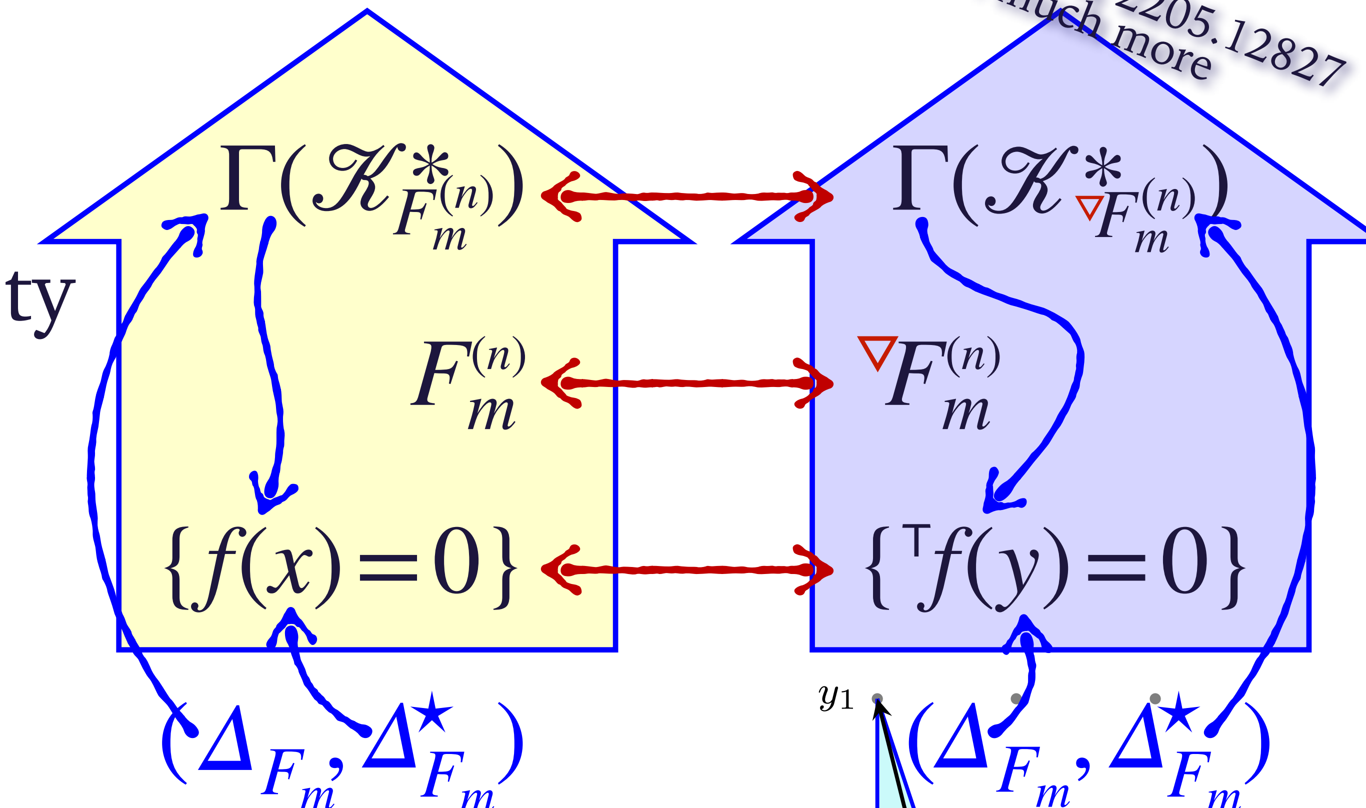

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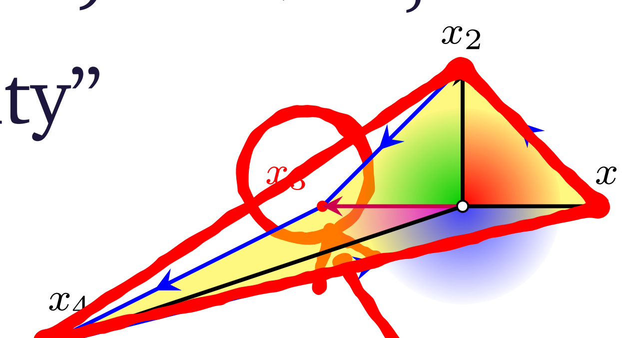

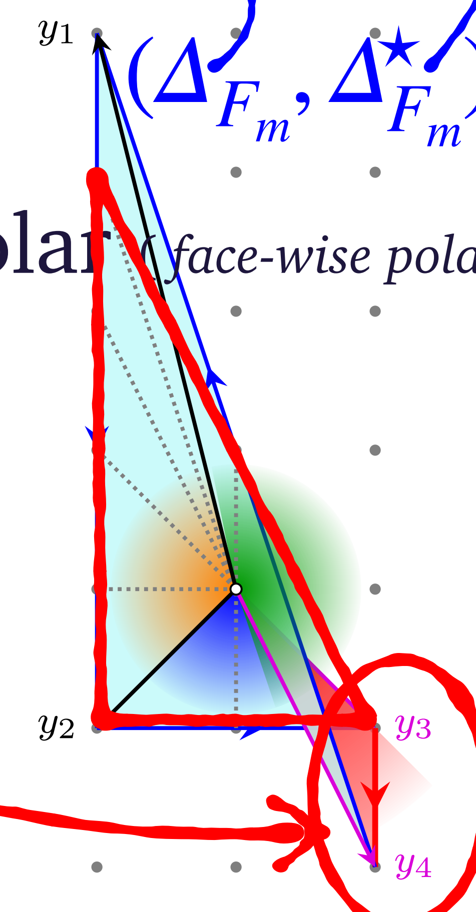

$$x_3^4 x_4 + x_4^5 + \frac{x_2^2}{x_4}$$

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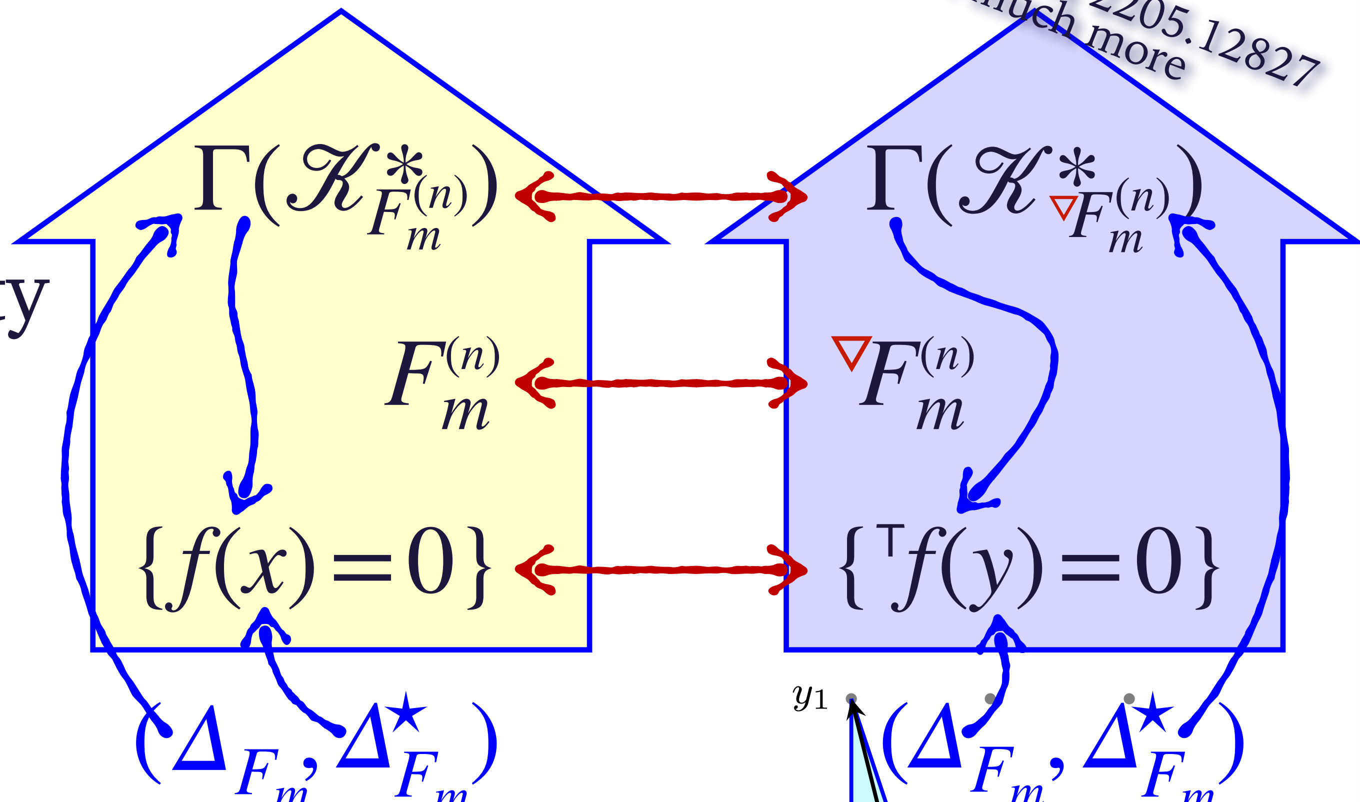

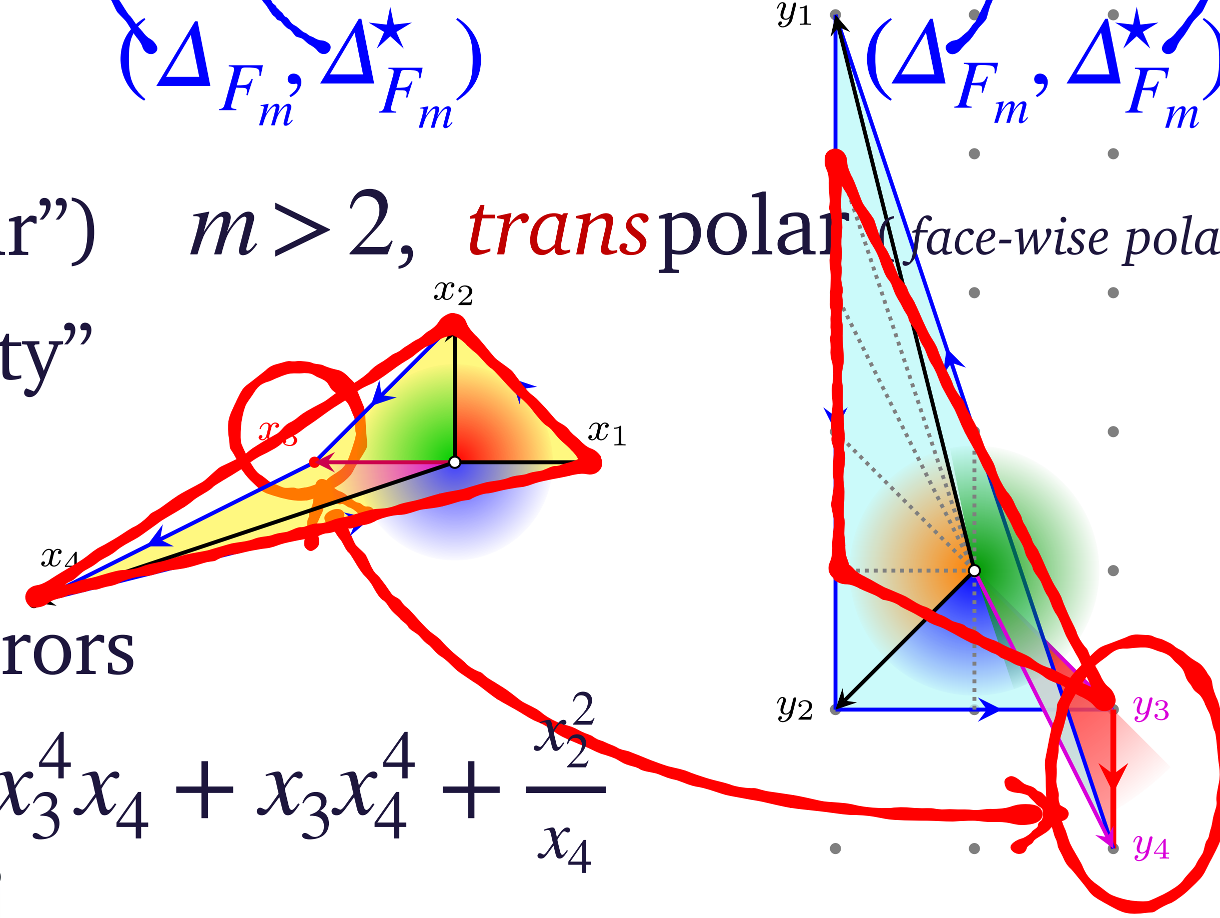

$$x_3^4 x_4 + x_3 x_4^4 + \frac{x_2^2}{x_4}$$

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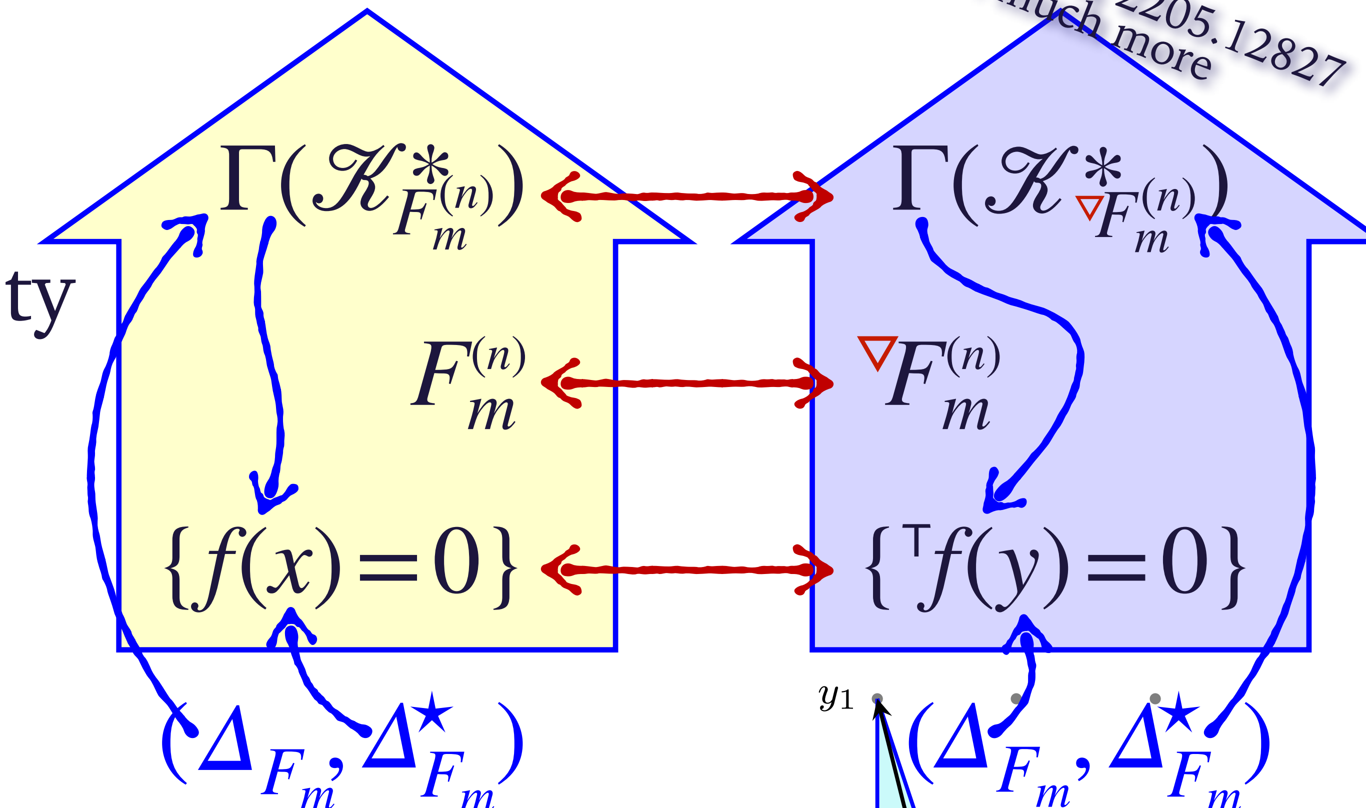

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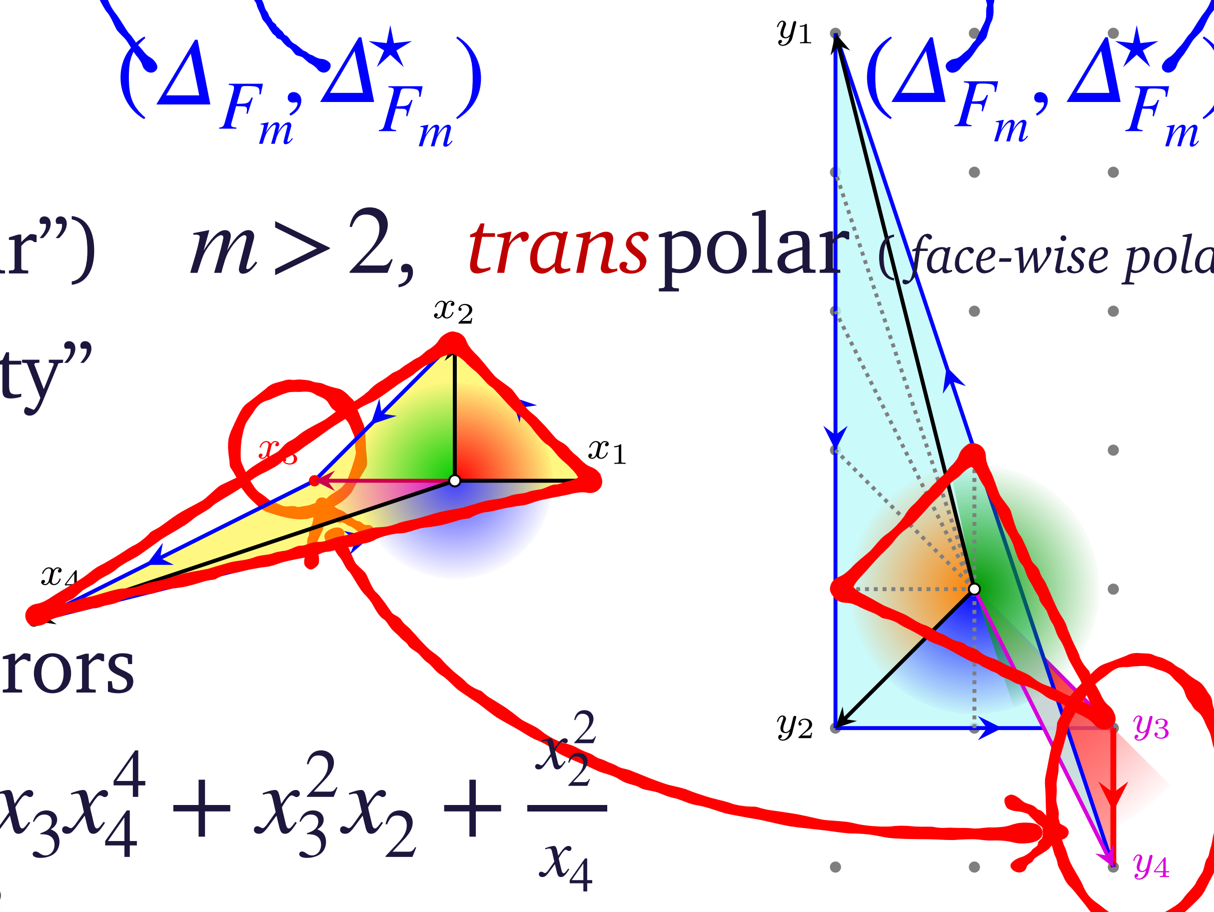

$$x_3x_4^4 + x_3^2x_2 + \frac{x_2^2}{x_4}$$

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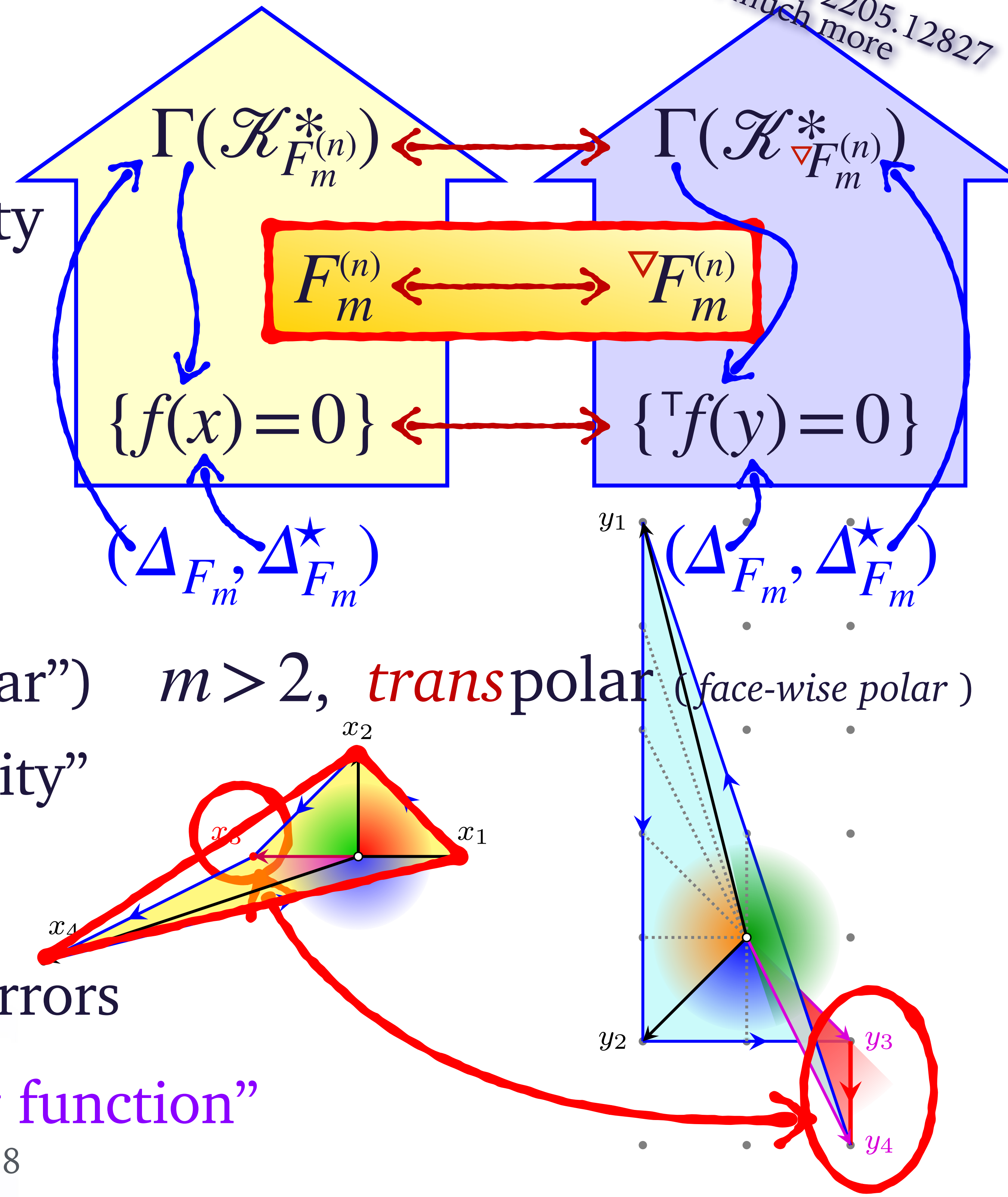
  - This “big picture” $\stackrel{?}{=}$ “generating function”



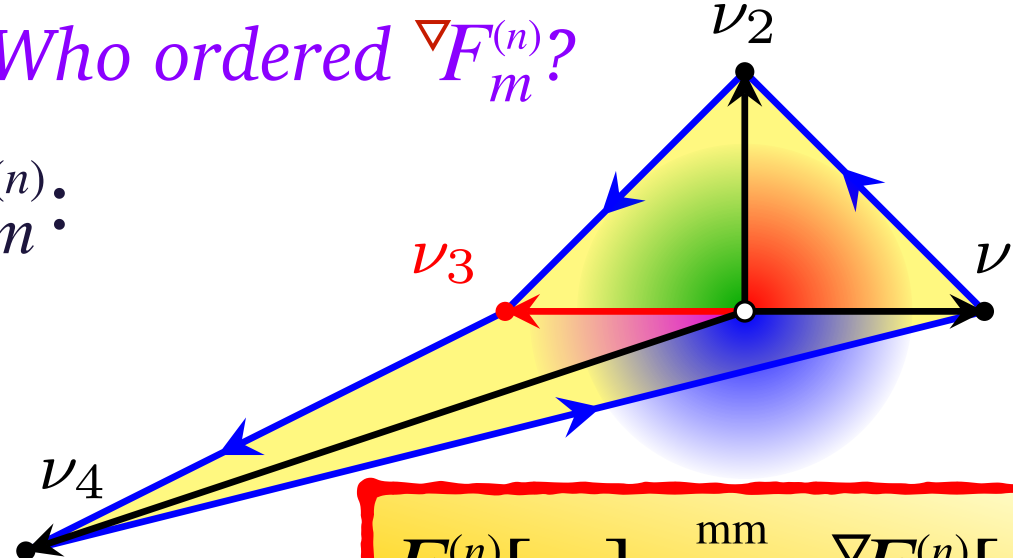
$F_m^{(n)} \longleftrightarrow {}^{\nabla}F_m^{(n)}$

1611.10300 & 2205.12827
+much more

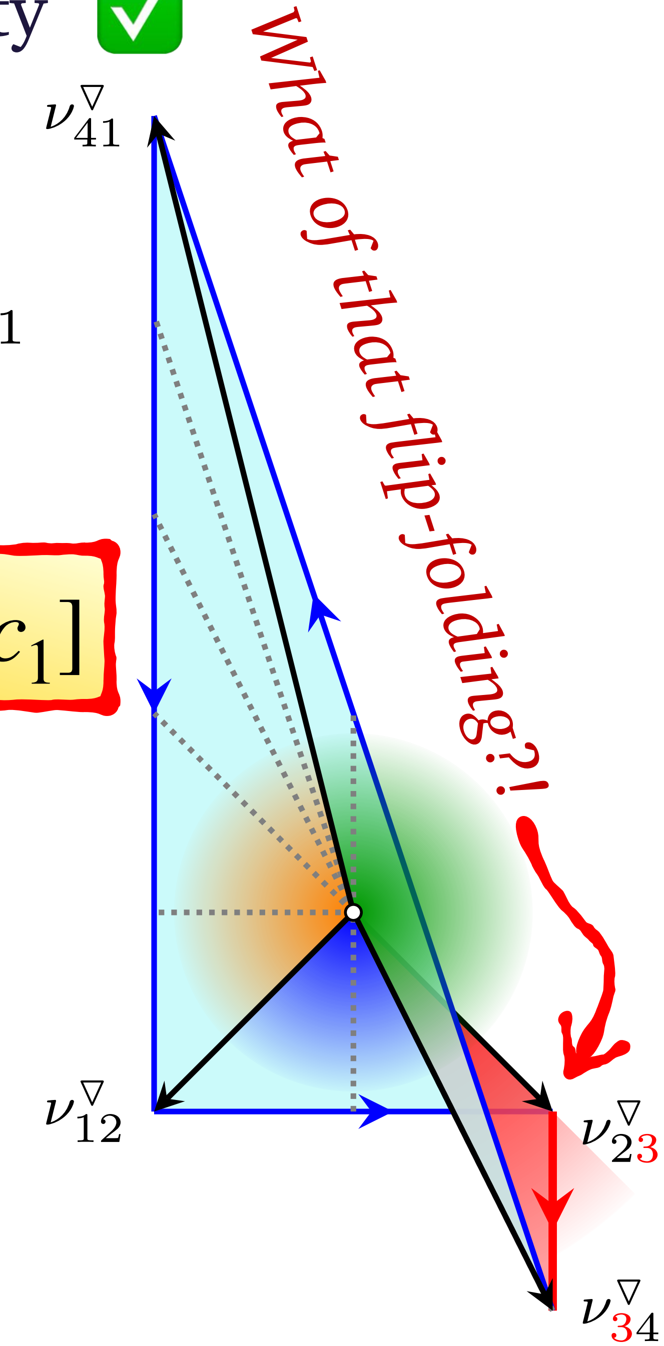
New? Toric Spaces

Sit Tight and Assess

- GLSM: $U(1)^n$-gauge symmetry; worldsheet SuSy: $U(1)^n \to (\mathbb{C}^*)^n$
  - Regular monomials ↔ toric (complex algebraic) variety ✅
  - which ${}^{\nabla}F_m^{(n)}$ …isn't. — *Who ordered ${}^{\nabla}F_m^{(n)}$?*
- Just as $\Sigma_{F_m^{(n)}}$ encodes $F_m^{(n)}$:
  - top cone = local chart;
  - codim-1-cone = gluing
- so does its *trans*polar
  - a $2n$-dim *manifold* w/$U(1)^n$-action
  - the …transpolar of $F_m^{(n)}$, denoted ${}^{\nabla}F_m^{(n)}$



$$F_m^{(n)}[c_1] \overset{\text{mm}}{\longleftrightarrow} {}^{\nabla}F_m^{(n)}[c_1]$$



- General multifans (& multitopes) correspond to
  - torus manifolds = ***real $2n$-dim mflds w/$U(1)^n$-action***
    [Masuda, 1999, 2000; Hattori+Masuda, 2003]

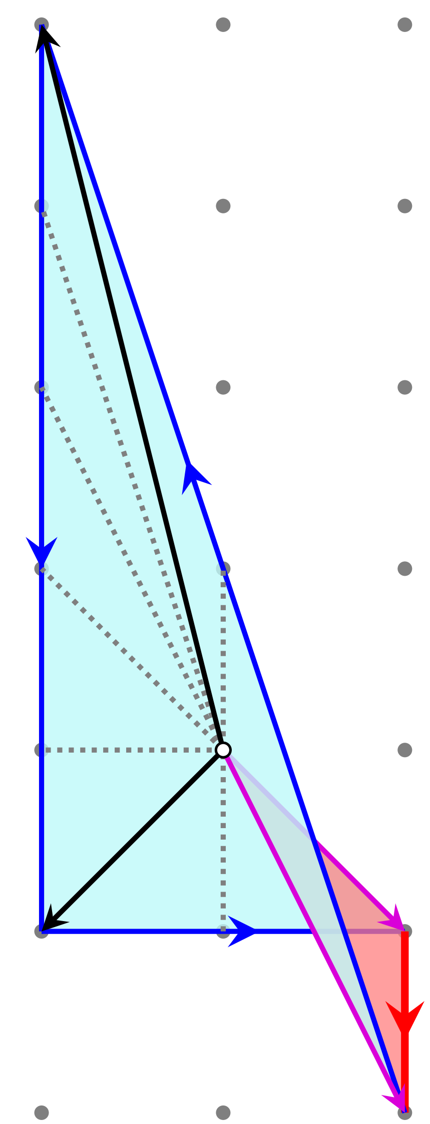
New? Toric Spaces

1611.10300 & 2205.12827
+much more

Sit Tight and Assess

Can we now use *all* of it?!

- What *is* this “${}^{\nabla}\!F_m^{(n)}$” ? (Such that ${}^{\nabla}\!F_m^{(n)}[c_1] \overset{\text{mm}}{\longleftrightarrow} F_m^{(n)}[c_1]$?)
  - Fan $\{\sigma_i;\prec\}$ of $\Delta_{F_m^{(n)}} \leftrightarrow$ atlas of charts $U_{\sigma_i} \approx \mathbb{C}^n$, $\dim \sigma_i = n$
  - But one chart is oriented reversely…



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+much more

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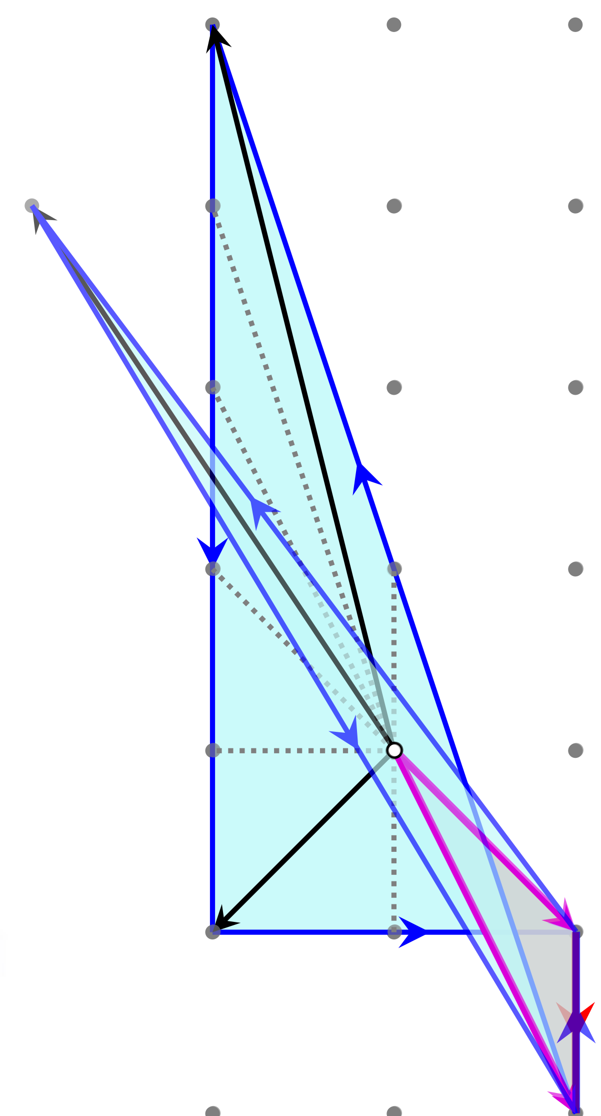
- Every flip-folded cone/facet can be surgically rev.-engineered



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+much more

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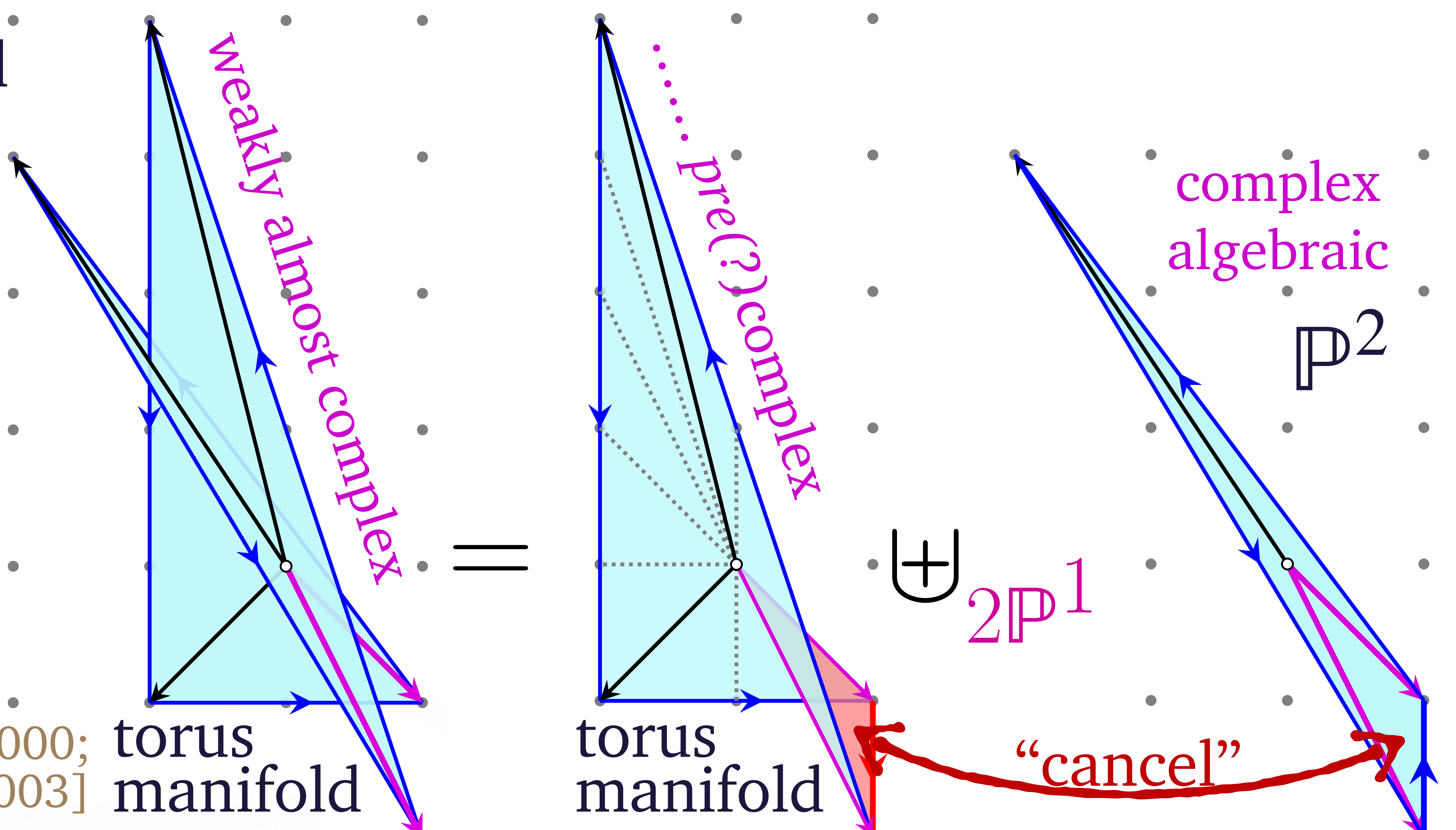
  - Fan $\{\sigma_i;\prec\}$ of $\Delta_{F_m^{(n)}} \leftrightarrow$ atlas of charts $U_{\sigma_i} \approx \mathbb{C}^n$, $\dim \sigma_i = n$
  - But one chart is oriented reversely…
- Every flip-folded cone/facet can be surgically rev.-engineered
- …from regular (cpx. alg.) toric varieties and (non-algebraic) torus manifolds

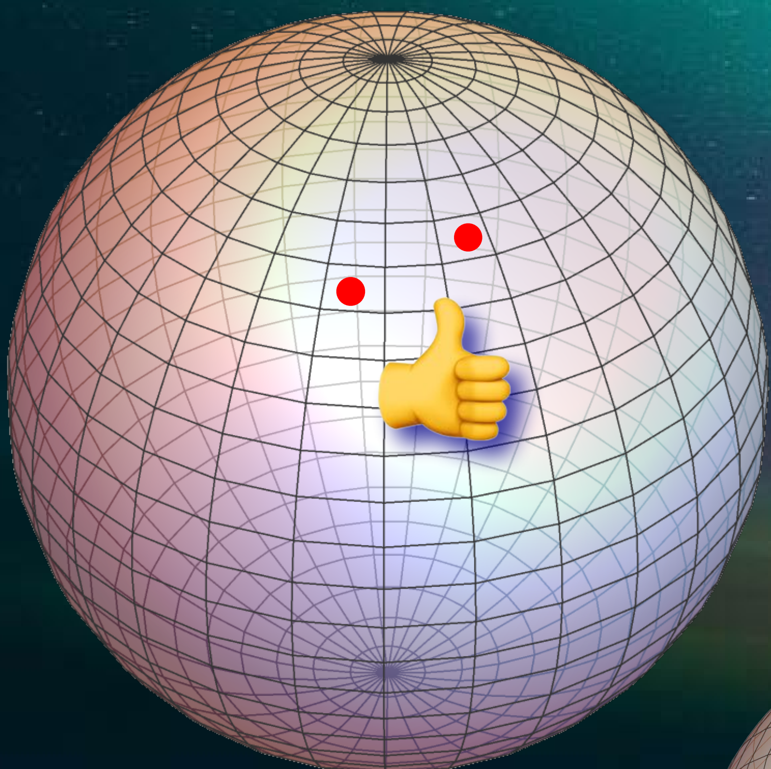
[Masuda, 1999, 2000; Hattori+Masuda, 2003]

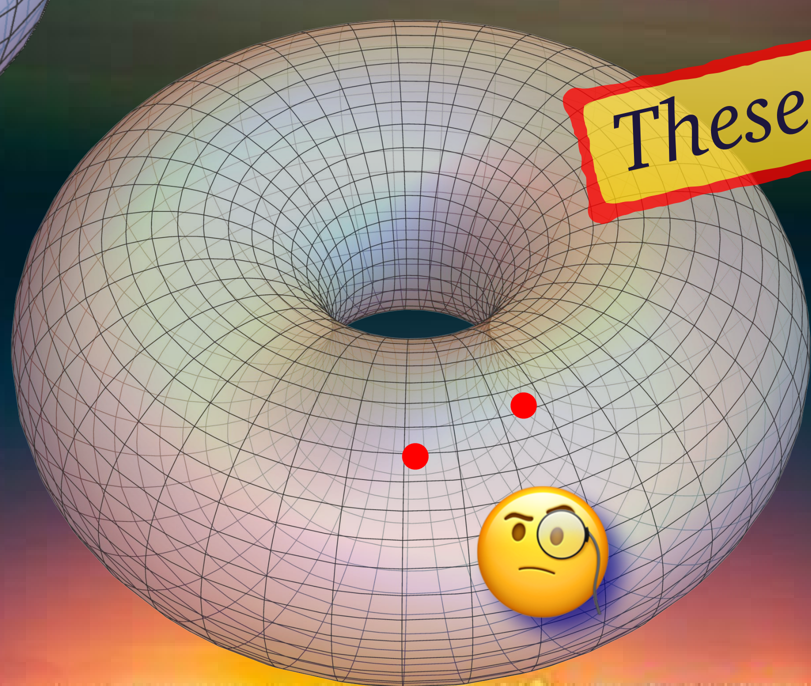


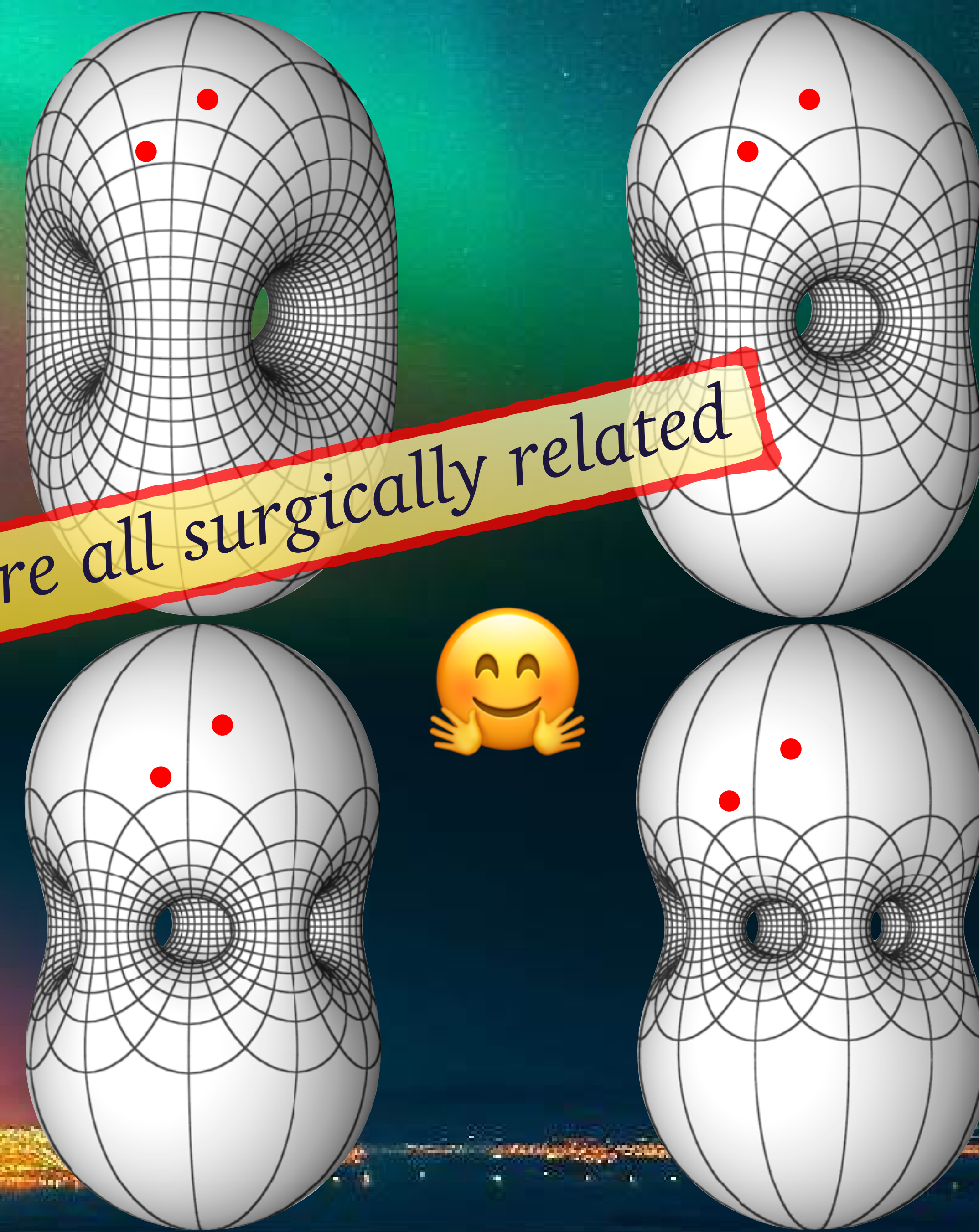
How Hard Can it Be?

Constructing CY ⊂ Some "Nice" Ambient Space

- Reduce to 0 dimensions: $\mathbb{P}^4[5] \to \mathbb{P}^3[4] \to \mathbb{P}^2[3] \to \mathbb{P}^1[2]$







Thank You!

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