An Introduction to Classical Electromagnetism: Physics 208, The University of Delaware

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Contents

Part0: Preface

The laws of electromagnetism contradict the formulation of Newton's laws one learns in introductory courses. Thus in a sense what you will learn in this course is a partial nullification of your previous physics course! However this should inspire you, because electromagnetism takes Newton's laws and gives rise to relativity. We will see how the postulates of relativity are really derivable from the electromagnetism we will learn.

Learning physics is not an easy task. One can learn the defintions and memorize the equations without understanding the concepts, and one can understand the concepts without being able to solve problems. I highly suggest that you read the concepts over carefully, and try many, many problems. I have tried to write concise, clear notes to help you learn the material without too much stress. Certainly you can and should look at other materials if you think it will help you. I have other materials I would be happy to let you borrow.

PartI: Mathematical Review

We begin with a review of vectors and calculus that you will need to understand classical electromagnetism (EM). First, we turn our attention to vectors–an indispensible concept in EM and indeed in physics in general.

Vectors

Suppose you are at point A and want me to direct you to a different point B. For argument's sake, say A and B are a distance 10 meters from one another. If I only give you this information, can you then move to point B? Obviously no, since I have not told you the direction you must travel; there are an infinite number of point a distance 10 meters from you–in 2 dimensions they form a cicle where you are at the center.

So I must give you a *distance and an angle*. This is a *vector*. Equivalently, I could tell you how many steps to move up or down and how many to move left or right. This is the most common and concise way of writing a vector, and we translate it into the mathematical language as

$$\vec{r} = a\hat{\imath} + b\hat{\jmath}$$

where the arrow above r indicates that it is a vector, and $\hat{\imath}$ and $\hat{\jmath}$ are unit vectors–which we define in a moment. The above equation reads, translated from math to english, “move 'a' meters in the $\hat{\imath}$ direction and 'b' meters in the $\hat{\jmath}$ direction”. This notation is known as unit vector notation. An equivalent way or writing the same thing is

$$\vec{r} = < a, b >$$

where the distance in the $\hat{\imath}$ direction is 'a' and the distance in the $\hat{\jmath}$ direction is 'b', just like before. We can extend this to any number of directions by listing more distances, each after a comma.

Before we define a unit vector, let us talk about some properties of vectors that we will need.

Properties of Vectors

A vector can always for our purposes be thought of as an arrow–and an arrow has a certain length. Also, for our purposes, the pythagorean theorem holds. Therefore, we can always draw our vector as an arrow and pencil in two sides to make a right triangle. Then the length of the vector is the hypoteneuse of that triangle; therefore

$$|\vec{r}|^2 = a^2 + b^2 = r^2$$

$$\Rightarrow |\vec{r}| = \sqrt{a^2 + b^2} = r$$

so 'r' is the length of the vector (also, the hypotenuese of the triangle). The symbol '$\Rightarrow$' means 'implies that'. It's a useful symbol and I'll use it a lot. Note that r and $|\vec{r}|$ are two equivalent symbols meaning the length of a vector $\vec{r}$. For simplicity I will usually use just r.

Example 1-1

Let's take the example of the vector
$\vec{r} =< 8, 6 >$
What is the length, or magnitude, of the vector?

Answer: $|\vec{r}| = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$

In addition to length, vectors have a direction. This can be described by specifying the angle a vector makes with some axis, and since all vectors for our purposes can be thought of as the hypotenuese of a right triangle, we can always use trignometry to find this angle. If I draw a vector $\vec{r} = a\hat{\imath} + b\hat{\jmath}$, starting from the origin and going up and to the right, then the angle ϕ it makes with the +x axis is

$$\tan\phi = \tfrac{b}{a} \Rightarrow \phi = \arctan\tfrac{b}{a}$$

But equivalently:

$$\sin\phi = \tfrac{b}{\sqrt{a^2+b^2}} = \tfrac{b}{r} \Rightarrow \phi = \arcsin\tfrac{b}{r}$$

$$\cos\phi = \tfrac{a}{\sqrt{a^2+b^2}} = \tfrac{a}{r} \Rightarrow \phi = \arccos\tfrac{a}{r}$$

These are the two defining properties of vectors: *magnitude (which is a* length*)* and *direction*. A scalar only has a magnitude and no direction. You must be very careful about distinguising between vectors and scalars; they are different entities and cannot be treated the same.

Unit Vectors (important!!)

Now we can define a unit vector: a unit vector is a vector that has a length (magnitude) of 1. Often they are directed entirely along an axis. For example, $\hat{\imath}$ is a vector of length 1 that is entirely along the x-axis. Therefore the vector $a\hat{\imath}$ is a vector of length 'a' that is along the x-axis. So if I wanted to tell you to move 10 meters along the x-axis (to test if you're drunk, perhaps?), I would tell you '$a\hat{\imath}$'; the famed 'white line' would be the x-axis (think about this if you're ever pulled over for drunk driving).

It is very important to understand and remember that a unit vector in the direction of any arbitrary vector can be found. If we take any arbitrary vector $\vec{A}$ and divide it by its magnitude $\left|\vec{A}\right| = A$, then we'll get a vector in the direction of $\vec{A}$ but with a length of 1. Thus,

$$\hat{A} = \frac{\vec{A}}{A}$$

is that unit vector. If we write $\vec{A}$ as

$$\vec{A} = a\hat{\imath} + b\hat{\jmath}$$

then $A = \sqrt{a^2 + b^2}$ and

$$\hat{A} = \frac{a\hat{i} + b\hat{j}}{\sqrt{a^2 + b^2}} = \frac{a}{\sqrt{a^2 + b^2}}\hat{i} + \frac{b}{\sqrt{a^2 + b^2}}\hat{j}$$

is the unit vector. This is true in general, we will use it frequently. Let's ensure that our unit vector has a magntide it 1:

$$\left|\hat{A}\right| = \left|\frac{a}{\sqrt{a^2 + b^2}}\hat{i} + \frac{b}{\sqrt{a^2 + b^2}}\hat{j}\right| = (\frac{a}{\sqrt{a^2 + b^2}})^2 + (\frac{b}{\sqrt{a^2 + b^2}})^2$$

$$= \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = 1$$

This is for *any* vector $\vec{A}$.

Example 1-2

PartI: What is the angle that the vector $\vec{r} =< 8, 6 >$ makes with the x axis?

Answer: We don't know the length of the vector, so the easiest function to use is Tangent:
$\tan\phi = \frac{6}{8} = \frac{3}{4} \Rightarrow \arctan\frac{3}{4} = \phi$
Now use a calculator and find the arctan of 3/4. You aren't expected to know this off the top of your head (I don't).

Part II: What is the unit vector in the direction of $\vec{r}$?

Answer: We apply our definition: $\hat{r} = \frac{8\hat{i}+6\hat{j}}{\sqrt{8^2+6^2}} = \frac{8\hat{i}+6\hat{j}}{10} = \frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}$. If you find the magnitude of this, you will see that it is exactly 1.

Adding Vectors

We can see how to add vectors by looking at the geometrical interpretation of vectors. Say we have two vectors $\vec{A}$ and $\vec{B}$:

$$\vec{A} =< 3, 4 >= 3\hat{i} + 4\hat{j}$$

$$\vec{B} =< -1, 2 >= -\hat{i} + 2\hat{j}$$

Now if we want to add them, we could first translate them into english. $\vec{A}$ says 'move 3 units along +x axis, and then 4 units along +y axis'. $\vec{B}$ says 'move 1 units along -x axis, and then 4 units along +y axis'. Adding both of these vectors is like following both of these directions. So by moving 3 units along the +x-axis and then 1 along the -x axis, we are moving 2 units along the +x axis. And by moving 4 units along the +y axis and then 2 units along the +y axis, we are moving 6 units along the +y axis. Therefore

$$\vec{A} + \vec{B} =< (3-1), (4+2) >= (3-1)\hat{i} + (4+2)\hat{j}$$

$$=< 2,6 >= 2\hat{\imath} + 6\hat{\jmath}$$

Do you see how to add vectors now? We add all the $\hat{\imath}$ coefficients and stick an $\hat{\imath}$ on it, and *then* we add the $\hat{\jmath}$ coefficients and stick a $\hat{\jmath}$ on it. We do *not* add $\hat{\imath}$ to $\hat{\jmath}$. Why should we? They represent different directions.

A quick word: I've been describing vectors in terms of distances so far; this is for ease of explanation. However, a vector can describe more than a distance–we will see soon that it can be generalized to represent velocities and accelerations (which you know about already), as well as more abstract things like fields. This is for the next chapter, though.

Multiplying Vectors: Intro

Adding vectors is natural, but multiplying them is not so natural. How does one multiply two directions? However since we are interested in physics, we can define the multiplication of vectors to fit concepts in physics. First, we may want a way to multiply vectors that produces a scalar (i.e., a number). This is called the dot (or scalar) product. Later in the course we will find that the cross (or vector) product is more useful; this produces a vector, as the name implies.

Multiplying Vectors: Dot (or scalar) products

An example of a dot prodcut is the definition of work. Recall that if I apply a downward force on a book that is sitting on a flat table, I do no work. I have to apply a force with a component parallel to the table in order to do work. Also remember that work is a scalar: it has no direction, it's only a number. Therefore what we get must be a scalar. If I apply a force with only a tiny component parallel to the table, then the work I will do will be small. If I apply a force almost completely parallel to the table, then I will do a lot of work. So if we take the Cosine of the angle between the force and the direction parallel to the table, we will reproduce this effect (Cosine of 0 is 1; Cosine of $\frac{\pi}{2}$ is 0). Thus a 'natural' definition is

$$\vec{A} \cdot \vec{B} = \left|\vec{A}\right| \left|\vec{B}\right| \cos\phi$$

where ϕ is the angle between $\vec{A}$ and $\vec{B}$. The ' $\cdot$ ' indicates that this is a dot product, for obvious reasons. This defines the dot product. It is not only true of work; I was merely using this concept to justify the definition. It is true in general, for all dot products.

We can now see why $\hat{\imath}$ and $\hat{\jmath}$ need to be unit vectors by finding the magnitude of a vector $\vec{r}$ by using the dot product. All of our definitions must be consistent, of course–so what happens if we dot $\vec{r}$ into itself? We get

$$\vec{r} \cdot \vec{r} = |\vec{r}|\, |\vec{r}| \cos(0) = r^2$$

$$= \vec{r} \cdot \vec{r} = (a\hat{\imath} + b\hat{\jmath}) \cdot (a\hat{\imath} + b\hat{\jmath}) = a\hat{\imath} \cdot a\hat{\imath} + a\hat{\imath} \cdot b\hat{\jmath} + b\hat{\jmath} \cdot a\hat{\imath} + b\hat{\jmath} \cdot b\hat{\jmath}$$

Remember that $\hat{\imath}$ and $\hat{\jmath}$ are perpendicular–since we define $\hat{\imath}$ to be along the x axis and $\hat{\jmath}$ to be along the y axis. This the angle between them is $\frac{\pi}{2}$, and so

$$\hat{\imath} \cdot \hat{\jmath} = \left|\hat{\imath}\right| \left|\hat{\jmath}\right| \cos(\frac{\pi}{2}) = 0 = \hat{\jmath} \cdot \hat{\imath}$$

The magnitude of both unit vectors is, of course, 1; but this is irrelevant since the Cosine of a right angle is 0. What about $\hat{\imath}$ into itself? Well

$$\hat{\imath} \cdot \hat{\imath} = \left|\hat{\imath}\right| \left|\hat{\imath}\right| \cos(0) = 1$$

since the magnitude of $\hat{\imath}$ is 1. The same is true of $\hat{\jmath}$. Thus

$$\vec{r} \cdot \vec{r} = (a\hat{\imath} + b\hat{\jmath}) \cdot (a\hat{\imath} + b\hat{\jmath}) = a\hat{\imath} \cdot a\hat{\imath} + a\hat{\imath} \cdot b\hat{\jmath} + b\hat{\jmath} \cdot a\hat{\imath} + b\hat{\jmath} \cdot b\hat{\jmath} = a^2\hat{\imath} \cdot \hat{\imath} + ab\hat{\imath} \cdot \hat{\jmath} + ba\hat{\jmath} \cdot \hat{\imath} + b^2\hat{\jmath} \cdot \hat{\jmath} = a^2 + b^2 = r^2$$

just as expected. If the magnitudes of $\hat{\imath}$ and $\hat{\jmath}$ had been different than 1, our defintions would not have been consistent and we would not have reproduced the same results here.

Now we should discuss the multiplication of two arbitrary vectors by the dot product a bit more. Say I have two vectors $\vec{A}$ and $\vec{B}$ such that

$$\vec{A} = a_x\hat{\imath} + a_y\hat{\jmath}$$

$$\vec{B} = b_x\hat{\imath} + b_y\hat{\jmath}$$

The dot product is

$$\vec{A} \cdot \vec{B} = (a_x\hat{\imath} + a_y\hat{\jmath}) \cdot (b_x\hat{\imath} + b_y\hat{\jmath}) = a_x b_x + a_y b_y$$

because, as we saw before, the dot product of $\hat{\imath}$ with itself (and of $\hat{\jmath}$ with itself) is 1, and the dot product of $\hat{\imath}$ with $\hat{\jmath}$ is 0. So we now have an alternative and equivalent way of calculating the dot product of two vectors. If we don't know the angle between the vectors, but we have the vectors in unit vector notation (or the backet notation, $<,>$), then using this is less work.

Example 1-3
What is the angle between the vectors

$$\vec{A} = 8\hat{\imath} + 6\hat{\jmath}$$

$$\vec{B} = \hat{\imath} - \hat{\jmath}$$

Answer:
We can first use the 'alternative' method of finding the dot product:

$$\vec{A} \cdot \vec{B} = (8\hat{\imath} + 6\hat{\jmath}) \cdot (\hat{\imath} - \hat{\jmath}) = (8)(1) + (6)(-1) = 8 - 6 = 2$$

Then we can use the first ('formal') definition, to find the dot product:

$$\vec{A} \cdot \vec{B} = \left|\vec{A}\right| \left|\vec{B}\right| \cos\phi = \sqrt{8^2 + 6^2}\sqrt{1^2 + 1^2}\cos\phi = \sqrt{100}\sqrt{2}\cos\phi = 10\sqrt{2}\cos\phi$$

Therefore, setting these two results equal, we get

$$10\sqrt{2}\cos\phi = 2 \Rightarrow \cos\phi = \frac{2}{10\sqrt{2}} \Rightarrow \arccos\frac{2}{10\sqrt{2}} = \phi$$

It looks like magic! But it's not.

MULTIPLYING VECTORS: CROSS (OR VECTOR) PRODUCTS

Sometimes in physics we need to define the multiplication of two vectors such that it produces another vector. An example of this in intro mechanics is torque. Recall (if you can!) that if a force acts on a door, then a torque is induced in the door–hence the door rotates around the hinge. As a reminder, torque is given by

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Also recall that when the force is directed along a line that goes through the hinge, the torque is zero. For example, if you hold your hand on the thin part of the door and push directly toward the hinge, the door will not move–the torque is zero! The vector $\vec{r}$ is a position vector going from the hinge to the point at which the force $\vec{F}$ acts. If I push toward the hinge, $\vec{F}$ and $\vec{r}$ are parallel. So we want our definition of the cross product to be such that if the vectors are parallel, the product is zero. But we also need the product to be greatest when $\vec{r}$ and $\vec{F}$ are perpendicular. Thus we should use the Sine of the angle between the vectors. The direction of the new vector is perpendicular to both $\vec{r}$ and $\vec{F}$, which is part of the definition but not so obvious from this example. So our definition is

$$\vec{A} \times \vec{B} = \left|\vec{A}\right| \left|\vec{B}\right| \hat{e} \sin\phi$$

where $\hat{e}$ is a unit vector that is perpendicular to both $\vec{A}$ and $\vec{B}$. Again, this is true in general; I have used torque as an example, to help justify the cross product–but this is the definition of all cross products. If you picture what this looks like, you'll notice that the direction of $\hat{e}$ is ambiguous: there are two unit vectors that are perpendicular to both $\vec{A}$ and $\vec{B}$. Let's think about this in terms of unit vectors. Say I draw $\hat{\imath}$ horizontally to the right (along the x axis) on a piece of paper, and $\hat{\jmath}$ vertically and upwards (along the y axis). What is $\hat{\imath} \times \hat{\jmath}$? Using our definition, we have

$$\hat{\imath} \times \hat{\jmath} = \left|\hat{\imath}\right| \left|\hat{\jmath}\right| \hat{e} \sin\frac{\pi}{2} = \hat{e}$$

because the magnitudes of $\hat{\imath}$ and $\hat{\jmath}$ are both 1, and the Sine of 90 degrees is 1. So now $\hat{e}$ is defined to be a unit vector perpendicular to both $\hat{\imath}$ and $\hat{\jmath}$. But the two unit vectors directly into and out of the paper both satisfy this definition. Thus we need to make up some convention so that we're using the same vectors. This convention is called the right-hand rule.

RIGHT-HAND RULE

The right-hand rule is simple. If we're taking the cross product $\hat{\imath} \times \hat{\jmath}$, then keep your four fingers together and direct them along $\hat{\imath}$, with the face of your palm facing $\hat{\jmath}$. Then curl your four fingers toward your palm (in other words, towards $\hat{\jmath}$). All the while, keep your thumb extended so that it's perpendicular to your four fingers, as if you're hitchiking (I don't recommend this). Now the direction of $\hat{e}$ is the direction of your thumb. Thus the cross product $\hat{\imath} \times \hat{\jmath}$ yields a unit vector that is out of the page and perpendicular to it. Typically this vector is called $\hat{k}$. This defines the right-handed coordinate system. Thus, by convention

$$\hat{\imath} \times \hat{\jmath} = \hat{k}$$

We can cycle through different cross products of the unit vectors that define 3 dimensional euclidean space $(\hat{\imath}, \hat{\jmath}, \hat{k})$. For instance, if we cross $\hat{\imath}$ and $\hat{\jmath}$ in the other direction, we have to turn our palms around so that our four fingers are in the diection of $\hat{j}$ and then we curl toward $\hat{\imath}$. Therefore we get a vector pointing into the page. This takes some practice, and you should get used to using the right-hand rule, even though it is somewhat embarrasing to use in public. This is the main reason for the right-hand rule, actually. Anyway, we have

$$\hat{j} \times \hat{\imath} = -\hat{k}$$

this is true in general: interchanging the order of vectors in a cross product produces a negative sign.

Calculus

We now review some calculus that you will need. In the interest of time I must keep this brief. But if you still feel too rusty after we've discussed this, I strongly suggest you go back to your calculus books and do more problems, and/or come see me and we can expand on this review.

Basic Differentiation and Integration

To take the derivative of a monomial, we use the following rule:

$$\frac{d}{dx}x^n = nx^{n-1}$$

To integrate a monomial, we use

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

the 'C' is necessary because if we take the derivative of $\frac{x^{n+1}}{n+1} + C$, we will get x^n, regardless of the value of C. Thus C is totally arbitrary and must be added when the integral is indefinite (indefinite meaning that the integral has no limits).

You will also be integrating and differentiating some trigonometric functions, and u-substitution will also be needed to solve some integrals. First, you will need to get familiar with these, if you aren't already:

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

Now let's review u-substitution. The idea is that we can use

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

to solve integrals that look somewhat more complicated, by making a substitution. For example, say we have the integral

$$\int \frac{xdx}{(1+x^2)^{1/2}}$$

This looks daunting, but if we make the substitution $u = 1 + x^2$, then $du = 2xdx$ and we can say that

$$\int \frac{xdx}{(1+x^2)^{1/2}} = \frac{1}{2}\int \frac{2xdx}{(1+x^2)^{1/2}} = \frac{1}{2}\int \frac{du}{(u)^{1/2}}$$

$$= \frac{1}{2}\int u^{-1/2}du = \frac{1}{2}\frac{(u)^{1/2}}{1/2} + C = \sqrt{u} + C = \sqrt{1+x^2} + C = \int \frac{xdx}{(1+x^2)^{1/2}}$$

Section II-1: Coulomb's Law

Many years ago, it was discovered that there was a force other than gravity. We now know that there are at least three others: the weak nuclear force, the strong nuclear force, and the electromagnetic force. The first two, however, were discovered decades after the theory of classical electromagnetism was formulated, and the connection between electricity and magnetism was not known when the electric force was first understood quantitatively. The process of discovering and correctly describing electricity took many millenia, but it was Charles Coulomb who, in 1783, discovered a law describing the force between two electric charges. He discovered, over many bottles of fancy french wine and baguettes, that the force between two electric charges is

$$F_{12} = k\frac{q_1 q_2}{r^2}$$

where q_1 and q_2 are two charges, k is a constant, and r is the distance between the two charges. The notation I use is that the 'F_{12}' means the force on q_1 from q_2. However, force is a vector, and I've written the law here as a scalar. To have a useful equation, therefore, we have to write Coloumb's Law in terms of vectors. Through experimentation, it was discovered that the electric force between two charges is directed along the line that connects the charges. We now put our knowledge of vectors to good use and define a vector that goes from one charge to the other. This vector we will call $\vec{r}$. The magnitude of this vector is $r = |\vec{r}|$. Then, $\hat{r} = \frac{\vec{r}}{r}$. Now we can write a vector (but abstract) form of Coulomb's Law:

$$\overrightarrow{F_{12}} = k\frac{q_1 q_2}{r^2}\hat{r}$$

I want to be clear that these charges (q_1 and q_2) are point charges, meaning that they have no volume at all. They are centralized charges at particular points. Now before we begin using Coulomb's Law to solve some problems, let's discuss some similarities and differences between Coulomb's Law and Newton's Law of gravity. The force of gravity between two masses is, in Newton's theory,

$$\overrightarrow{F_G} = G\frac{m_1 m_2}{r^2}\hat{r}$$

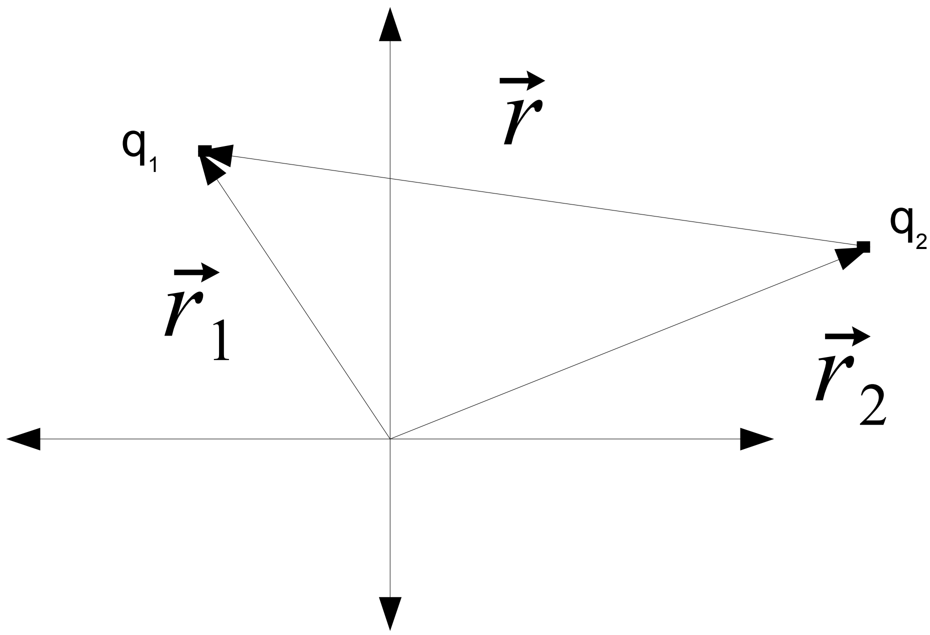
Obviously the equations look very similar: both forces are inversely proportional to the square of the distance between the charges or masses, both are proportional to the product of the charges or masses, and both forces are directed along the line connecting the charges or masses. There are, however, two key differences I want to talk about:

1. The gravitational force is only attractive–masses only attract, they never repel (as far as we know). However, in nature there are some charge configurations that attract and some that repel. This can be stated in the following way: masses are only positive, while charges can be negative or positive. One negative and one positive charge will attract each other. However, two negative charges will repel, and two positive charges will repel.

2. The gravitational force is many, many orders of magnitude less powerful than the electrical force. If I rub a balloon on my hair for less than a minute, I can induce enough charge on it that it will stick to a wall without falling, overcoming the gravitational force of the entire Earth.

In SI units, which we will use for this course, the unit of charge is a Coulomb (C). The constant k is $8.99 * 10^9 \frac{Nm^2}{C^2} = 8.99 * 10^9 \frac{Kgm^3}{s^2C^2}$.

Let us now discuss how to apply Coulomb's Law to problems. To start, let's say that we have two charges oriented as shown below:



Obviously it would be easier if we made one of the charges at the origin, but let's take a more general approach; then, if we wish to add more charges we can more easily generalize. First, the vector that describes how to get from q_2 to q_1 is $\vec{r}$, and we can find this like so:

$$\overrightarrow{r_2} + \overrightarrow{r} = \overrightarrow{r_1}$$

$$\Rightarrow \overrightarrow{r} = \overrightarrow{r_1} - \overrightarrow{r_2}$$

Now, the force on 1 from 2 is

$$\overrightarrow{F_{12}} = k\frac{q_1 q_2}{|\overrightarrow{r}|^2}\hat{r} = k\frac{q_1 q_2}{|\overrightarrow{r_1} - \overrightarrow{r_2}|^2}\hat{r}$$

Therefore if we know the positions of q_1 and q_2, we can find the distance between them and then, directly from this equation, the force q_2 exerts on q_1. But what about $\hat{r}$? Well remember that the defintion of the unit vector is

$$\frac{\overrightarrow{r}}{|\overrightarrow{r}|} = \hat{r} = \frac{\overrightarrow{r_1} - \overrightarrow{r_2}}{|\overrightarrow{r_1} - \overrightarrow{r_2}|}$$

Putting this all together, we get

$$\overrightarrow{F_{12}} = k\frac{q_1q_2}{|\overrightarrow{r_1}-\overrightarrow{r_2}|^2}\frac{\overrightarrow{r_1}-\overrightarrow{r_2}}{|\overrightarrow{r_1}-\overrightarrow{r_2}|} = k\frac{q_1q_2(\overrightarrow{r_1}-\overrightarrow{r_2})}{|\overrightarrow{r_1}-\overrightarrow{r_2}|^3} = k\frac{q_1q_2\overrightarrow{r}}{|\overrightarrow{r_1}-\overrightarrow{r_2}|^3}$$

Notice that the subscript notation on F matches the order of subtraction of the position vectors$\overrightarrow{r_1}$ and $\overrightarrow{r_2}$. The notation is consistent, so is easier to remember (hopefully). Also notice that if $q_1q_2 > 0$, then the force on q_1 from q_2 is in the direction of $\overrightarrow{r}$. In other words, the force on q_1 is directed from q_2 to q_1. This is what we need from our theory, because like charges repel. So what force would we get if either q_1 or q_2 were negative? Then, $q_1q_2 < 0$ and

$$q_1q_2 = -|q_1q_2|$$

$$\Rightarrow \overrightarrow{F_{12}} = -k\frac{|q_1q_2|(\overrightarrow{r_1}-\overrightarrow{r_2})}{|\overrightarrow{r_1}-\overrightarrow{r_2}|^3} = k\frac{|q_1q_2|(\overrightarrow{r_2}-\overrightarrow{r_1})}{|\overrightarrow{r_1}-\overrightarrow{r_2}|^3} = k\frac{|q_1q_2|(-\overrightarrow{r})}{|\overrightarrow{r_1}-\overrightarrow{r_2}|^3}$$

Now, the force on q_1 from q_2 is direct toward q_2. Therefore the force is now attractive, as expected.

Let's continue with a simple example:

Example 2-1 (Chapter 19, # 17, p.636 in Serway and Jewett, Ed. 4)

Take a charge q_1 at the origin and a charge q_2 a distance 0.3 meters from it, on the +x axis. What is the force on q_1 from q_2? Take $q_1 = 12 * 10^{-9}C$ and $q_2 = -18 * 10^{-9}C$.

Answer: The position vector from q_2 to q_1 is $\overrightarrow{r} =< -\frac{3}{10}, 0 >$.

The force on q_1 from q_2 is

$$\overrightarrow{F_{12}} = k\frac{q_1q_2}{|\overrightarrow{r}|^2}\hat{r} = k\frac{q_1q_2}{|\overrightarrow{r}|^2}\frac{\overrightarrow{r}}{|\overrightarrow{r}|} = k\frac{q_1q_2\overrightarrow{r}}{|\overrightarrow{r}|^3} = k\frac{q_1q_2 < -\frac{3}{10}, 0 >}{((\frac{3}{10})^2+0)^{3/2}}$$

because $|\overrightarrow{r}| = ((\frac{3}{10})^2+0)^{1/2} = \frac{3}{10}$. Therefore we have

$$\overrightarrow{F_{12}} = k\frac{q_1q_2 < -\frac{3}{10}, 0 >}{((\frac{3}{10})^2+0)^{3/2}} = k\frac{q_1q_2 < -\frac{3}{10}, 0 >}{(\frac{3}{10})^3} = (8.99{*}10^9)(12{*}10^{-9})(-18{*}10^{-9}) < -(\frac{3}{10})^{-2}, 0 >= 2.16{*}10^{-5} < 1, 0 >$$

which is in newtons (N). The force is attractive, as we expect.

In the previous example, we neglected to find the force on q_2 from q_1. However, if we remember Newton's Laws we do not have to do any work to find it. Recall that the Third Law states that for every action, there is an equal and opposite reaction. Thus if we know the force on q_1 from q_2, we know right away that the force on q_2 from q_1 must be equal in magnitude but in the opposite direction. So:

$$\overrightarrow{F_{21}} = 2.16 * 10^{-5} < -1, 0 >$$

If we were to go through the same steps as in example 2-1 to find this, we would get the same answer. You should do this as practice.

Let's proceed with a slightly more complex problem, and then another.

Example 2-2 (Chapter 19, # 7, p.636 in Serway and Jewett, Ed. 4)

We have two charges: $q_1 = 12 * 10^{-9}C$, at (-3/20, 2); and $q_2 = -18 * 10^{-9}C$ at (3/20, 2). What is the force on q_1 from q_2, and the force on q_2 from q_1?

Answer: First, let's find the r vectors:

$\overrightarrow{r_1} =< -3/20, 2 >, \overrightarrow{r_2} =< 3/20, 2 >$

Then

$\overrightarrow{F_{12}} = k\frac{q_1 q_2(\overrightarrow{r_1}-\overrightarrow{r_2})}{|\overrightarrow{r_1}-\overrightarrow{r_2}|^3} = k\frac{q_1 q_2(<-3/20,2>-<3/20,2>)}{|<-3/20,2>-<3/20,2>|^3} = k\frac{q_1 q_2<-3/10,0>}{|<-3/10,0>|^3}$

$|< -3/10, 0 >| = \sqrt{\frac{3}{10}^2 + 0} = \frac{3}{10} \Rightarrow |< -3/10, 0 >|^3 = \frac{3}{10}^3 = \frac{27}{10^3}$

$\Rightarrow k\frac{q_1 q_2<-3/10,0>}{|<-3/10,0>|^3} = k\frac{q_1 q_2 10^2<-3,0>}{27} = 8.99 * 10^9 \frac{(12*10^{-9})*(-18*10^{-9})10^2<-3,0>}{27}$

$= 2.16 * 10^{-5} < 1, 0 >= \overrightarrow{F_{12}}$

This is really the same problem as in Example 2-1. In this case it's a waste of time, since we've done more work than we had to. However this serves as a relatively simple example of the Coulomb Force formula we just derived. We will see in a moment that we can apply this equation over and over in much more complicated problems, making them straightforward to solve.

What happens if we have more than two charges in a region of space? How do we go about applying Coulomb's Law to each charge? Well in gravity, we add the individual contributions to get the total gravitational force. For example, the total gravitational force on you is the sum of that from the Earth, all 8 other planets, the Sun, and the rest of the matter in the universe. We can find the gravitational force on you from each mass in the universe while ignoring the others, and then add up all the contributions. I can find the gravitational force on you from just the sun by ignoring every other mass in the universe. Then I can do the same for the Earth. If I repeat this for every mass in the universe, I can find all the contributions to the total gravitational force on you. When I am done, I can add all of the contributions (which are vectors), and I get the total. A fancy way of saying this is that the gravitational force (in Newton's theory) obeys the Principle of Superposition.

Luckily for us, the electric force also obeys this princple. The Superposition Principle states that if there are multiple forces from different charges q on a charge Q, the total force on the charge Q is the sum of all the individual forces from each q. That's it–just find each force and add them up.

Let's try a more complicated example now.

Example 2-3

Say we have four charges at the following positions:

$q_1 = +q = q_3; q_2 = q_4 = -q$
$q_1 : (3, 4)$
$q_2 : (5, 1)$
$q_3 : (3, -3)$
$q_1 : (-4, 2)$

What is the total force on q_1?

Answer: First, the position vectors of the charges are

$\overrightarrow{r_1} =< 3, 4 >$
$\overrightarrow{r_2} =< 5, 1 >$
$\overrightarrow{r_3} =< 3, -3 >$
$\overrightarrow{r_4} =< -4, 2 >$

The force on charge 1 from charge 2 is

$$\overrightarrow{F_{12}} = k\frac{q_1 q_2(\overrightarrow{r_1}-\overrightarrow{r_2})}{|\overrightarrow{r_1}-\overrightarrow{r_2}|^3}$$
$$= k\frac{-q^2(<3,4>-<5,1>)}{|<3,4>-<5,1>|^3}$$
$$= k\frac{-q^2<-2,3>}{|<-2,3>|^3} = k\frac{-q^2<-2,3>}{(2^2+3^2)^{3/2}} = q^2k\frac{<2,-3>}{13^{3/2}} = \overrightarrow{F_{12}}$$

this is directed from q_1 to q_2, and therefore is attractive. This is what we should expect, because the charges are of opposite sign (they are 'unlike' charges). We can now find the force on charge 1 from charge 3:

$$\overrightarrow{F_{13}} = k\frac{q_1 q_3(\overrightarrow{r_1}-\overrightarrow{r_3})}{|\overrightarrow{r_1}-\overrightarrow{r_3}|^3}$$
$$= k\frac{q^2(<3,4>-<3,-3>)}{|<3,4>-<3,-3>|^3}$$
$$= k\frac{q^2<0,7>}{|<0,7>|^3} = k\frac{q^2<0,7>}{7^3} = k\frac{q^2<0,1>}{49} = \overrightarrow{F_{13}}$$

This is a repulsive force.
Then:

$$\overrightarrow{F_{14}} = k\frac{q_1 q_4(\overrightarrow{r_1}-\overrightarrow{r_4})}{|\overrightarrow{r_1}-\overrightarrow{r_4}|^3}$$
$$= k\frac{-q^2(<3,4>-<-4,2>)}{|<3,4>-<-4,2>|^3}$$
$$= k\frac{-q^2<7,2>}{|<7,2>|^3} = k\frac{-q^2<7,2>}{(7^2+2^2)^{3/2}} = k\frac{-q^2<7,2>}{53^{3/2}} = \overrightarrow{F_{14}}$$

This is an attractive force.
Now the total force is, by the Superposition Principle, just the sum of the individual forces we're found:

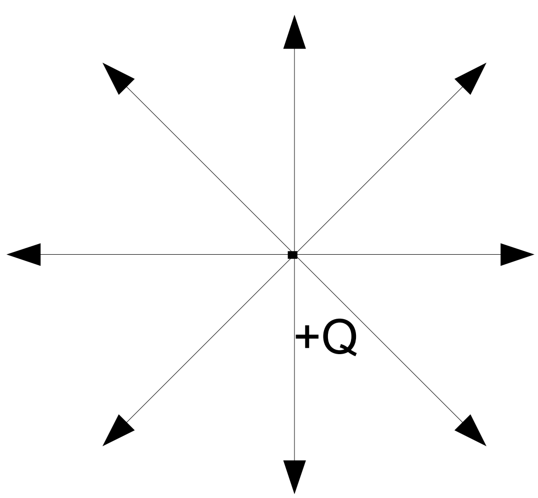
$$\overrightarrow{F_1} = \overrightarrow{F_{12}} + \overrightarrow{F_{13}} + \overrightarrow{F_{14}}$$

We've covered the electric force and some examples of how to apply it to problems, but we've neglected something: just how does a charge exert a force on another charge that is at a distance from it? This is the problem of 'action at a distance', one which Newton pondered in light of his gravitational theory. The modern solution to this can be traced back to Faraday, who first proposed the concept of a field in the context of electromagnetism. In the early 20th century, Einstein explained how masses exert forces at a distance via the interaction of space-time and mass-energy, asnwering the 'action at a distance' problem through geometry. But the concept of fields remained and remains the basis of classical electromagnetism.

Section II-2: The Electric Field

If we're going to postulate the existence of a field to explain the electric force, we need to carefully define it so that we can reproduce all of the effects of Coulomb's Law that we observe. Let's try a thought experiment; say I have a single positive charge $+Q$ in a region of space. Let's keep the definition of this field as simple as possible and say that positive charges follow along lines in the field. But we know that charges are very particular–the direction of movement of charges is important. So we need a vector field: at each point in space, there is a arrow that tells us the direction a positive charge will flow. And of course this arrow must be in the right direction.

So we have our positive charge $+Q$ in a region of space. For argument's sake, let's say this charge is glued in place on a table. Now in which direction would a positive charge $+q$ move if it were placed on the table? From observation we know that it would move away from the $+Q$, along the line connecting the charges. The $+q$ charge will behave this way wherever we put it on the table. So therefore, the 'electric field' must flow outward from $+Q$ in straight lines. Then, all positive charges in the vicinity will follow along these electric field lines. The electric field outside of a positive charge looks like



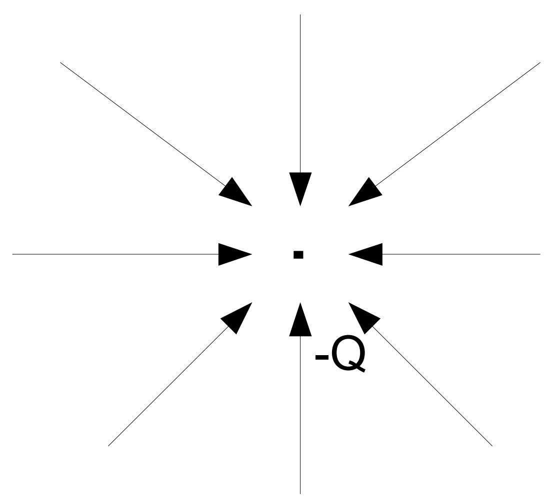
The electric field consists of the arrows shown. Of course, I've only drawn a few of them here. We have therefore defined our electric field: it eminates outward from positive charges. But how do we write it down mathematically? We need to add to our definition now. We've said that a charge $+q$ follows along electric field lines; but what is the magnitide of the force on $+q$ from the electric field? Let us define it simply: the force on $+q$ from an electric field $\overrightarrow{E}$ is just the product of the two. So

$$\overrightarrow{F_e} = q\overrightarrow{E}$$

defines the electric field, where $\overrightarrow{F_e}$ is the electric force (in other words, the Coulomb force). The charge q will move in the direction of $\overrightarrow{E}$, as we need. Thus we now have a simple definition of the electric field, which we postulate carries the electric force. Two charges can therefore exert forces on each other without touching, through this electric field.

I want to make an important point here: the electric field $\overrightarrow{E}$ that induces a force on q is the field from charges other than q. This is for the simple reason that if we have just one charge q, it will produce an electric field but there will be NO electric force on q. Charges don't induce forces on themselves, they exert forces on other charges. Thus, if we have just one charge q in a region of space, there will be no external electric field–only the field from q. Thus $\overrightarrow{E} = 0$ here. This better be part of the definition of $\overrightarrow{E}$, because a charge cannot exert a force on itself.

What should be the electric field outside of a negative charge, $-Q$? Well we've defined the electric field such that positive charges flow along electric field lines, and we know that negative charges move toward positive charges. Thus the electric field outside of a negative charge $(-Q)$ must be directed toward the charge, in straight lines. It looks like



where, again, the electric field consists of the lines shown–though I've only drawn a few of them. The $-Q$ is the dot in the middle. So remember, the idea is that if I place a positive charge $(+q)$ near $-Q$, it will follow the electric field lines towards $-Q$. Thus $+q$ is attracted to $-Q$, which is what we would observe in nature.

We have still not discussed how to write down the electric field mathematically. If we can find out how to do this, then we can find the force on charges in a region by just multiplying the charge by the field–which, remember, is a vector. The electric force between point charges q and Q is, by Coulomb's Law

$$\overrightarrow{F_e} = k\frac{qQ}{r^2}\hat{r}$$

But we've just defined the electric field:

$$\overrightarrow{F_e} = q\overrightarrow{E}$$

So now we can combine these two (isn't this exciting!?):

$$\overrightarrow{F_e} = q\overrightarrow{E} = k\frac{qQ}{r^2}\hat{r}$$

Which tells us that the electric field outside of a point charge Q is

$$\overrightarrow{E} = k\frac{Q\hat{r}}{r^2}$$

where $\hat{r}$ is a unit vector that is directed radially. Remember that r is the distance from the charge Q to the point at which we're finding the electric field. This equation is true only for point charges, since Coulomb's Law is only true for point charges and we used Coulomb's Law in our derivation. So now we have a vector equation telling us the value of the electric field outside a point charge, based on our definition of the electric field.

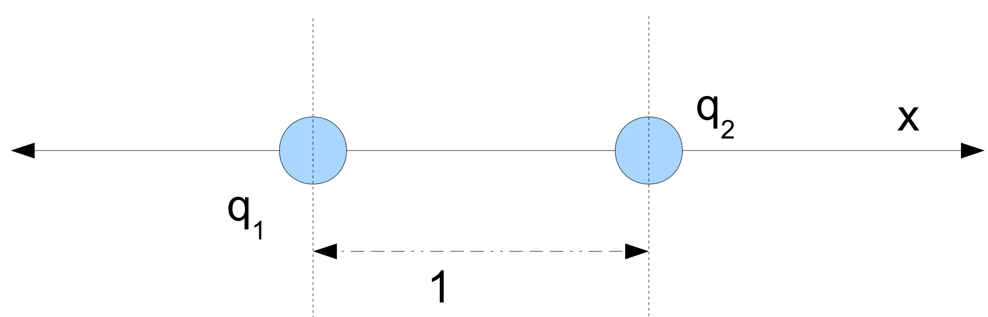
We need to discuss units. Being cognizant of units is a great way to check one's work, so getting into the habit of checking units is wise. The unit for charge is the Coulomb (C), and the unit for force is the Newton (N). Thus the unit for the electric field is

$$\frac{N}{C}$$

This is important to keep in mind. Let's discuss an example of how to apply this.

Example 2-4 (Chapter 19, # 11, p.635 in Serway and Jewett, Ed. 4)

Say we have two charges that are 1m apart, as shown below. At what point or points along the x axis is the electric field zero? Assume $q_1 = -2.5 * 10^{-6}C$ and $q_2 = 6 * 10^{-6}C$.



Answer: First let's think about whether there are certain regions along the x axis where the electric field cannot be zero. Inbetween the charges, the electric field from q_1 is to the left, and the electric field from q_2 is also to the left. How can two vectors in the same direction cancel? The electric field cannot be zero inbetween the charges. To the right of q_2, the electric field from q_2 is to the right and the field from q_1 is to the left. The reverse is true to the left of q_1. So we cannot rule out either of these regions yet.

Let's say, for argument's sake, that the place at which the electric field is 0 is to the left of q_1, a distance d away from it. Then, the electric field from q_1 is

$$\overrightarrow{E_1} = k\frac{|q_1|}{d^2}\hat{i}$$

and then the field from q_2 is

$$\overrightarrow{E_2} = -k\frac{q_2}{(1+d)^2}\hat{i}$$

Then the condition we want is

$$\overrightarrow{E_2} + \overrightarrow{E_1} = 0 = k\frac{|q_1|}{d^2}\hat{i} - k\frac{q_2}{(1+d)^2}\hat{i}$$

$$\Rightarrow k\frac{|q_1|}{d^2}\hat{i} = k\frac{q_2}{(1+d)^2}\hat{i}$$

$$\Rightarrow \frac{|q_1|}{d^2} = \frac{q_2}{(1+d)^2}$$

$$\Rightarrow \left|\frac{q_1}{q_2}\right| = \frac{d^2}{(1+d)^2} = \frac{2.5}{6} = \frac{5}{12}$$

$$\Rightarrow \frac{5}{12}(1+d)^2 = d^2 \Rightarrow (1+d)\sqrt{\frac{5}{12}} = \pm d$$

$$\Rightarrow \sqrt{\frac{5}{12}} = d(\pm 1 - \sqrt{\frac{5}{12}}) = d(\frac{\pm\sqrt{12} - \sqrt{5}}{\sqrt{12}})$$

$$\Rightarrow \sqrt{5} = d(\pm\sqrt{12} - \sqrt{5})$$

$$\Rightarrow \frac{1}{\pm\sqrt{\frac{12}{5}} - 1} = d$$

Let's check each solution. First the negative solution:

$$d = \frac{1}{-\sqrt{\frac{12}{5}} - 1} = -0.39$$

Notice that this is negative, and so represents a point to the right of q_1, inbetween q_1 and q_2. But we've already pointed out that it is impossible for the electric fields to cancel in this region.

The positive solution gives

$$d = \frac{1}{\sqrt{\frac{12}{5}} - 1} = 1.82$$

which is a reasonable solution at first sight. Now let's check to see if it is in fact a solution.

$$\Rightarrow d^2 = \frac{1}{(\sqrt{\frac{12}{5}} - 1)^2}$$

and

$$1 + d = 1 + \frac{1}{\sqrt{\frac{12}{5}} - 1} = \frac{\sqrt{\frac{12}{5}} - 1 + 1}{\sqrt{\frac{12}{5}} - 1} = \frac{\sqrt{\frac{12}{5}}}{\sqrt{\frac{12}{5}} - 1}$$

Substituting this back in, we get

$$\frac{d^2}{(1+d)^2} = \frac{\frac{1}{(\sqrt{\frac{12}{5}}-1)^2}}{(\frac{\sqrt{\frac{12}{5}}}{\sqrt{\frac{12}{5}}+1})^2} = (\sqrt{\frac{5}{12}})^2 = \frac{5}{12}$$

and therefore it is in fact a solution. Thus the electric field is zero at a distance $d = 1.82$ meters to the left of q_1.

Up to now we've discussed our theory only in the context of point charges. But often we will want to use our theory to make predictions when we have non-point charges. What is the electric field outside of a charged duck? I'm sure those of you studying duck engineering have already asked this question. Let's turn now toward answering it.

The Electric Field outside of Continuous Charge Distributions

To do this we need to make an observation: a continuous charge distribution is just an infinite number of point charges stuck together in some shape. To be precise, there does not appear to be any charge which is truly 'continuous'. For some reason, charges in nature are discretized–they come in small packets. It used to be thought that the charge of an electron, let's call it e, is the fundamental charge, out of which all other charges are made. In other words, we can have a charge of 10,000,000e, 3e, 9e, etc.–but we cannot have a charge of 0.2e. We either add a whole e (or integer multiples of e), or subtract a whole e (or integer multiples of e). In most textbooks that I have seen, e is cited as the fundamental charge. However there has been in recent decades substantial evidence for fundamental particles called quarks, which come in 6 types and can carry 3 colors (often labeled as 'red', 'green', and 'blue'; but these have nothing to do with the electromagnetic radiation of the same names). The smallest charge a quark can carry is e/3; this is now thought to be the fundamental charge. Anyway, this is a side note.

Let's go back to the observation that a continuous charge distribution is really an infinite series of point charges, which is, as I pointed out, not precisely true. However it's a good enough approximation if we're dealing with systems on a 'macroscopic' scale, meaning, say, a scale large enough for us to see. It is probably a very good approximation for systems somewhat smaller than this, too. So let's make that assumption, which will considerably simplify matters. We can now use our knowledge of calculus by applying it to the present problem. Say we have an infintesimal charge dq that is emitting an infintesimal electric field $\overrightarrow{dE}$. This infintesimal charge is a point charge, by definition, and so we can use our recently-derived formula for the electric field outside of a point charge, but this time we rewrite it in the language of calculus:

$$\overrightarrow{dE} = k\frac{dq}{r^2}\hat{r}$$

I used Q before, but from now on I will use the more standard notation that the charge creating the electric field is q, or in calculus language dq. So now we can think of our macroscopic charged object as being built our of these dqs. So then, the total electric field can be written as a sum of all the individual contributions from each of the dqs. If we had a finite number of them we could just sum them by hand, but we instead have an infinite number. Summing over an infinite number of infintesimally small things is call integration; so the total field is

$$\overrightarrow{E} = \int \overrightarrow{dE}$$

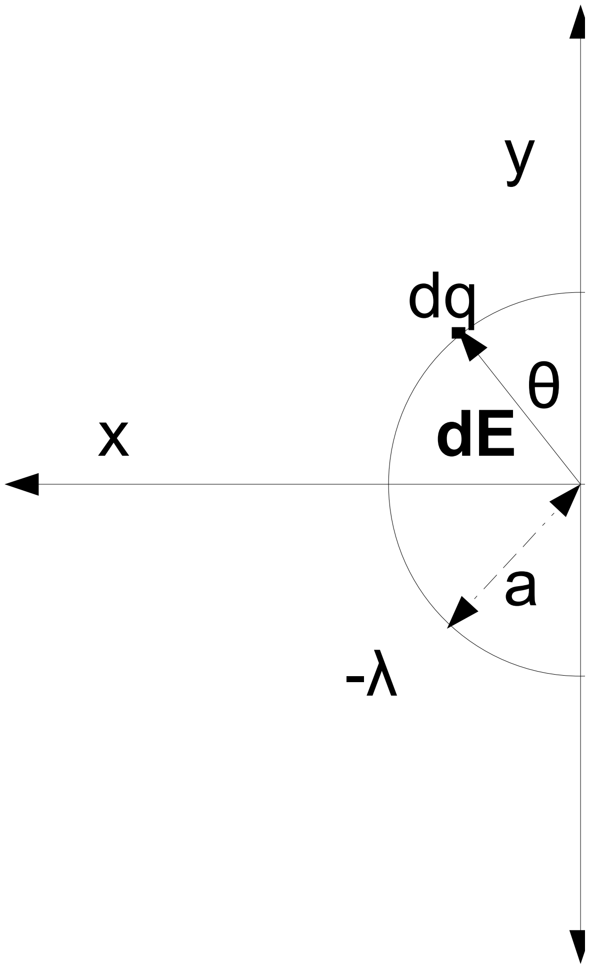
This should be familiar notation to you. We can then replace the $\overrightarrow{dE}$ with its value as a function of r:

$$\overrightarrow{E} = \int k\frac{dq}{r^2}\hat{r}$$

You may not be used to integrating vectors, but don't be intimidated: the unit vector $\hat{r}$ just comes along for the ride; you only really integrate the scalar part while leaving the unit vectors alone. Let us immediately now turn toward examples of how to apply this equation to solve problems.

Example 2-5 (Chapter 19, # 21, p.636 in Serway and Jewett, Ed. 4)

We have a charged semicircle, as shown below. What is the electric field at the center (the origin)? Take the total charge to be $-7.5 * 10^{-6}C$, the total length of the semicircle to be $l = 14cm$, and the radius of the semicricle to be a. The charge per length of the semicircle let's call $-\lambda$, and we will assume this is constant. The 'dE' in the below diagram, by the way, is a vector. Unfortunately I had trouble adding a vector arrow over it, so instead I bolded it.



Answer:

We need to chose an infintesimal charge on the semicircle, called dq as shown. Then we can apply the equation we just derived:

$$\overrightarrow{dE} = k\frac{dq}{r^2}\hat{r}$$

The field at the origin from dq is $\overrightarrow{dE}$, and it points toward dq because of the way we defined the electric field (it points toward negative charges). But how do we write down $\hat{r}$ in terms of $\hat{i}$ and $\hat{j}$? Well we can decompose the vector into x and y components like any other vector. The x component of $\hat{r}$ is

$$|\hat{r}|\sin\theta$$

and the y component is

$$|\hat{r}|\cos\theta$$

Thus we can write

$$\hat{r} = -\left|\hat{r}\right| \sin\theta \hat{\imath} + \left|\hat{r}\right| \cos\theta \hat{\jmath}$$

Let's take the left direction to be negative, for argument's sake. Obviously, this is arbitrary; we could take it to be positive and get the same physics result.

We should be easily able to simplify this since we know the value of $|\hat{r}|$:

$$|\hat{r}| = 1$$

And thus

$$\hat{r} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$$

Now r is just the radius of the semicircle, which is a. So we have

$$\overrightarrow{dE} = k\frac{dq}{a^2}(-\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta)$$

So we now must integrate over dq if we're going to solve for the total electric field at the origin. We might be tempted to just integrate dq to get the total charge Q, but this would be wrong. Let's think about the answer we would get if we did that: the $\sin\theta$ and $\cos\theta$ terms would remain unchanged, since if we integrated directly over dq all terms that depended on θ would be constant in the integral. So our electric field would depend on θ; but this is ridiculous, because θ is the angle of an arbitrary infintesimal charge dq along the semicricle. It makes no sense for the total electric field to depend on this angle. The key is that dq can be written as a function of θ, so we must replace dq with that function. The way to do this is to use the concept of charge density, which is extremly useful. I've already defined the charge per length as λ. What is the length of the part of the semicircle that contains dq? Well the length of an arc is $s = R\theta$, and so an infintesimal arc length is $ds = Rd\theta$ or, in our case,

$$ds = ad\theta$$

So the charge per length at dq is

$$\lambda = \frac{dq}{ds} = \frac{dq}{ad\theta}$$

Therefore

$$a\lambda d\theta = dq$$

$$\Rightarrow \overrightarrow{dE} = k\frac{a\lambda d\theta}{a^2}(-\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta)$$

And now we can integrate over θ (with limits) and our electric field will therefore not depend on this arbitrary angle. Finally we have

$$\overrightarrow{E} = \int_{Allcharges} \overrightarrow{dE} = \int_0^\pi k\frac{a\lambda d\theta}{a^2}(-\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta) = \int_0^\pi k\frac{\lambda d\theta}{a}(\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta)$$

The limits are from 0 to π because we need to integrate over all the charges, which start from the +y axis and swoop down to the -y axis. If you look at the diagram I've drawn, you will see that this corresponds to a change in the angle θ from 0 to π. Now remember that we're assuming λ is constant–or more precisely, that it is not a function of θ. If it were, we would need to find this function and integrate over it. But luckily we will not concern ourselves with that here, and thus we can safely remove λ from the integral:

$$\overrightarrow{E} = \int_0^\pi k\frac{\lambda d\theta}{a}(-\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta) = k\frac{\lambda}{a}(-\int_0^\pi \hat{\imath}d\theta\sin\theta + \int_0^\pi \hat{\jmath}d\theta\cos\theta)$$

$$= k\frac{\lambda}{a}(\hat{\imath}|_0^\pi\cos\theta + \hat{\jmath}|_0^\pi\sin\theta) = k\frac{\lambda}{a}(\hat{\imath}(-1-1) + \hat{\jmath}(0-0)) = -2k\frac{\lambda}{a}\hat{\imath} = -2k\frac{Q}{al}\hat{\imath}$$

Let's think about this answer carefully. First, the vector is in the $-\hat{\imath}$ direction. This is what we should expect, because the electric field points toward negative charges, and our charged semicircle is negatively charged and is to the left of the origin. Now what should happen to the strength of the field if a is increased? If a is extremely large, then the charges are a great distance away from the origin, so we should expect the electric field to be weaker. Our electric field is inversely proportional to a, so our answer does give the right behavior. But what about λ? If λ is larger (if the electric charge density increases), then the electric field should be larger; and since our answer is proportional to λ, our answer has the right behavior here too. For completeness, the electric field has the numerical value (after substituting the given values for Q, a, l, and k):

$$\overrightarrow{E} = -2.16 * 10^6\hat{\imath}\frac{N}{C}$$

You should try to answer a few questions regarding this example:

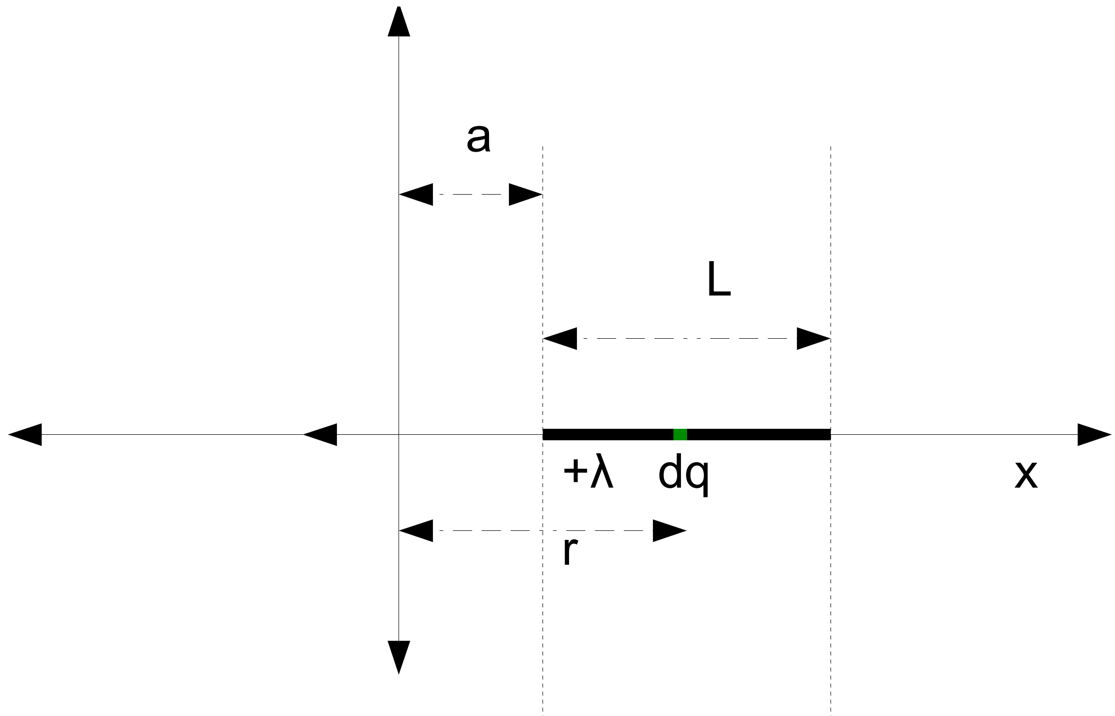
1. What would you expect the electric field to be at the center of a full circle (instead of a semicircle) that had a uniform charge density? Using the method above, derive explicitly what the electric field is at the center of a charged circle. Is the answer what you expected? It should be!

2. What would the electric field at the center of a circle be if the left half were negatively charged (like in example 2-4) but the right half were positively charged?

Let's turn to another example.

Example 2-6

Say we have a thin rod along the x axis, which has a linear charge density of λ as shown below.



What is the electric field at the origin?

Answer: We should use the same basic method as before. Again we pick an arbitrary point dq and find its electric field, and then sum (integrate) over all the dqs to get the total field. I've marked an arbitrary dq along the line charge with green. The electric field due to this (and any) dq is

$$\overrightarrow{dE} = k\frac{dq}{r^2}(-\hat{i})$$

where r, remember, is always defined as the distance of the charge dq from the point at which we're finding the electric field–in this case, the origin. Thus the distance of dq from the origin is r, and I've labeled it as such in the diagram. Now we have to use our two main tricks: replace dq with a function of distance, and then integrate over all the charges. For our first trick we use the defintion of the linear charge density:

$$\lambda = \frac{dq}{dx}$$

$$\Rightarrow \lambda dx = dq$$

And if we think about it, r is really just the distance of dq along the x axis–thus it's really just x. So now we can write

$$\overrightarrow{dE} = k\frac{\lambda dx}{x^2}(-\hat{i})$$

We're almost done, but before we integrate we need to determine the limits of integration. We must integrate over all the charges, and the charges start at $x = a$ and end at $x = a + L$. Therefore, the limits of integration must be from a to $a + L$:

$$\overrightarrow{E} = \int_{Allcharges} \overrightarrow{dE} = \int_a^{a+L} k\frac{\lambda dx}{x^2}(-\hat{i})$$

$$\Rightarrow \overrightarrow{E} = -k\lambda\hat{i}\int_a^{a+L} x^{-2}dx = -k\lambda\hat{i}|_a^{a+L}(\frac{x^{-1}}{-1}) = k\lambda\hat{i}(\frac{1}{a+L} - \frac{1}{a})$$

Let's think about our answer. First, what is the direction of the electric field? Well notice that $a + L$ is larger than a, and so $\frac{1}{a+L} < \frac{1}{a}$ and thus $\frac{1}{a+L} - \frac{1}{a} < 0$. So there is a negative sign that comes from this. Hence the electric field is to the left (in the -$\hat{i}$ direction).This is what we should expect, because the line charge is positively charged and electric field lines point away from positive charges. So far, so good.

What would happen, intuitively, if the line charge were to shrink to zero? Well there would then be no charge, and the electric field would disappear! The line charge shrinking to zero corresponds to $L \rightarrow 0$. If that happens, our equation tells us that

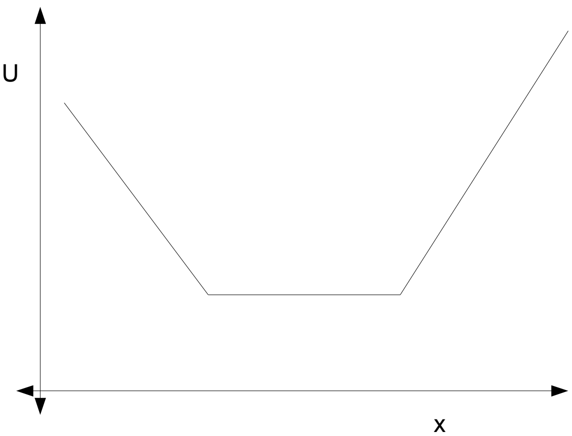
$$\overrightarrow{E} = k\lambda\hat{i}(\frac{1}{a+0} - \frac{1}{a}) = 0$$

just as we expect. If we got an answer other than this, we would know that we'd made a mistake.

Section II-3: The Electric Potential and Potential Energy

So far we have defined an electric 'field' to help explain the coulomb force. You might guess that somewhere along the way, someone decided it might be a good idea to try to explain the electric field by defining another quantity. There are at least two reasons why: first, there are already concepts developed from Newtonian physics that can applied to our new theory of the electric field, and in physics we like to connect all the dots; and secondly,vector fields are not so easy to find and use.

Let's begin by reviewing some basic mechanics. In mechanics, you learned that a force is related to a potential energy. The force on a mass is related to the slope of the potential energy graph, much like a ball rolling up or down a hill: the steeper the hill, the larger the force on the ball. When the ball reaches a flat portion of the hill–say at the top–there is no more force. These graphs looked like



In what direction would the force on a ball be if the ball were on the left slope? Well if we drew a graph with 2 spacial dimensions, the gravitational force would be down, and the component perpendicular to the slope would be equal and opposite to the normal force on the ball from the slope. But the graph drawn has one spacial dimension; and since the ball moves along the graph to the right, we say the force is to the right. The slope of the graph here is negative, and so the force must be in the opposite direction–to the right, which is positive. Thus

$$F = -\frac{dU}{dx}$$

When applied to gravity, we get

$$F = -mg = -\frac{dU}{dx}$$

$$\Rightarrow mgdx = dU$$

$$\Rightarrow \int mgdx = \int dU = U = mgh + C$$

which is the gravitational potential energy. Let's apply this now directly to the electric force, and see what we get. Instead of using x, I will use the more conventional r:

$$F = -\frac{dU}{dr}$$

$$\Rightarrow -dU = Fdr$$

Now we're going to do something slightly different than what you did in your previous mechanics course: we're going to define an electric potential V, that is related to the electric potential energy U by

$$U = qV$$

$$\Rightarrow dU = qdV$$

This is useful because electric potential energy is contained only between multiple charges (2 or more), whereas a single charge has associated with it an electric potential. So

$$-dU = Fdr = qEdr = -qdV \Rightarrow Edr = -dV$$

$$\Rightarrow E = -\frac{dV}{dr}$$

I want to tie in another concept from mechanics at this point: the concept of work. Remember that work is a force times a distance; so if we integrate $-dU = Fdr$ we will get work. However, remember also that only forces in the direction of motion do work. Thus we need to take the dot product of the force and the direction of motion:

$$W = \int -dU = \int \overrightarrow{F} \cdot \overrightarrow{dr}$$

$$= -\Delta U = W = \int \overrightarrow{F} \cdot \overrightarrow{dr}$$

So now let's substitute our concept of electric potential into this equation:

$$-\Delta U = -q\Delta V = \int \overrightarrow{F} \cdot \overrightarrow{dr} = \int q\overrightarrow{E} \cdot \overrightarrow{dr}$$

Thus we come to a very important equation, which defines the electric potential, by canceling a q from both sides:

$$\int_{trajectory} \overrightarrow{E} \cdot \overrightarrow{dr} = \Delta V = V_{final} - V_{initial}$$

where we can integrate over any trajectory we like[1]. This, so far, is completely general. How do these equations look if we have point charges? Recall that, in scalar form, $F = qE$ and $F = k\frac{qQ}{r^2}$. Thus

$$-dU = Fdr = k\frac{qQ}{r^2}dr$$

and, taking the integral of both sides we get

$$-\int dU = \int Fdr = \int k\frac{qQ}{r^2}dr$$

But remember that, in scalar form, $F = qE$; hence

[1] This is not trivial and is due to the fact that the electric field is path-independent, or conservative. We will discuss this later in the course; for now it won't affect us.

$$\int Edr = \int \frac{kQ}{r^2}dr = -\int \frac{dU}{q} = -\int dV \Rightarrow V = k\frac{Q}{r} + C$$

Hence the electric potential of a single charge Q is

$$V = k\frac{Q}{r} + C$$

and the electric potential energy between two charges q and Q is, if we integrate $Fdr = k\frac{qQ}{r^2}dr$ directly,

$$U_{qQ} = -\int Fdr = -\int k\frac{qQ}{r^2}dr$$

$$\Rightarrow U_{qQ} = k\frac{qQ}{r} + C$$

We will see examples of how to use these equations, but first let us discuss units. The unit of electric potential energy, like any energy, is the Joule (J). The unit of electric potential is the Volt (V); the relationship between them is

$$U = qV$$

$$\Rightarrow Joule = Coulomb * Volts$$

Thus

$$1J = 1C * 1V$$

This is the definition of a volt: it is $\frac{1J}{1C}$.

Let us discuss this more intuitively now that we've done some derivations and made some mathematical definitions. What we are saying here is this: that there is a function V everywhere in space, which is created by charges in space, that assigns a number to every point in space. Furthermore, the derivative of this function with respect to position gives the magnitude of the electic field at that point. Of course the electric field is a vector field, and so we need some way of assigning a direction; but this is a bit tricky and we won't concern ourselves with that here.[2] So the electric field arises from the change in electric potential over space. Notice that, just like in mechanics, the electric potential has a constant $+C$ stuck at the end, from the fact that our integrals were indefinite. This may seem weird, but you have seen this before. Remember that when you were solving mechanics problems using conservation of energy, you had to define a 'zero potential line', which usually was the ground. But it didn't have to be–you could have defined the point at which there is zero gravitational potential energy to be at the top of a roller coaster or something.

[2]This is done by generalizing the definition of V. We've said that V is related to $\overrightarrow{E}$ by $E = -\frac{dV}{dr}$. Now we can assign a direction to E by defining an operator, which has a meaning only when it operates (as the name implies) on some function. The operator we need is called the del operator, defined as $\overrightarrow{\nabla} = \hat{\imath}\frac{d}{dx} + \hat{\jmath}\frac{d}{dy} + \hat{k}\frac{d}{dz}$ in rectangular coordinates (it has different definitions in different coordinate systems); and now, the definition of V becomes $\overrightarrow{E} = -(\hat{\imath}\frac{d}{dx} + \hat{\jmath}\frac{d}{dy} + \hat{k}\frac{d}{dz})V = -\overrightarrow{\nabla}V$. So here we're taking the derivative of V in all three directions and sticking the associated unit vector on each term.

This is because only the change in gravitational potential energy had any physical significance; in other words, only the change in it affected the physical answer you'd get in the problem. This is the same thing–the point or points in space where there is zero electric potential is or are arbitrary. We must, however, be consistent! Once you choose where the zero electric potential is, you cannot change it in the middle of the problem. You can't go hog wild here.

The convention usually is to choose points at infinity to have zero electric potential. We can see, then, that since

$$V = k\frac{q}{r} + C$$

we get

$$V = C = 0$$

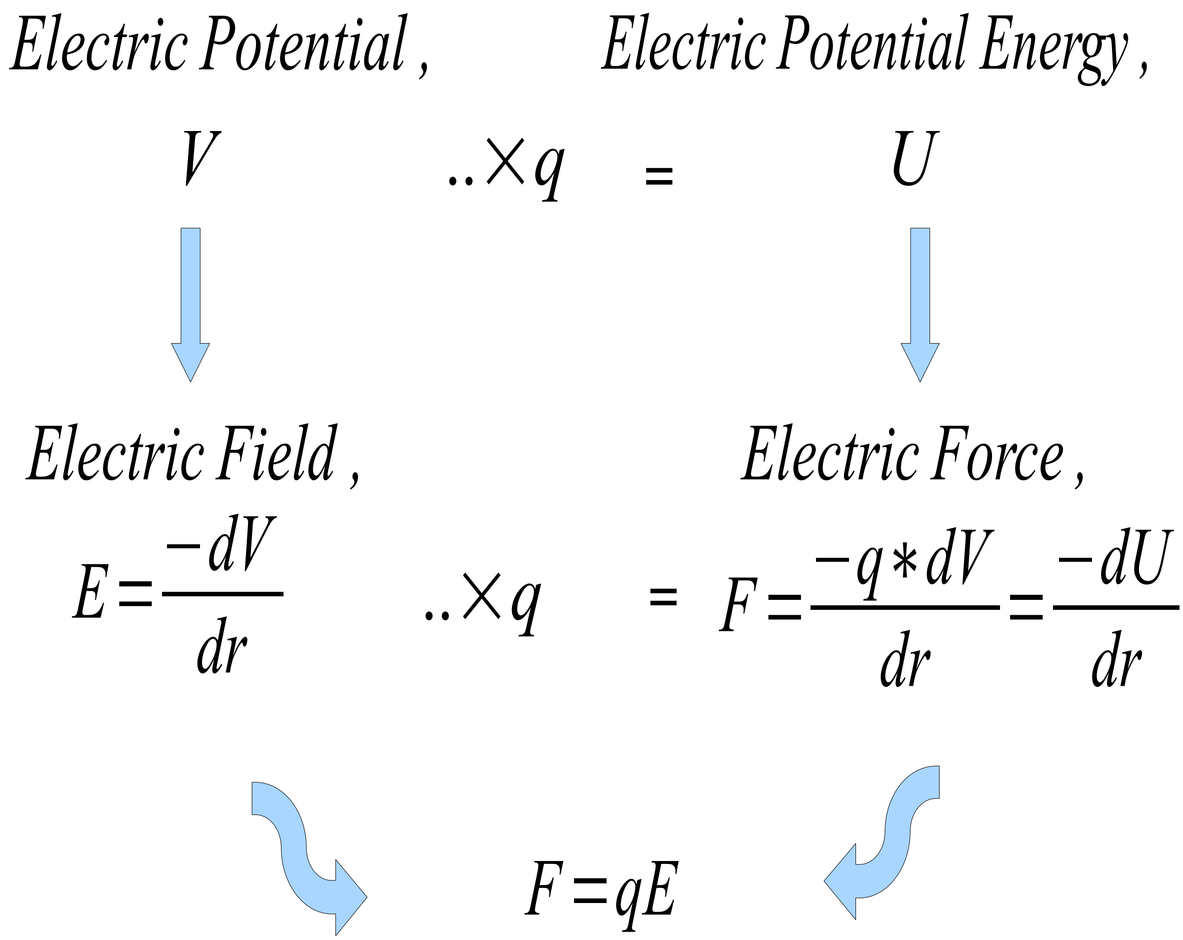
at $r = \infty$. Thus the usualy convention is to set $C = 0$. The difference in potential between two points a and b is, in the presence of a point charge q, is

$$\Delta V = kq(\frac{1}{r_b} - \frac{1}{r_a})$$

which is true regardless of one's convention. However, the electric potential at a point is, with our convention,

$$V = k\frac{q}{r}$$

If there were several point charges, when we'd have to add the potential due to each one at any particular point. Before we discuss some examples, let's think about how all these concepts fit together. To that end I've made a map for you:



These quantities are related by a charge q in a region of space that is experiencing an electric force from an electric field. Let us know turn toward a few examples.

Example 2-7

Say we have two charges, q_1 at (0,0) and q_2 at (0,4). What is the electric potential at the point (2,2)?

Answer: The equation for the potential due to a point charge is

$$V = k\frac{q}{r}$$

where r is the distance from the charge q to the point at which one is finding the potential. We have to two charges, so let's start with q_1:

$$V_1 = k\frac{q_1}{r_1}$$

where r_1 is the distance between (0,0) and (2,2). This is

$$r_1 = \sqrt{(2-0)^2 + (2-0)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{3}$$

Hence

$$V_1 = k\frac{q_1}{2\sqrt{3}}$$

Then the potential due to q_2 is

$$V_2 = k\frac{q_2}{r_2}$$

Now r_2 is the distance from (2,2) and (0,4):

$$r_2 = \sqrt{(2-0)^2 + (4-2)^2} = \sqrt{4+4} = 2\sqrt{3}$$

and

$$V_2 = k\frac{q_2}{r_2} = k\frac{q_2}{2\sqrt{3}}$$

Thus

$$V = V_1 + V_2 = k\frac{q_2}{2\sqrt{3}} + k\frac{q_1}{2\sqrt{3}} = \frac{k}{2\sqrt{3}}(q_1 + q_2)$$

Let us now discuss a problem involving electric potential energy.

Example 2-8 (similar to Chapter 20, # 18, p.676 in Serway and Jewett, Ed. 4)

Say we have 4 charges with the following configuration (length unit is meter):

$20 * 10^{-9}C = q_1 : (0, 0.04)$
$-20 * 10^{-9}C = q_2 : (0, -0.04)$
$10 * 10^{-9}C = q_3 : (0, 0)$
$40 * 10^{-9}C = q_4 : (0.03, 0)$

q_1, q_2, q_3 are glued in place, while I am holding q_4 in place. What is the total electric potential energy of the configuration? If I let go of q_4, in what direction will it move? After an infinite amount of time, what will be its speed?

Answer: Think of this as a series of springs. A spring has potential energy because if it is compressed or elongated and then let go, it will exert a force. The same is true of these charges. Charge q_1 will move due to q_2 if it becomes unglued; it will also move due to q_3 and q_4. Therefore it is as though there are 4 elongated or compressed springs containing potential energy. Hence:

$$U_{12} = k\frac{q_1 q_2}{r_{12}}$$

is the potential energy contained between q_2 and q_1. Without doing an work we can see that $r_{12} = 0.08$ meter. Can you see what the other potential energies are going to be? They will be

$$U_{13} = k\frac{q_1 q_3}{r_{13}}; U_{12} = k\frac{q_1 q_4}{r_{14}}$$

where $r_{13} = 0.04$ and $r_{14} = \sqrt{0.03^2 + 0.04^2} = \sqrt{(10^{-2})^2(3^2 + 4^4)} = 10^{-2}\sqrt{25} = 0.05$ meter. This we could guess by remembering our 3-4-5 triangles. This gives

$$U_1 = U_{12} + U_{13} + U_{14} = k\frac{q_1 q_2}{r_{12}} + k\frac{q_1 q_3}{r_{13}} + k\frac{q_1 q_4}{r_{14}}$$

$$\Rightarrow U_1 = k\frac{q_1 q_2}{r_{12}} + k\frac{q_1 q_3}{r_{13}} + k\frac{q_1 q_4}{r_{14}}$$

$$= k\frac{-20 * 20 * (10^{-9})^2}{8 * 10^{-2}} + k\frac{20 * 10 * (10^{-9})^2}{4 * 10^{-2}} + k\frac{20 * 40 * (10^{-9})^2}{5 * 10^{-2}} = 1.44 * 10^{-4} J$$

This is not the total electric potential energy–what about the enrgy stored between q_2 and q_4? We need now to add all the combinations:

$$U_{total} = U_{12} + U_{13} + U_{14} + U_{23} + U_{24} + U_{34}$$

We don't need to add U_{21} to this, because there is one value for the electric potential energy stored between q_1 and q_2–interchanging the order of the indices on U doesn't produce extra energy. So one way to solve for the total electric potential energy is to sum U_{ij} over all combinations of charges i and j, and then multiply by $\frac{1}{2}$ because we're overcounting by a factor of 2 by summing over all potentials where ij is just interchanged. This is in fact just what we've done above. So remember the useful formula:

$$U_{total} = \frac{1}{2}\sum_{ij} U_{ij} = \frac{1}{2}\sum_{ij} k\frac{q_i q_j}{r_{ij}}$$

So $r_{23} = 0.04$, $r_{24} = 0.05$, and $r_{34} = 0.03$ meter; thus

$$U_{total} = 1.44 * 10^{-4} J + k\frac{q_3 q_2}{r_{32}} + k\frac{q_2 q_4}{r_{24}} + k\frac{q_3 q_4}{r_{34}}$$

$$= 1.44 * 10^{-4} J + 8.99 * 10^9 (\frac{-20 * 10 * (10^{-9})^2}{4 * 10^{-2}} + \frac{-20 * 40 * (10^{-9})^2}{5 * 10^{-2}} + \frac{40 * 10 * (10^{-9})^2}{3 * 10^{-2}})$$

$$= 1.44 * 10^{-4} J - 6.89 * 10^{-5} = 7.51 * 10^{-5} J$$

Now if I let go of q_4, in which direction will it move? We only have one negative charge and it's along the y axis, and a positive charge along the y axis that is of the same magnitude and the same distance from the origin, in the other direction. So without doing any calculations we can state that the y components from these two charges cancel, and therefore q_4 will move solely along the x axis–away from the other charges.

The charge q_4 will move along the x axis forever; but what will its speed be after an infinite period of time? The key is to use conservation of energy here. When q_4 is extremely far away from the other charges, it has no potential energy, since our convention is that the electric potential is

0 at $r = \infty$. Its energy is solely kinetic, then. So over this period, electric potential energy is converted into kinetic energy. We can say this in another way: the change in energy of q_4 is zero. Thus

$$\Delta E = 0 = \Delta U + \Delta K = U_f - U_i + K_f - K_i = -U_i + K_f$$

$$= -U_i + K_f = 0 \Rightarrow U_i = K_f$$

The initial electric potential energy is

$$U_i = K_f = U_4 = U_{14} + U_{24} + U_{34} = k(\frac{q_1 q_4}{r_{14}} + \frac{q_2 q_4}{r_{24}} + \frac{q_3 q_4}{r_{34}}) = 1.2 * 10^{-4} J$$

$$\Rightarrow \frac{1}{2} m v^2 = K_f = 1.2 * 10^{-4} = v = \sqrt{\frac{2 * 1.2 * 10^{-4}}{m}}$$

in meters per second. Where m is the mass of q_4.

I want to highlight an equation noted in this previous example. A straightforward way of calculating the electric potential energy of a bunch of charges is via

$$U_{total} = \frac{1}{2} \sum_{ij} U_{ij} = \frac{1}{2} \sum_{ij} k \frac{q_i q_j}{r_{ij}}$$

Finally, let's discuss how to calculate the electric potential due to continuous charges.

The Electric Potential outside of Continuous Charge Distributions

The method for doing this is fundamentally the same as that we used for finding the electric field due to continuous charge distributions. A continuous charge distribution is formed from an infinite number of infintesimal charges, as an approximation. So we choose an arbitrary point charge dq and use the equation for the potential of a point charge:

$$dV = k \frac{dq}{r}$$

and then the total potential is

$$\int dV = V = \int k \frac{dq}{r}$$

The best way to learn how to perform such calculations is to discuss an example.

Example 2-9

I give you a 2-D disk of radius a with a constant charge density on its surface. The y axis perpendicular to the disk and is going through its center. The z axis is up and the x axis points out of the page. Find the potential along the y axis due to the disk.

Answer: We start with the equation for the potential due to a continuous charge distribution:

$$V = \int k \frac{dq}{r}$$

Let's start with dq. Remember that we have to rewrite this using the charge density. The surface charge density is

$$\sigma = \frac{charge}{area} = \frac{dq}{dA}$$

Since the disk is really a circle, we should use polar coordinates. In these coordinates, the differential element of area is

$$dA = RdRd\theta$$

where θ is the angle around the disk and R is the distance of a point on the disk from the center of the disk. Now we can substitute this into our equation for the charge density:

$$\sigma = \frac{dq}{dA} = \frac{dq}{RdRd\theta} \Rightarrow R\sigma dRd\theta = dq$$

Thus we see that dq is related to the differential angle $d\theta$; it would therefore be incorrect to just integrate over dq directly. We must replace dq with its function in terms of $d\theta$, and only then can we integrate. Substituting, we get

$$V = \int_{wholedisk} k \frac{R\sigma d\theta dR}{r}$$

Remember that r is the distance from the point at which we're finding the potential to the charge dq. The distance from some point on the disk to a point on the x axis is

$$r = \sqrt{x^2 + R^2}$$

We therefore have

$$V = \int_{wholedisk} k \frac{R\sigma d\theta dR}{\sqrt{x^2 + R^2}}$$

First let's note that when integrating over θ, everything in the integral is constant. In other words, there is no function of θ in the integral. We can therefore integrate over θ right away. But what should our limits of integration be? We need integrate over the entire disk to find the potential of the entire disk; the angle around the whole disk is 2π, and the distance from the center of all points on the disk goes from 0 to a. Our limits of integration must then be from 0 to 2π and 0 to a. We can integrate over θ to get[3]

$$V = (\int_0^{2\pi} d\theta)(\int_0^a k \frac{R\sigma dR}{\sqrt{x^2 + R^2}})$$

The integral over θ will give 2π:

$$V = |_0^{2\pi} \theta \int_0^a k \frac{R\sigma dR}{\sqrt{x^2 + R^2}} = 2\pi \int_0^a k \frac{R\sigma dR}{\sqrt{x^2 + R^2}}$$

[3]We can seperate thes two integrals like this because of a mathematical theorem called Fubini's Theorem.

Now we have to integrate over R. We must use u-substitution: define $u \equiv x^2 + R^2$. Then we need to find the relationship between du and dR, because ultimately we need to integrate over du. Remember that x^2 is actually a constant in the integral; thus

$$d(u) = d(x^2 + R^2) = du = 2RdR$$

Now we replace $u = x^2 + R^2$ and $du = 2RdR$:

$$V = 2\pi k \int_{limits} \frac{\frac{1}{2}\sigma du}{u^{1/2}}$$

The limits must be altered, because we've changed variables. However we can ignore this for now, integrate, and then substitute the original variable back in. Now we integrate:

$$V = \sigma\pi k \int_{limits} \frac{du}{u^{1/2}} = \sigma\pi k \int_{limits} u^{-1/2} du$$

$$= \sigma\pi k|_{limits} \frac{u^{1/2}}{1/2} = \sigma\pi k|_0^a \frac{(x^2 + R^2)^{1/2}}{1/2}$$

$$= 2\sigma\pi k[(x^2 + a^2)^{1/2} - x]$$

Let's think about our answer. What would happen if the radius of the disk shrunk to 0? Then there would be no disk! If there were no disk, there would be no charges and therefore no electric potential. The disk radius shrinking to 0 corresponds to $a = 0$. What does our answer yield in this case? We get

$$V = 2\sigma\pi k[x - x] = 0$$

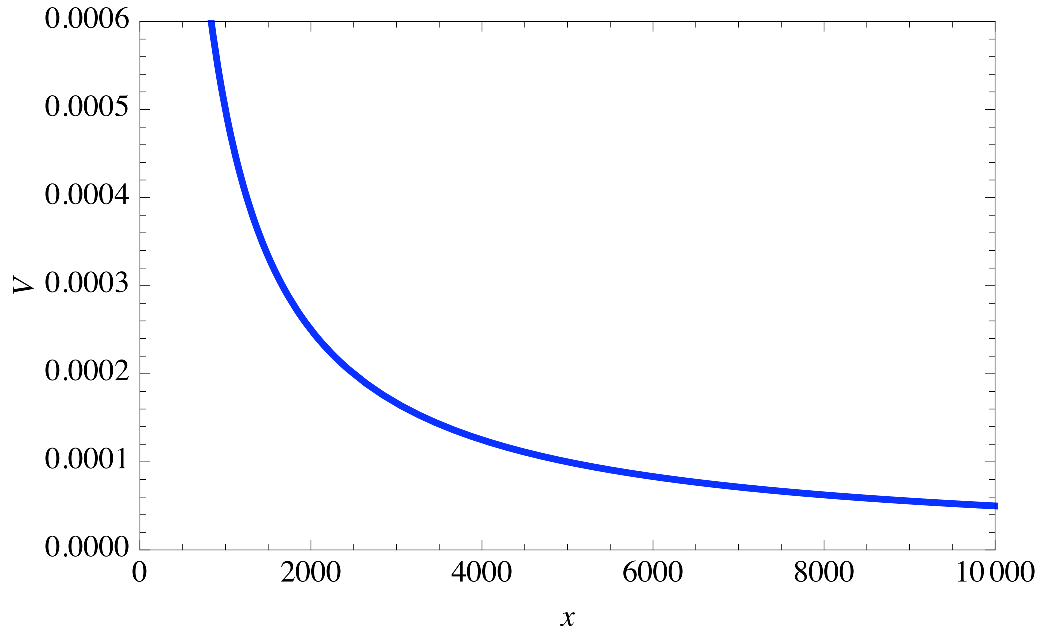
So far, so good. Now, x is the distnace along the x axis from the center of the disk. What should the potential be if we moved along the x axis to infinity? But remember our convention that the potential at infinity is 0. Thus $V(x \to \infty) = 0$; our equation yields

$$V(x \to \infty) = \lim_{x\to\infty} 2\sigma\pi k[(x^2 + a^2)^{1/2} - x]$$

$$= 2\sigma\pi k \lim_{x\to\infty} [x(1 + \frac{a^2}{x^2})^{1/2} - x]$$

$$= 0$$

The function $(1 + \frac{a^2}{x^2})^{1/2}$ will approach 1 as $x \to \infty$. Thus $x(1 + \frac{a^2}{x^2})^{1/2} - x$ should approach 0 as $x \to \infty$. This is 'hand-waving', but we can be more precise by graphing V:



Here I've set $2\sigma\pi k = 1$ and $a = 1$ for technical simplicity (mathematica doesn't want to plot functions with undetermined constants). But we can see that this will go to zero as $r \to \infty$, for all finite values of a and $2\sigma\pi k$.

Now what if we wanted to find the electric field along the x axis near this disk? We we can use the relation

$$E = -\frac{dV}{dr}$$

Finding the electric field everwhere is beyond the scope of this course; so we can just find the derivative of the electric field along only the x axis:

$$E = -\frac{dV}{dx} = -2\pi\sigma k[\frac{1}{2}(x^2 + a^2)^{-1/2}2x - 1] = \pi\sigma k[x(x^2 + a^2)^{-1/2} - 1]$$

Check this answer by finding the electric field by integrating the equation

$$E = k\frac{dq}{r^2}$$

over the whole disk.

Before we continue, I'd like to discuss an important property of certain electric fields.

Conservative and Nonconservative Electric Fields

The potential we have defined is a function: it assigns a number to every point in space. If there were two potentials associated with a single point in space, there would be no unique potential and no potential function (using the precise definition). Of course, the potential is only defined up to an arbitrary constant. But once we define what that constant is, then we have a definite function that assigns one potential value to every point in space.

Now remember that relationship between the electric field and the potential:

$$\Delta V = -\int \overrightarrow{E} \cdot \overrightarrow{ds}$$

What happens if we use this to find the change in potential between the same point–in other words, integrate the electric field over some trajectory that begins at one point (say, point a) and ends at the same point? Well the change in potential must be zero in this case. We have

$$\Delta V = V_a - V_a = -\oint \overrightarrow{E} \cdot \overrightarrow{ds} = 0$$

where the circle on the integral means that it is over a closed trajectory (i.e., one that begins and ends at the same point). So therefore the electric fields we've been discussing so far have the property that

$$\oint \overrightarrow{E} \cdot \overrightarrow{ds} = 0$$

for *any* closed trajectory. Another way of saying this is that this kind of electric field is *conservative*. I say 'this kind' because not all electric fields are conservative; we will come across nonconservative fields when discussing Faraday's law.

A conservative electric field, by the way, indicates a conservative electric force. The work done by electric fields around a closed loop must be zero, since

$$\oint \overrightarrow{E} \cdot \overrightarrow{ds} = 0 \Rightarrow 0 = \oint q\overrightarrow{E} \cdot \overrightarrow{ds} = \oint \overrightarrow{F} \cdot \overrightarrow{ds} = Work$$

Furthermore, a conservative field is necessarily one which is path-independent. This means that the difference in potential between two points,

$$\Delta V = -\int \overrightarrow{E} \cdot \overrightarrow{ds}$$

does not depend on the path over which we integrate $\int \overrightarrow{E} \cdot \overrightarrow{ds}$.

I can't help but mention that this is true of any vector field that can be written as the gradient of a function, where the gradient of a function V is defined to be

$$gradV \equiv \overrightarrow{\nabla} V$$

where

$$\overrightarrow{\nabla} \equiv < \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} >$$

in rectangular coordinates. The electric field we have been discussing so far is in general defined to be

$$\overrightarrow{E} = -\overrightarrow{\nabla} V$$

This automatically implies that $\overrightarrow{E}$ is a conservative vector field. One way to show this is via Stoke's Theorem, which states that

$$\oint \overrightarrow{A} \cdot \overrightarrow{ds} = \int_{area} (\overrightarrow{\nabla} \times \overrightarrow{A}) \cdot \overrightarrow{da}$$

So if there were a vector field $\overrightarrow{A}$ such that $\overrightarrow{\nabla} \times \overrightarrow{A} = 0$, then it must be true that $\oint \overrightarrow{A} \cdot \overrightarrow{ds} = 0$. In other words, the field must be conservative. In our case, we're writing our field as the gradient of a function, but it is true that for any function V,

$$\overrightarrow{\nabla} \times \overrightarrow{\nabla} V = 0$$

So therefore

$$\overrightarrow{\nabla} \times (-\overrightarrow{E}) = 0 \Rightarrow \overrightarrow{\nabla} \times \overrightarrow{E} = 0 \Rightarrow \oint \overrightarrow{E} \cdot \overrightarrow{ds}$$

and the electric field is necessarily conservative. However this is *not true in general*. It is only true for special cases.

Section II-4: Gauss' Law

We now have two ways of calculating the electric field due to charges: by integrating $k\frac{dq}{r^2}$ over all the charges, and by integrating $k\frac{dq}{r}$ to find the electric potential and then using the relationship $E = -\frac{dV}{dr}$. I want to discuss another very important and often easy way of calculating the electric field. This method is based on a mathematical theorem–the Divergence Theorem–that is also often called Gauss' Law. A thorough discussion of the divergence theorem won't conern us here; we will just focus on its application to electricity. [4]

Let's go back to the concept of electric field lines. The more charges are in an area, the greater number of electric field lines that are emitted. Thus the number of electric field lines is related to the magnitude of charge. But how do we quantify the number of electric field lines? We could ask an analogous question: how can we quantify the amount of water passing through a net in a pool? We would do this by multiplying the differntial element of area at a point on the net by the amount of water passing through that differential element of area, and then summing over all of these terms. If dA is the differential element of area, and W is the amount of water passing through dA, then the flux of water passing through the total net area is

$$Flux = \int W dA$$

The 'flux', then, is the total amount of water passing through the net. But what about the direction of the flow of water? What if the water is flowing parallel to the net? If this were the case, there would be no water flowing through the net, and so no flux. We can use our dot product

[4]The divergnce theorem states: $\int_{volume} \overrightarrow{\nabla} \cdot \overrightarrow{A} dV = \int_{surface} \overrightarrow{A} \cdot \overrightarrow{ds}$. Here dV means a differential element of volume. The surface integral is over a completely closed surface, and the volume integral is over the entire volume enclosed by that surface. This was first published by Carl Gauss.

trick, and take the dot product of the amount of water $\overrightarrow{W}$ and the differential element $\overrightarrow{dA}$. The vector $\overrightarrow{W}$ is in the direction of the water, and therefore we need $\overrightarrow{dA}$ to be perpendicular to the suface of the net. Thus when $\overrightarrow{W}$ is perpendicular to the surface of the net, it will be parallel to $\overrightarrow{dA}$ and the dot product will be just WdA. And when $\overrightarrow{W}$ is parallel to the net–i.e., when it is perpendicular to $\overrightarrow{dA}$–the dot product will be 0. Thus a more general definition is

$$Flux = \int \overrightarrow{W} \cdot \overrightarrow{dA}$$

Now let's construct a similar equation by analogy for electricity. Instead of a flux of water, we need to write down an equation for the flux of electric field lines. So instead of talking about the magnitude and direction of water passing through a tiny bit of area, we need to talk about the magnitude and direction of the electric field passing through a tiny bit of area. Thus the electric flux is

$$electricflux = \int_{area} \overrightarrow{E} \cdot \overrightarrow{dA} = \Phi$$

The symbol Φ is widely used for flux. So now we have half of Gauss' Law. The other half is not so easy to follow–it comes from a mathematical theorm, as I've mentioned. But Gauss asserted that the electric flux through any closed surface is proportional to the amount of charge enclosed by the surface. A way to picture this is to think of a light bulb in a box with transparent sides. The total flux of light through the sides of the box is proportional to the brightness of the bulb. The shape of the box is irrelevant; if I replaced the box with a large transparent sphere, the total amount of light passing through the sphere would be the same, right? How could it be different? Gauss' Law operates in the same way. Imprecisely, Gauss' Law states:

$$\Phi = c * q_e = \oint \overrightarrow{E} \cdot \overrightarrow{dA}$$

where c is some proportionality constant, and the integral symbol with the circle means that the integral is over an entire surface that (1) is completely closed and (2) contains a total charge q_e. The vector $\overrightarrow{dA}$ is perpendicular to the surface, and its magnitude is just dA. Notice that I didn't say 'the' entire surface; I said 'an' entire surface. Just like the light bulb contained in a transparent surface, I can choose from an infinite number of surfaces. If I put the light bulb in a closed surface that's in the shape of a velociraptor, will the total flux change? Of course not–unless the raptor has been shot, in which case it's no longer a closed surface and therefore doesn't count.

This equation is of little practical use, however, unless we can determine what this constant is. Well let's compare this to the equation for the electric field outside of a point charge q. It is

$$\overrightarrow{E} = k\frac{q}{r^2}\hat{r}$$

which we get from the definition of $\overrightarrow{E}$ and the equation for $\overrightarrow{F}$ that is experimentally determined. Let's try to derive this from Gauss' Law. We need to choose a surface, though. We will have to evaluate the integral $\oint \overrightarrow{E} \cdot \overrightarrow{dA}$ over the entire surface, so we should try to ensure that the surface we choose has two properties:

1. The vector $\overrightarrow{E}$ is perpendicular to the surface; then, $\overrightarrow{E} \cdot \overrightarrow{dA} = EdA$ because $\overrightarrow{E}$ is parallel to $\overrightarrow{dA}$. This makes integration much easier.

2. This is the most important part. The electric field $\overrightarrow{E}$ should be constant on the entire surface that encloses the charge q. If its constant, we can pull it out of the integral like so: $\oint \overrightarrow{E} \cdot \overrightarrow{dA} = \overrightarrow{E} \cdot \oint \overrightarrow{dA}$. If we can't do this, we must integrate over $\overrightarrow{E}$, which we don't know. How can we evaluate this integral? We can't–and so Gauss' Law, while still valid, is pretty useless. If #1 is also true, we can easily evaluate the dot product to get $\overrightarrow{E} \cdot \oint \overrightarrow{dA} = E \oint dA = EA$.

Thus if the above two conditions are met, then we can immediately say $\oint \overrightarrow{E} \cdot \overrightarrow{dA} = EA$. Thus, according to Gauss' Law,

$$\oint \overrightarrow{E} \cdot \overrightarrow{dA} = c * q_e = EA$$

where A is the area of the surface which encloses the charge q. Thus solving for E becomes quite simple, if we know A.

A Very Important Point About Gauss' Law.

Notice that only the charges enclosed by the surface contribute to the flux. Any charge outside the surface contributes percisely nothing to the total flux through the surface. But think about this: say there were a point charge outside of a surface, and no charges inside of it; what would the electric field on this surface be? From our previous discussion of the electric field of a point charge, we know that there should be a nonzero electric field on this surface. However, Gauss' Law tells us that the flux through this surface would be zero. How can this be, if there are electric field lines passing through the surface? This can only be, of course, if all the fluxes across the surface cancel for all charges outside the surface. This is not intuitive–at least it wasn't for me when I first learned Gauss' Law.

So remember this key fact:

Charges outside of a closed surface contribute nothing to the total electric flux through the surface.

Now, back to our regularly scheduled programming.

So what type of surface should we choose? Well we know that the electric field lines radiate directly outward from positive charges (let's take $q > 0$), and the magnitude of the electric field depends only on the distance from the charge. Thus we need a surface that is the same distance away from a point. Also, we need a surface where the electric field vectors are perpendicular to it. Can you guess which surface we should choose? A hollow sphere! Ok so,

$$\oint \overrightarrow{E} \cdot \overrightarrow{dA} = c * q_e = \int_{sphere} EdA = E \int_{sphere} dA$$

$$= EA_{sphere} = 4\pi r^2 E$$

since the surface area of a hollow (or filled) sphere is $4\pi r^2$. The enclosed charge q_e is q. Thus,

$$c * q_e = 4\pi r^2 E \Rightarrow \overrightarrow{E} = \frac{cq\hat{r}}{4\pi r^2}$$

So now,

$$\overrightarrow{E} = k\frac{q}{r^2}\hat{r} = \frac{cq\hat{r}}{4\pi r^2}$$

$$\Rightarrow k = \frac{c}{4\pi}$$

This must be the constant c such that Gauss' Law is consistent with our previous equations for the electric field. It is customary to define a new constant ϵ_0 such that

$$k = \frac{1}{4\pi\epsilon_0}$$

Thus

$$\frac{1}{4\pi\epsilon_0} = \frac{c}{4\pi} \Rightarrow c = \frac{1}{\epsilon_0}$$

This is the customary way of writing Gauss' Law (this is in SI units; there are many different types of units):

$$\oint \overrightarrow{E} \cdot \overrightarrow{dA} = \frac{q_e}{\epsilon_0}$$

Remember: q_e is the total enclosed charge by a closed surface, and the integral is over this enclosed surface. This surface, by the way, is often called the *Gaussian surface*.

You might ask why the heck we'd want to define yet a new constant. Well it is useful because Gauss' Law must be altered when being used to find the electric field in materials; in particular, different materials have different constants in place of ϵ_0.

So we have Gauss' Law. Let's apply this to problems, to better understand it.

Example 2-10: Electric field outside of a charges sphere

What is the electric field outside of a sphere containing a charge q that is homogenously (evenly) distributed inside of it?

Answer: First, we must determine what Gaussian surface to use. We can in principle use any surface, so long as its closed. But not every surface is useful, for the reasons discussed above. Since we have a sphere with a charge evenly distributed in it, we have symmetry. If we take the sphere and rotate it about its center, should the electric field around it change? It would not make sense for it to do so, because the sphere looks exactly the same from all directions. In fact, if you didn't see me rotate the sphere, you wouldn't be able to determine if I had done so. Thus, the electric field around the sphere should only have a radial dependence–it should not change if the distance from the center of the sphere does not change. Furthermore, the direction of the electric field should be radial–it should point directly outward from the sphere.

We therefore need a Gaussian surface such that this kind of electric field is constant on the surface, and such that the electric field is perpendicular to the surface. How about a surface where each point is the same distance away from the center of the sphere? This would work perfectly. So we should choose our gaussian surface to be a hollow sphere. Since the hollow sphere satisfies the two conditions listed above, we can write

$$\oint \overrightarrow{E} \cdot \overrightarrow{dA} = \frac{q_e}{\epsilon_0} = EA$$

A is the surface area of the Gaussian surface–in this case, a hollow sphere. The surface area is $A = 4\pi r^2$, and thus

$$\frac{q_e}{\epsilon_0} = 4\pi r^2 E \Rightarrow E = \frac{q}{4\pi\epsilon_0 r^2}$$

is the magnitude of the electric field outside of the charged sphere. Since the Gaussian surface encloses the entire charged sphere, the total enclosed charge, q_e, is q. We have already asserted that the electric field lines radiate directly outward, so the electric field is

$$\overrightarrow{E} = \frac{q\hat{r}}{4\pi\epsilon_0 r^2}$$

This is the same electric field outside of a point charge! Gauss' Law tells us that the electric field outside of a sphere with a charge evenly distributed in it, is the same as that of a point charge.

Example 2-11: Electric field inside of an evenly charged sphere

Now find the electric field inside a sphere with an evenly distributed charge Q, and a radius R.

From the same symmetry arguments we used above, we can conclude again that the best Gaussian surface (in fact, the only good one to use) is a hollow sphere. Thus, as before,

$$\oint \overrightarrow{E} \cdot \overrightarrow{dA} = \frac{q_e}{\epsilon_0} = EA$$

But we must be careful here–q_e is the total charge enclosed by the surface. But the surface is no longer enclosing the entire charge Q; it is only enclosing part of it. But how much is it enclosing? To asnwer this question we use the same concept that saved us before: charge density.

The volumetric charge density is $\frac{charge}{volume}$. This is not necessarily constant, but in our problem we've assumed that the charge is distributed evenly, and so here the density is in fact constant. Usually the greek letter ρ is used to denote volumetric charge density:

$$\rho = \frac{charge}{volume} = \frac{Q}{\frac{4}{3}\pi R^3}$$

where the volume of our charged sphere is $\frac{4}{3}\pi R^3$. So if we multiply the volume of the sphere by its charge density, we get the total charge:

$$\rho\frac{4}{3}\pi R^3 = Q$$

How about we do this for the part of the charged sphere enclosed by our spherical Gaussian? The total charge enclosed by the Gaussian is, then,

$$\rho\frac{4}{3}\pi r^3 = q_e$$

But we can now replace ρ with $\frac{Q}{\frac{4}{3}\pi R^3}$:

$$\rho \frac{4}{3}\pi r^3 = q_e = \frac{Q}{\frac{4}{3}\pi R^3} * \frac{4}{3}\pi r^3 = \frac{Qr^3}{R^3}$$

We can now substitute this into Gauss' Law:

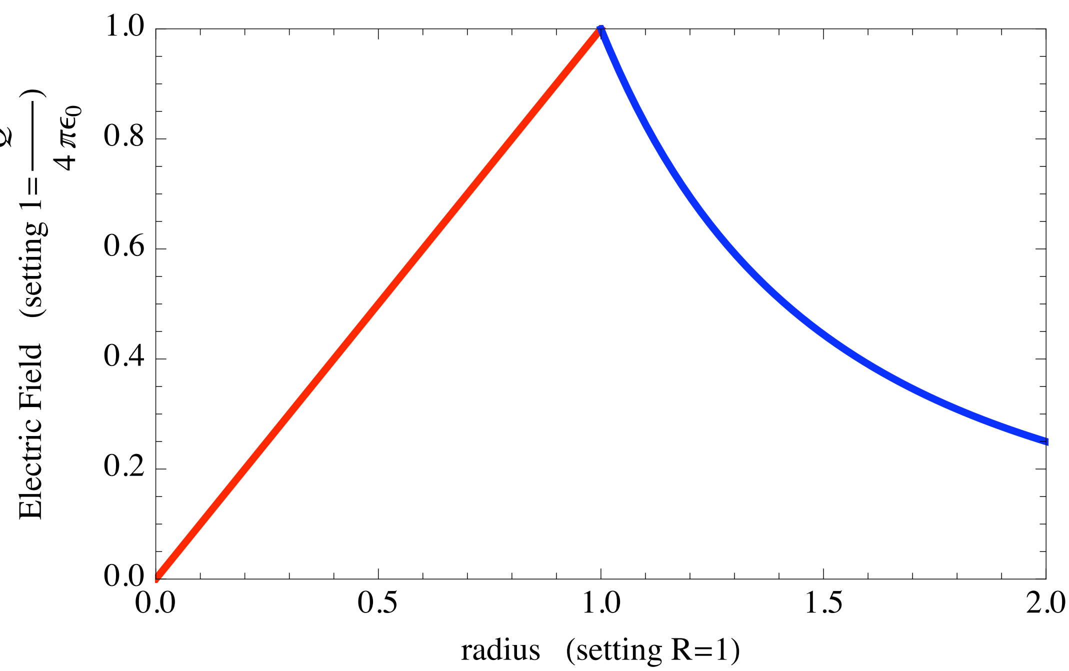
$$\frac{\frac{Qr^3}{R^3}}{\epsilon_0} = EA = \frac{Qr^3}{\epsilon_0 R^3}$$

Remember that A is the surface area of the Gaussian; here our Gaussian is a hollow sphere, and its surface area is, like before, $4\pi r^2$. Thus

$$EA = \frac{Qr^3}{\epsilon_0 R^3} = 4\pi r^2 E \Rightarrow \overrightarrow{E} = \frac{Qr^3 \hat{r}}{4\pi\epsilon_0 r^2 R^3} = \frac{Qr\hat{r}}{4\pi\epsilon_0 R^3}$$

Again, we've already presumed that the direction of the electric field is radially outward from the center of the charged sphere, so we ca just stick a $\hat{r}$ on the magnitude and be done with it.

We therefore have our answer. So now let's put them together; what does the electric field look like inside and outside an evenly charged sphere? Here I've (qualitatively) graphed the solutions for you:



The red part of the graph represents the electric field inside the charged sphere, whereas the blue part represents the electric field outside. I've set some arbitrary constants to 1 purely for simplicity;

they can take an infinite number of arbitrary values, and if so the plot would look qualitatively the same.

I want to point out that it is only because ρ is constant that we can write $\rho\frac{4}{3}\pi R^3 = Q$. If ρ were not constant, we would need to define ρ at particular points in the sphere:

$$\rho = \frac{dq}{dV}$$

were dV is a differential element of volume (NOT potential). Therefore, the total charge in an object is

$$\int dq = Q = \int \rho dV$$

If ρ were constant, we could take it out of the integral:

$$Q = \int \rho dV = \rho \int dV = \rho V$$

which is precisely the result we wrote down before.

Example 2-12: Electric field outside of a cylinder

What is the electric field outside of a cylinder with an evenly distributed charge Q?

Answer: The very first step is to choose a Gaussian surface that satisfies our two conditions. A cylinder, like a sphere, has a symmetry: if I rotate the cylinder around the axis that goes through its center, the electric field around it should not change. Thus the electric field should be entirely radial, with the exception that this is not true near the edges of the cylinder. The edges of a cylinder destroy the symmetry, and so if we are to use Gauss' Law we must only find the electric field near the center of the cylinder.

So our Gaussian surface is therefore a hollow cylinder. The length of the Gaussian, let' say, is l. What is the charge enclosed by the Gaussian? We need the lineral charge density λ, $\lambda = \frac{Q}{L}$. Thus the enclosed charge is

$$q_e = \lambda l$$

Thus

$$\oint \overrightarrow{E} \cdot \overrightarrow{dA} = \frac{q_e}{\epsilon_0} = EA_{Gaussian} = E * 2\pi r l$$

$$\Rightarrow E * 2\pi r l = \frac{q_e}{\epsilon_0} = \frac{\lambda l}{\epsilon_0}$$

$$\Rightarrow \overrightarrow{E} = \frac{\lambda l \hat{r}}{2\pi r l \epsilon_0} = \frac{\lambda \hat{r}}{2\pi r \epsilon_0}$$

Note that the answer here doesn't depend on the length l of the Gaussian. This is good, because l is totally arbitrary and thus should cancel.

A Few Words About Conductors And Insulators

We have not discussed properties of materials up to now. We have found the electric field inside a charged sphere–but of what material is the sphere made? Are electric fields affected by materials? The answer is yes–but this subject is far too complex and deep to discuss it in its entirely. The electric field in the sphere that we found, by the way, is not quite correct; the field is altered by the presence of the material, a fact we completely ignored. I thus just want to talk about two categories of materials: conductors and insulators. You probably already know the distinction between these two materials, so let's dive right in.

A conductor is a material that has an abundance of relatively free electrons. By 'relatively free', I mean that they can, with little resistance, move around the conductor. Electrons have a very tiny mass, so they tend to move rather quickly. Say there is a conductor that overall is electrically neutral. If there is an electric field inside this conductor due to a buildup of charges inside it, then the electrons will rapidly move to around the conductor until they 'neutralize' the charge. This will make the electric field vanish. If I then throw excess electrons on the conductor, they will very quickly move away from each other and eventually occupy positions that maximize the distance between them. Therefore charge on a conductor will exist on its surface; and the electric field inside a conductor is zero. An insulator, as you might guess, is a material with the opposite property: electrons are 'stuck' in place and cannot move freely throughout the material. So remember the two basic properties of conductors:

1. The electric field inside a conductor is zero, and, equivalently, there is no buildup of charges inside it.
2. If a conductor has an overall charge, all charges are spread out on the surface of the conductor.

We will not discuss these properties too frequently or thoroughly, but they are integral to more advanced studies of EM.

Let's apply this to a simple problem–finding the electric field outside of the surface of a conductor.

Example 2-13: Electric field outside of a conductor's surface

There shouldn't be a build-up of charges on a particular part of the surface. So if we zoom in on a very small part of the surface, it would look flat. If this seems like an odd statement, think of our experience on the Earth. The Earth is spherical, but since we inhabit only a tiny portion of the surface of the Earth, we see it as flat.

So the electric field on a tiny flat portion should be perpendicular to the surface, because all parallel components should cancel. This is very 'hand-waving' and not very scientific, but a full explanation is byond the scope of this course.

Anyway, we should choose a Gaussian where the electric field is constant on, and perpendicular to, the surface. Our surface is a cylinder that we stick over the surface of the conductor. Since there is no electric field inside a conductor, the bottom of the conductor has no flux through it, nor do the rounded sides because the electric field is perpendicular to the surface (and therefore parallel to the sides). So we can write

$$\oint \overrightarrow{E} \cdot \overrightarrow{dA} = \frac{q_e}{\epsilon_0} = EA$$

In general, A is the surface of the Gaussian–but in this case E only flows through the top of the cylinder, and therefore A is really just the area of the top of the cylinder. The area is

$$A = \pi r^2$$

though we won't need this. What is the enclosed charge? If we multiple the surface area by the area, we will get the charge over the area. If the surface charge is σ, then

$$q_e = \sigma A$$

Hence,

$$EA = \frac{q_e}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

$$\Rightarrow E = \frac{\sigma}{\epsilon_0}$$

This is a rather famous equation, and it gives the electric field just outside of a conductor. What is the material is not a conductor–in other words, if there is an electric field inside it? Then there would be an electric field passing through the top and bottom portions of the cylinder. Our equations are the same, except that the area of the Gaussian is $2A$:

$$2AE = \frac{\sigma A}{\epsilon_0} \Rightarrow E = \frac{\sigma}{2\epsilon_0}$$

This is the electric field outside of a nonconductor. (CHECK)

Let's take another example.

Example 2-14: Electric field inside and outside of an insulator surrounded by a conductor (similar to Ch.19, #65, pg. 641 in Serway and Jewett, Ed. 4)

Say we have an insulating sphere of radius a, surrounded by a conducting spherical shell of inner radius b and outer radius c. There is a charge Q on the inner sphere, and no charge on the conductor. Find the electric field everywhere.

Answer: There can be an electric field inside an insulator, and so we use the same Gauss' Law method to find it as before. Again, we ignore the effects of the material on the electric field. For sake of time and space I won't repeat it here. The answer we got before was

$$\overrightarrow{E} = \frac{Qr\hat{r}}{4\pi\epsilon_0 R^3}$$

What is the electric field inbetween the inner and out spheres? Again we have the same answer as before, which was

$$\overrightarrow{E} = \frac{Q\hat{r}}{4\pi\epsilon_0 r^2}$$

Since the outer spherical shell is a conductor, we know that the electric field inside of it is zero.

What about the electric field outside of the conducting shell? The total enclosed charge is the same, and we still need a spherical Gaussian surface, so the electric field is

$$\overrightarrow{E} = \frac{Q\hat{r}}{4\pi\epsilon_0 r^2}$$

the same as the electric field inbetween the two materials.

Section II-5: Application: Capacitors

In this last section of part II, we discuss a particular application of all the physics we've been learning. The ability to hold charge and then release it is an important property that is required in certain devices. For example, defibrillators require a rapid release of charge to have the desired effects. This ability to hold charge is called *capacitance.*

More precisely, capacitance is defined as the amount of charge that can be held or stored per volt:

$$C \equiv \frac{Q}{\Delta V}$$

Capacitance is always (for our purposes) positive; thus is we ever get a negative change in potential, we will just take the absolute value. What is the unit for capacitance? Well the unit of charge is the Coloumb, and that of potential is the Volt; so the unit of capacitance, which is the Farad (F), is

$$1F = \frac{1C}{1V}$$

A Farad is a very large unit of measurement. Many of the run of the mill capacitors are measured in 10^{-3} Farads.

Now, charge cannot be stored in one space–if positive and negative charges are not physically seperated, they will move together and neutralize. Thus to actually build a capacitor we need to seperate the charges on two different conductors. Thus Q is the amount of charge stored on one of the conductors and ΔV is the potential difference between the conductors. There are many different types of capacitors, including chemically based ones that don't consist of two seperated conductors. However this is meant only to be an introduction to capacitance, and we won't concern ourselves with that here. Let's start with the simplest capacitor–the parallel plate capacitor.

Parallel Plate Capacitors

A Parallel plate capacitor is just what it sounds: two parallel, conducting plates that are seperated by a distance that we'll call d. Let's calculate the capacitance of this capacitor. Let's place an arbitrary charge Q on each place–though one place has a $+Q$ and the other one has a $-Q$. Therefore there is an electric field between the plates, that is directed from the '$+$' plate to the '-' plate. Let's take the area of each of the plates to be A. What is the magnitude of the electric field between the plates? Well these plates are conductors, and we've already found the electric field outside of a conductor; it is

$$E = \frac{\sigma}{\epsilon_0}$$

where σ is the surface charge density of each plate. Ultimately, we need to find an equation for ΔV. But we know that potential is related to the electric field; recall the equation

$$\Delta V = -\int \overrightarrow{E} \cdot \overrightarrow{ds}$$

where ΔV is the difference in electric potential between two points, and where the integral is between those two points. So what is the electric potential difference between the two plates. Well $\overrightarrow{ds}$ is a differential element of length along the integration trajectory. The electric field inbetween the two plates does not really flow from one plate to another in straight lines, but if we stay away from the edges of the plates this is a good approximation. So if we integrate from from the '+' plate to the '-' plate in a straight line, $\overrightarrow{ds}$ will be in the same direction as $\overrightarrow{E}$; hence $\overrightarrow{E} \cdot \overrightarrow{ds} = Eds$. Also, we will approximate the magnitude of the electric field as that outside of a conductor, $E = \frac{\sigma}{\epsilon_0}$. We can now substitute these into the potential equation, and take the absolute value of ΔV:

$$\Delta V = \int Eds = \int \frac{\sigma}{\epsilon_0} ds = \frac{\sigma}{\epsilon_0} \int ds = \frac{\sigma d}{\epsilon_0}$$

The surface charge density is not a variable over the integral, and of course ϵ_0 is a constant. So we can take them out of the integral, as I've done. Now, the surface charge density is

$$\sigma = \frac{Q}{A}$$

Substituting this into our ΔV,

$$\Delta V = \frac{Qd}{A\epsilon_0}$$

Now we can substitute this into our capacitance equation:

$$C = \frac{Q}{\frac{Qd}{A\epsilon_0}} = \frac{A\epsilon_0}{d}$$

This is the capacitance of a parallel plate capacitor. It is a famous result, and a rather simple one. If we increase the area A of the plates, the capacitance increases. The opposite relation holds for the distance d between the plates.

Notice that the capacitance is a geometrical property; it depends on the area of the plates and the distance between then. It does not depend upon the charge on the plates, or the potential between them (the capacitance is the ratio of the two, it's not related to the actual values of these). This in general is true. We will see that the capacitance of all the capacitors we will study are based on the capacitors' geometry. There are capacitors that depend on things like voltage and current, but that is for more advanced courses.

We can continue to perform the same calculations for different capacitor shapes. Let us turn to a cylindrical capacitor now.

Cylindrical Capacitors

Say we have a capacitor that consists of two concentric cylinders. The inner cylinder has a radius of a and an arbitrary charge $+Q$, and the outer one has a radius b and am arbitrary charge $-Q$. What is the capacitance of this capacitor? We use the same basic method as before, by calculating the electric field and from this use the difference in potential between the charges. We have already found the electric field outside of a charged cylinder (using Gauss' Law); it is

$$\overrightarrow{E} = \frac{\lambda \hat{r}}{2\pi r \epsilon_0}$$

where, remember, λ is the linear charge density. We now find the difference in potential using

$$|\Delta V| = \int \overrightarrow{E} \cdot \overrightarrow{ds}$$

$$= \int_{limits} \frac{\lambda \hat{r}}{2\pi r \epsilon_0} \cdot \overrightarrow{ds}$$

We now integrate radially, from the inner cylinder to the outer one. Thus $\overrightarrow{dr} = \overrightarrow{ds}$ and $\hat{r} \cdot \overrightarrow{ds} = \hat{r} \cdot \overrightarrow{dr} = dr$. What are the limits of integration? They are from $r = a$ to $r = b$. We have

$$\int_{limits} \frac{\lambda \hat{r}}{2\pi r \epsilon_0} \cdot \overrightarrow{ds} = \int_a^b \frac{\lambda \hat{r}}{2\pi r \epsilon_0} \cdot \overrightarrow{ds} = \frac{\lambda}{2\pi \epsilon_0} \int_a^b \frac{1}{r} dr = \frac{\lambda}{2\pi \epsilon_0} |_a^b \ln r = \frac{\lambda}{2\pi \epsilon_0} \ln \frac{b}{a}$$

But since $\lambda = \frac{Q}{L}$, we can substitute this in and we will see that the Q in the definition of capacitance will cancel with this one. Here, L is the length of our Gaussian that enclosed a charge Q. Remember that we must stay away from the edges of the cylinders, because the assumptions we need to make to use Gauss' Law break down there. Subtituting, we have

$$|\Delta V| = \frac{Q}{2\pi L \epsilon_0} \ln \frac{b}{a}$$

Finally, we substitute this into our definition of capacitance:

$$C = \frac{Q}{\frac{Q}{2\pi L \epsilon_0} \ln \frac{b}{a}} = \frac{2\pi L \epsilon_0}{\ln \frac{b}{a}}$$

So once again, we see that capacitance is a geometrical property. The larger the ratio of $\frac{b}{a}$, the smaller the capacitance; the longer the capacitor, the larger the capacitance (Again, we're ignoring the capacitance of the edges of the cylinders.).

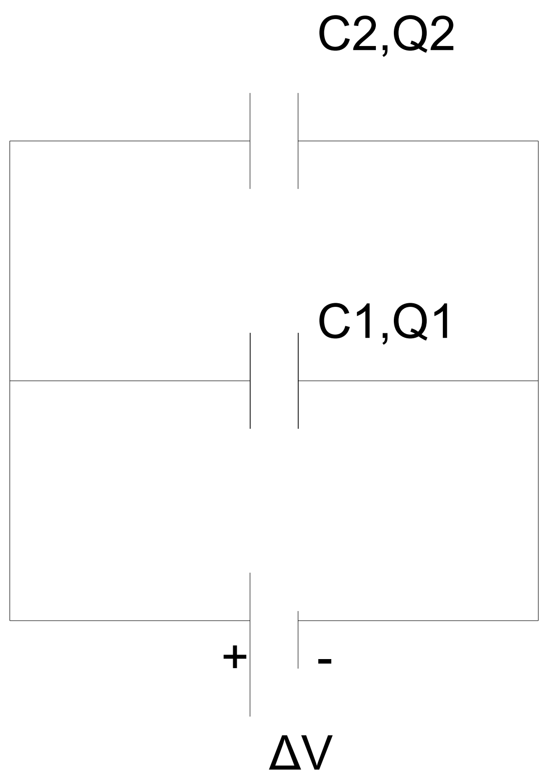
Let's now discuss how to place multiple capacitors in circuits.

Multiple Capacitors in Circuits

You already know what a circuit is, I'm sure. We haven't begun discussing circuits, but for now we will use this word to mean components that are connected by wires. For example, we could have a battery, which produces a voltage difference across it, connected to several capacitors. There are two distinct ways of adding capacitors–in and in parallel and series. We begin with parallel.

Parallel Capacitors

Circuit elements connected in parallel are those that have the same potential across them. For example, two capacitors C_1 and C_2 can be connected in parallel like so:



Ignore the fact that C_1 is darker than C_2; this is due to a lack of open office drawing skills on my part. Anyway, note the symbols here. A capacitor is denoted by two parallel lines, and a battery (in other words, a voltage source), is represented by two parallel lines of different lengths–the longer line is the higher voltage side of the battery, as you can see in the drawing.

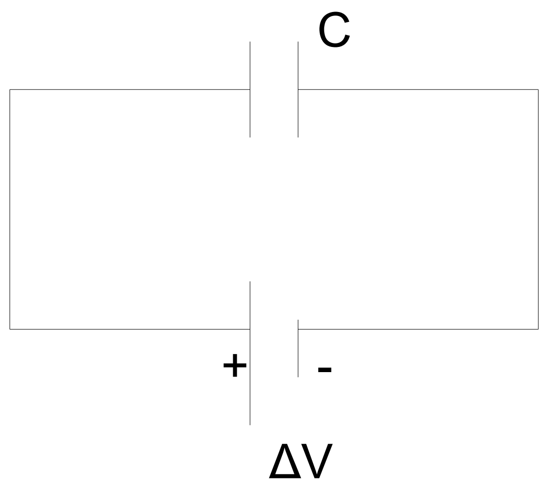
These two capacitors are parallel because the voltage difference between them is the same, ΔV. This is because there is no voltage drop along the wires connecting the capacitors. A circuit element that can reduce a voltage along a wire is called a resistor; we will discuss resistors soon. For now, it is not terribly relevant.

The capacitor C_1 has a charge Q_1 on it, and C_2 has a charge Q_2 on it. What if we wanted to replace these two capacitors with a single capacitor that had the equivalent capacitance? Then we'd have to learn how to add capacitors in parallel. To begin, let's write down the capacitance of these two:

$$C_1 = \frac{Q_1}{\Delta V}$$

$$C_2 = \frac{Q_2}{\Delta V}$$

Replacing the capacitors with an equivalent capacitor would make the circuit look like



The key to continuing from this point is to realize that if we replace the two capacitors by a single capacitor, the charge on it will be the sum of the charges on C_1 and C_2. If the charge on the equivalent capacitance is Q, then

$$Q = Q_1 + Q_2$$

The capacitance of the equivalent capacitor is

$$C = \frac{Q}{\Delta V} = \frac{Q_1 + Q_2}{\Delta V} = C_1 + C_2$$

Thus, when two capacitors C_1 and C_2 are in parallel, they add directly:

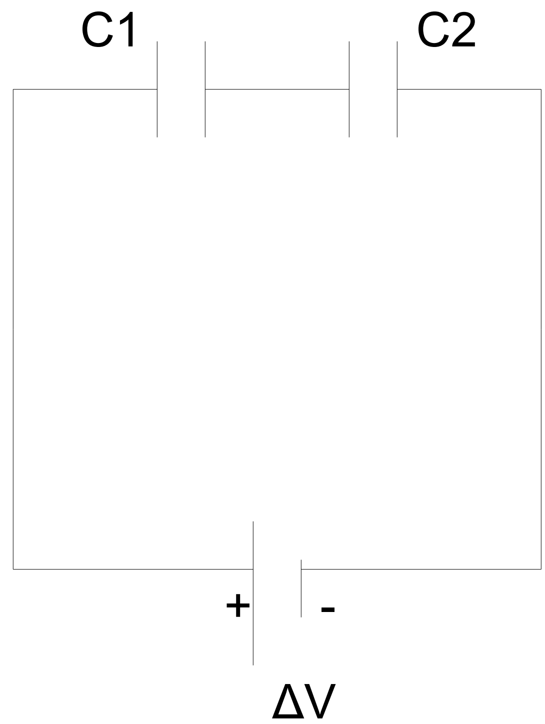
$$C = C_1 + C_2$$

We can easily generalize this to any number of capacitors connected in series:

$$C = C_1 + C_2 + C_3 + ...$$

Series Capacitors

The other way to add capacitors is by adding capacitors in series. Circuit elements are in series if the same charge is going through them. Circuit elements in series are connected by the same wire; for example, two capacitors connected in series looks like



So C_1 and C_2 have the same charge across them, but the voltages across them are not generally the same. The voltages across C_1 and C_2 are ΔV_1 and ΔV_2, respectively; and the charges on C_1 and $C_2$are Q_1 and Q_2, respectively. So

$$Q_1 = Q_2 = Q$$

The voltage across both capacitors is ΔV, and therefore

$$\Delta V = \Delta V_1 + \Delta V_2$$

Now let's write the defintion of capacitance for each capacitor:

$$C_1 = \frac{Q_1}{\Delta V_1} = \frac{Q}{\Delta V_1} \Rightarrow \Delta V_1 = \frac{Q}{C_1}$$

$$C_2 = \frac{Q_2}{\Delta V_2} = \frac{Q}{\Delta V_2} \Rightarrow \Delta V_2 = \frac{Q}{C_2}$$

Now we substitute both of these equations into our equation for ΔV:

$$\Delta V = \Delta V_1 + \Delta V_2$$

$$\Rightarrow \Delta V = \frac{Q}{C_1} + \frac{Q}{C_2} = Q(\frac{1}{C_1} + \frac{1}{C_2})$$

If we replaced the two capacitors with on equavalent capacitor of capacitance C, then

$$C = \frac{Q}{\Delta V} \Rightarrow \Delta V = \frac{Q}{C}$$

These two equations are equal; setting them equal, we get

$$\Delta V = Q(\frac{1}{C_1} + \frac{1}{C_2}) = \frac{Q}{C}$$

$$\Rightarrow \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{C}$$

This is how two capacitors add when in series.
We can extend this to any number of capacitors in series:

$$\Rightarrow \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + ... = \frac{1}{C}$$

Section II-6: Current, Resistance, and Circuits

Let's continue our examples of applications of what we've been learning by discussing circuits more generally.

You have heard of current, I'm sure. Current is the flow of electric charge over time; the definition is

$$I = \frac{dq}{dt}$$

I is commonly used for current. The unit of current we will use is the Ampere, abbreviated A.

Current doesn't flow equally well in all materials. We've already discussed that there are some materials in which electrons flow quite freely (known as conductors), and others in which electrons's flow is severely restricted (known as insulators). We can quantify this by introducing the concept of resistance, which is just what it sounds like: the resistance of a material to the flow of electrons. Electrons, or any other charges, will flow when there potential differences between points in the material; remember that the change in potential over distance is the electric field (or the negative of it). The greater the change in potential, the greater the current. There is a law that the relationship between these two variables is linear; this is called Ohm's Law:

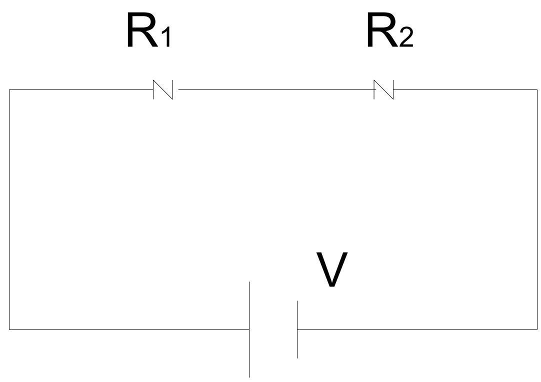
$$V = IR$$

where I is the current, R is the resistance, and V is the potential difference across the part of the circuit in which the current I exists. The unit of resistace we will use is the Ohm, which is abbreviated by the greek letter Ω.

How would we go about adding resistors in a circuit? Let's start with resistors in series.

Series Resistors

Two resistors in series look like



Let's use Ohm's Law to figure out how to add these two resistors, so that we could replace them with a single equivalent resistance. First of all, the current going through the resistors is the same, while the voltage differences across them are different–let's call them V_1 and V_2. Using Ohm's Law, we get

$$V_1 = IR_1$$

$$V_2 = IR_2$$

If we add the voltage differences across both resistors, we get the battery voltage V:

$$V = V_1 + V_2$$

If we replaced the two resistors with a single equivalent resistance R, we could apply Ohm's Law:

$$V = IR$$

Now, let's replace V_1 and V_2 using Ohm's Law:

$$V = V_1 + V_2 = IR_1 + IR_2 = I(R_1 + R_2)$$

But of course, V also is

$$V = IR = I(R_1 + R_2) \Rightarrow R = R_1 + R_2$$

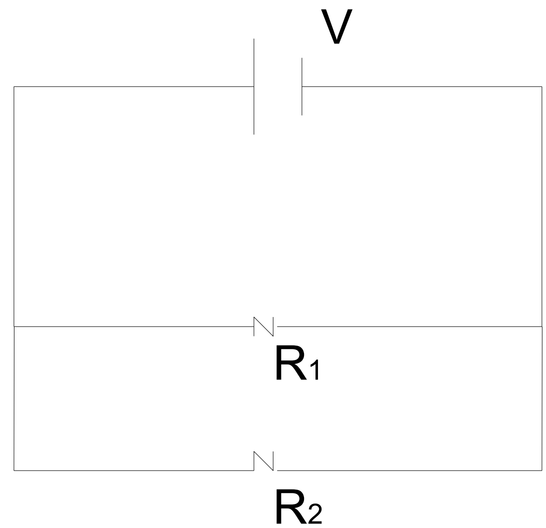
So when in series, resistors add directly. We can generalize this to any number of resistors in parallel:

$$R = R_1 + R_2 + R_3 + ...$$

Let's discuss resistors in parallel now.

Parallel Resistors

Two resistors in parallel look like



Two parallel resistors like these have the same voltage differences across them–specifically, V. However, the current through each resistor is in general different. Applying Ohm's Law, we get

$$V = I_1 R_1 \Rightarrow \frac{V}{R_1} = I_1$$

$$V = I_2 R_2 \Rightarrow \frac{V}{R_2} = I_2$$

I we replace these two resistors with a single equivalent resistance, we can apply Ohm's Law:

$$V = IR \Rightarrow \frac{V}{R} = I$$

where I is the current through the battery V. What is the relationship between I and I_1, I_2? Well if we replace the two resistors with a single resistance R, then the current through R must be $I_1 + I_2$. So

$$I = I_1 + I_2$$

Then, we can substitute our previous equations into this:

$$\frac{V}{R} = \frac{V}{R_1} + \frac{V}{R_2}$$

$$\Rightarrow \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

This is the way two resistors add when in series. We can generalize this to any number of resistors:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + ...$$

Power In Circuits

How do we apply energy considerations to resistors? It turns out that it's rather easy. Let's start with the equation for the electric potential energy:

$$U = qV$$

We need to recall a concept from introductory mechanics–power. Remember that power is the work done per time:

$$P = \frac{dW}{dt}$$

But the work done by a circuit element is equal to the potential energy it dissipates: $dW = dU = d(qV)$. Then we can write that

$$P = \frac{d(qV)}{dt} = V\frac{dq}{dt}$$

where V is the potential difference across a resistor, and we presume that it remains constant. The charge through the resistor, however, in general changes over time. Our definition of current is $I = \frac{dq}{dt}$; hence:

$$P = IV$$

This is the power dissipated in a resistor, through which there is a current I and across which there is a potential difference V. However by using Ohm's Law we can rewrite this in two ways:

$$P = I(IR) = I^2R$$

$$P = V\frac{V}{R} = \frac{V^2}{R}$$

Each of these three equations for the power in a resistor is equally valid. You should use the equation that is the most useful for the problem; if you have the current through the resistor and its resistance, then use I^2R; if instead you have the voltage across it and its resistance, use $\frac{V^2}{R}$.

Notice that I said P is the power *dissipated* in a resistor. Resistors don't add energy to circuits, they act like brakes to the flow of current and, like brakes, remove energy from a system in the form of heat.

Kirchoff's Laws

What if we have a circuit for which we have a limited about of information–like the voltage of a battery and the values of its resistors–and we want to know the current in each part of the circuit? We clearly need to apply some physics principles, which will allow us to write down some equations; then by solving them we will hopefully have solved for the currents and such. These 'physics principles' are in the form of two laws called 'Kirchoff's Laws'. Let's discuss them one by one.

1. ***Conservation of Charge***. It is known that charge is conserved–charges can be moved around but cannot be created nor destroyed. This, of course, applies to circuits. The point at which several wires merge or diverge in a circuit is called a *node*. At each node, charge is conserved. But current is just the movement of charge, and so at each node current is conserved. This is the first law:

$$\sum I_{initial} = \sum I_{final}$$

2. ***Conservation of Energy***. You already know about conservation of energy, and as you might guess it applies to circuits. If you were an electron, you could go on the 'circuit loop ride'–traveling around and entire loop in a circuit. If you could measure the electric potential around the loop, what should you get? Let's say you measured the potential at the start and got V_a, and then at the and of the ride (when you returned to the same point), you measured and got V_b. Then what you would be asserting is that the same point can have different potentials, which runs counter to the concept of a potential that we've discussed. It also cannot be true if the electric field in question is conservative, which ours is. So we require that the change in electric potential around an entire loop is zero:

$$\Delta V_{loop} = 0$$

   If I multiply the change in potential by the charge of the electron, I will get

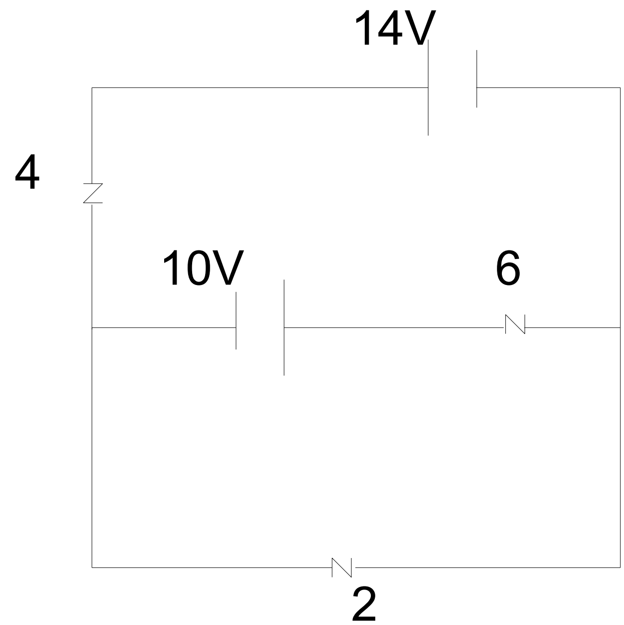
$$\Delta U_{loop} = 0$$

   for any charge that goes on the 'circuit loop ride'. So we can think of this as an application of the conservation of energy.

The best way to learn these is to apply them to problems. Let's start with this:

Example 2-15: Finding currents inside a circuit (Ch.21, Ex. 21.9, pg. 708 in Serway and Jewett, Ed. 4)

We have the following circuit:



Find all the currents in the resistor.

Answer: Before we being these problems, we must pick the directions of the currents. Of course we don't know the directions yet, but if we pick the wrong directions we will get negative currents–but the magnitude of the currents will be correct. So let's call the current that goes through the $10V$ battery I_1, and it is going from left to right. The current in the $14V$ battery is I_2, traveling from left to right through it. Finally the current through the 2Ω resistor is I_3, traveling from right to left.

Now we need to apply the laws we've just learned methodically, which will give us equations we can solve. First let's apply the conservation of charge. Let's look at the node where all three wires meet to the right of the 6Ω resistor. We know that all the currents going into it must be equal to the sum of the currents leaving it. I_2 and I_1 and coming into the node, and I_3 is leaving. Therefore:

$$I_1 + I_2 = I_3$$

Now let's apply the second law to the loops. We can apply it to any closed loop in a circuit. Let's start with the top loop consisting of the two batteries and the 6Ω and 4Ω resistors. Let's call this loop 1. We'll begin at a point inbetween the $10V$ battery and the 6Ω battery, as we'll sum in a counterclockwise direction. The current I_1 is going through the 6Ω resistor from left to right, and we are summing in this direction. Now resistors resist the flow of current and dissipate energy, so there should be a voltage drop in this direction–in other words, the voltage (or potential–same thing) to the left of the resistor should be greater than the voltage to the right. So as we travel through this resistor, the voltage difference we write down must be negative:

$$-6I_1$$

Then, we travel through the $14V$ battery. Remember that the short side is the negative terminal, and so we're traveling from the negative to the positive terminal and therefore experiencing an increase in potential. So we write this as a positive change in potential.

We then travel through the 4Ω resistor. I_2 is traveling upwards through this resistor, and yet we are 'traveling' downwards. Thus the change in potential should be positive:

$$4I_2$$

Finally, we 'travel' through the $10V$ battery, going from the negative to positive terminals. This change in potential is positive. So, summing over the entire loop,

$$0 = -6I_1 + 4I_2 + 14 + 10$$

$$0 = -6I_1 + 4I_2 + 24$$

We must do the same thing for the bottom loop. We get

$$0 = -2I_3 + 10 - 6I_1$$

Now we have three equations and three unknowns, so we should be able to solve for all the currents. Solving, we get

$$I_1 = 2A$$

$$I_2 = -3A$$

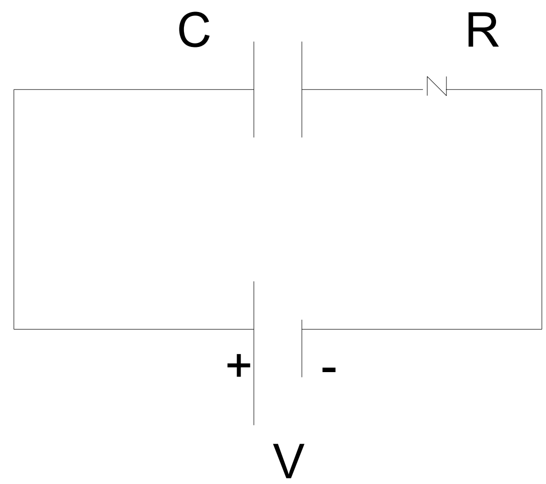
$$I_3 = -1A$$

You should be able to solve those three equations to get these currents.

Now let's turn to the physics of charging capacitors.

Charging Capacitors

You know that capacitors hold charge. But how do we quantify this charge? How can we predict what charge will be held by a capacitor after a specified time? Well let's start with a capacitor that initially has no charge on it, connected to a battery. For instance, say we have the following circuit:



This is referred to as an *RC circuit*, for the simple reason that it consists of a resistor and capacitor in series. Let's apply the second law. Summing over the whole circuit in a clockwise fashion,

$$0 = V - V_c - IR$$

V_c is the voltage across the capacitor. The voltage across the capacitor is negative because the left plate has a positive potential, while the right plate has a negative potential. We can use the definition of capacitance to replace V_c:

$$C = \frac{q}{V_c} \Rightarrow V_c = \frac{q}{C}$$

Since we want an equation for the charge q on the capacitor as a function of time, we should replace the current with its definition:

$$I = \frac{dq}{dt}$$

Now our equation is

$$0 = V - \frac{q}{C} - \frac{dq}{dt}R$$

R, V, and C are constant, and so we should be able to solve this differential equation. Let's do some rearranging:

$$-V = -\frac{q}{C} - \frac{dq}{dt}R$$

$$\Rightarrow -V + \frac{q}{C} = -\frac{dq}{dt}R$$

$$\Rightarrow (-\frac{V}{R} + \frac{q}{RC}) = -\frac{dq}{dt}$$

$$\Rightarrow dt = -\frac{dq}{(-\frac{V}{R} + \frac{q}{RC})}$$

$$\Rightarrow dt = -\frac{RCdq}{(-VC + q)} = \frac{RCdq}{(VC - q)}$$

Finally, we integrate:

$$\Rightarrow \int_{t_0}^{t} dt = -\int_{0}^{q} \frac{-RCdq}{(VC - q)}$$

$$\Rightarrow t - t_0 = -[RC\ln(VC - q) - RC\ln(VC)]$$

$$= -RC\ln(\frac{VC - q}{VC})$$

$$\Rightarrow e^{-(t-t_0)/RC} = \frac{VC - q}{VC}$$

Therefore, the charge on the capacitor as a function of time is, when the capacitor is charging,

$$\Rightarrow q(t) = VC(1 - e^{-(t-t_0)/RC})$$

For simplicity we can set $t_0 = 0$:

$$\Rightarrow q(t) = VC(1 - e^{-t/RC})$$

Let's think about whether this equation makes any sense. What if we were to allow this capacitor to charge forever? Well then the charge on the capacitor would be maximum, and the charge a capacitor can hold is

$$q = VC$$

So we should expect that our equation yields this for $t = \infty$. When $t = \infty$,

$$e^{-t/RC} \to 0$$

Therefore as $t \to \infty$, $q \to VC$. Good thing!

Of course, at $t = 0$ there should be no charge on the capacitor. So at $t = 0$,

$$e^{-t/RC} = 1$$

and therefore $q(t) = 0$.

Since we've been so successful at finding the charge on a charging capacitor, maybe we could find the current on it, too? Luckily, there is a simple relationship between the current and charge:

$$I = \frac{dq}{dt}$$

Therefore,

$$I(t) = VC(-(-\frac{1}{RC})e^{-t/RC}) = \frac{V}{R}e^{-t/RC}$$

The initial current is

$$I(0) = \frac{V}{R} = I_0$$

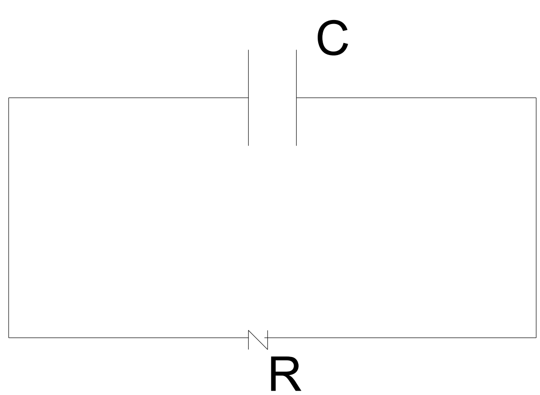
What should the current be after an infinite period of time, when the capacitor is fully charged? Well if there were a flow of charge, then the capacitor would accumulate more charge. But this cannot be, because the capacitor is fully charged. So our equation should yield no current as $t \to \infty$. We get

$$I(t \to \infty) \to 0$$

from the equation we just derived.

Discharging Capacitors

What is the charge held by a capacitor that is discharging? What if we charged a capacitor and put it in series with a resistor? Our circuit would look like



We can apply the same method as we did for a charging capacitor, by starting with the Kirchoff's second law:

$$0 = IR + V_c$$

So we have

$$0 = \frac{dq}{dt}R + \frac{q}{C}$$

which is a bit easier to solve. We get

$$dt = -\frac{dq}{q}RC$$

$$\Rightarrow \int_{t_0}^{t} dt = \int_{q_0}^{q} \frac{-dq}{q}RC = -RC(\ln q - \ln q_0)$$

where q_0 is the initial charge of the capacitor, and q is the charge after some time t.

$$-(t - t_0) = RC \ln \frac{q}{q_0}$$

$$\Rightarrow e^{-(t-t_0)/RC} = \frac{q}{q_0}$$

$$\Rightarrow q_0 e^{-(t-t_0)/RC} = q(t)$$

Let's see if this makes sense. The initial charge on the capacitor we've defined as q_0. So when, $t = t_0$,

$$q_0 e^0 = q_0 = q(t_0)$$

We also know that after the capacitor has discharged for an infinite period of time, there should be no charge left. So as $t \to \infty$,

$$q \to q_0 e^{-\infty} \to 0$$

as we expect.

Section III-1: Of Magnets and Monopoles, Magnetic Fields and Forces

For several millenia it has been known that there exists some phenomenon, involving forces acting at a distance, that is distinct from the electric force. For instance, sailors for hundreds of years have known that some materials will automatically direct themselves toward certain directions on the Earth. The first compasses were made around the 13th century "A.D.", and I believe the first reference to the use of magnetic materials for navigation was in 1119 "A.D.", in China. Eventually, for better or (more likely) for worse, Europeans realized its usefulness and it became invaluable to European explorers.

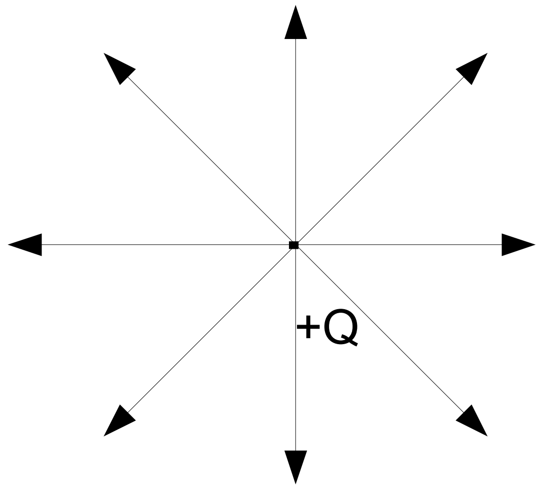
It has only been in the past 180 years or so, however, that magnetism has been understood more precisely. There is an entire history to be written about this, but we don't have time to delve into that here. So let's start with some of the similarities and differences between electricity and magnetism.

Electric forces occur between individual electric charges, called monopoles–meaning that each charge is either negative or positive. But it is a fact that no magnetic charges like this have ever been observed. Magnetic forces occur between individual packets that contain two poles–one negative and one positive.

These 'packets' are called magnets. You might think that you can 'trick' nature by just cutting a magnet in half, but you'd be unnerved to discover that instead of having two magnetic monopoles, you would have two new magnets! Why magnets always have two poles is not known, and many physicists believe that magnetic monopoles do in fact exist, but have for some reason become exceptionally rare. One leading explaination is that monopoles were common in the early universe, but their density was then enormously depressed by a rapid and intense period of cosmic inflation, when the universe expanded by a factor of about e^{55} ($e \simeq 2.7$). A period of 'reheating' followed, where matter densities were increased by the production of particles; monopoles, obviously, were excluded from production. These are just hypotheses, however.

Anyway, the important thing to remember is that the smallest units between which magnetic forces act are magnets, which contain two poles–one negative, one positive. Let's start with some similarities and differences between magnetism and electricity.

First, the electric field around a positive charge is



whereas the magnetic field around a magnet is

Just like electric field lines, magnetic field lines emanate from positive charges. However, since magnetic charges are tied to negative ones in magnets, magnetic fields start from the positive pole and wrap around to the negative pole. Let's turn to magnetic forces now.

Magnetic Forces

The force exerted on charges from magnetic fields is dependent upon the velocity of the charges. By experiment the force on a charge q, that has a velocity $\overrightarrow{v}$, in a magnetic field $\overrightarrow{B}$ has been determined to be

$$\overrightarrow{F} = q\overrightarrow{v} \times \overrightarrow{B}$$

This can be determined from more advanced studies on electromagnetism, but that is beyond this course.

What is the magnetic force on a charge q that is moving in the same direction as the magnetic field; this is because in that case $\overrightarrow{v}$ would be parallel to $\overrightarrow{B}$, and the cross product of two parallel vectors is always zero:

$$\overrightarrow{F} = q\overrightarrow{v} \times \overrightarrow{B} = q\left|\overrightarrow{v}\right|\left|\overrightarrow{B}\right|\hat{e}\sin(0) = 0$$

if $\overrightarrow{v}$ is parallel to $\overrightarrow{B}$. And when is the magnetic force maximum? Well sin is maximum when the angle is $\frac{\pi}{2}$; in other words, when the vectors are perpendicular. Thus when $\overrightarrow{v}$ is perpendicular to $\overrightarrow{B}$, the magnetic foce is maximum. In this case:

$$\overrightarrow{F} = q\overrightarrow{v} \times \overrightarrow{B} = q\left|\overrightarrow{v}\right|\left|\overrightarrow{B}\right|\hat{e}\sin(\frac{\pi}{2}) = q\left|\overrightarrow{v}\right|\left|\overrightarrow{B}\right|\hat{e}$$

where $\hat{e}$ is a unit vector determined through the right hand rule. It is perpenicular to both $\overrightarrow{v}$ and $\overrightarrow{B}$.

So we see that the magnetic force is not as simple as the electric force. The unit of the magnetic field is:

$$force = charge * speed * magneticfield$$

$$\Rightarrow Newton = Coloub * \frac{meter}{second} * B$$

$$\Rightarrow B = \frac{1Ns}{Cm} = 1T$$

which is a $Tesla$, a Newton second per Coloumb meter. Or, replacing a Newton with a $\frac{kgm}{s^2}$, we have

$$1T = \frac{kgms}{Cms^2} = \frac{kg}{Cs}$$

which is an equivalent defintion of a Tesla. A Tesla is a very large unit, and often magnetic fields are measured in Gauss, defined as

$$10^4 Gauss = 1Tesla$$

The magnetic field of the Earth ranges from about 0.3 Gauss to about 0.6 Gauss, depending upon where on the Earth one measures it.

Let's discuss an example of how to use the magnetic force. Say we have an electron of charge q moving in the $\hat{i}$ direction with a velocity v, and we have a magnetic field B in the $\hat{j}$ direction. What is the magnetic force on this electron? It is

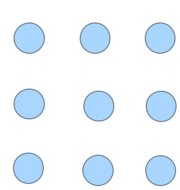
$$\overrightarrow{F} = q\overrightarrow{v} \times \overrightarrow{B}$$

$$= qv\overrightarrow{i} \times B\overrightarrow{j}$$

$$= qBv\overrightarrow{k} = \overrightarrow{F}$$

What if the electron is moving in the $\hat{k}$ direction and the magnetic field is in the $\hat{j}$ direction?

By convention, if the magnetic field is out of the page we use dots as a symbol:



If a magnetic field is into the page, by convention we use Xs:

X X X X
X X X X
X X X X
X X X X

Let's now look at an example.

Example 3-1: Determining the initial direction of deflection (Ch.22, #1, pg. 757 in Serway and Jewett, Ed. 4)

a. Determine the initial direction of deflection of a positively charged particle moving to the right that enters a region where the magnetic field is into the page.

Answer: The velocity is $\overrightarrow{v} = v\hat{\imath}$; the magnetic field is $\overrightarrow{B} = -B\hat{k}$. Thus

$$\overrightarrow{F} = q\overrightarrow{v} \times \overrightarrow{B} = -qvB\hat{\imath} \times \hat{k} = -qvB(-\hat{\jmath}) = qvB\hat{\jmath}$$

The initial deflection is up.

b. Ditto for a negatively charged particle traveling to the left, that enters a region where the magnetic field is up.

Answer: The velocity is $\overrightarrow{v} = -v\hat{\imath}$; the magnetic field is $\overrightarrow{B} = B\hat{\jmath}$. Thus

$$\overrightarrow{F} = q\overrightarrow{v} \times \overrightarrow{B} = -qvB\hat{\imath} \times \hat{\jmath} = -qvB(-\hat{k}) = qvB\hat{k}$$

c. Ditto for a positively charged particle traveling to the left, that enters a region where the magnetic field is to the right.

Answer: The velocity is $\overrightarrow{v} = -v\hat{\imath}$; the magnetic field is $\overrightarrow{B} = B\hat{\imath}$. Thus

$$\overrightarrow{F} = q\overrightarrow{v} \times \overrightarrow{B} = -qvB\hat{\imath} \times \hat{\imath} = 0$$

there is no force!

d. Ditto for a positively charged particle traveling to the left, that enters a region where the magnetic field directed up and to the left, making a 45 degree angle with the horizontal.

Answer: The velocity is $\overrightarrow{v} = v\hat{j}$; the magnetic field is $\overrightarrow{B} = B(\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{j})$. Thus

$$\overrightarrow{F} = q\overrightarrow{v} \times \overrightarrow{B} = qvB\hat{j} \times (\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{j}) = \frac{qvB}{\sqrt{2}}(\hat{j} \times \hat{\imath} + \hat{j} \times \hat{j}) = -\frac{qvB}{\sqrt{2}}\hat{k}$$

So we have an equation for the magnetic force on a charges particle. Therefore the total force on a charged particle, from both magnetic and electric fields, is

$$\overrightarrow{F} = q\overrightarrow{v} \times \overrightarrow{B} + q\overrightarrow{E}$$

$$= q(\overrightarrow{v} \times \overrightarrow{B} + \overrightarrow{E})$$

This is the Lorentz force.

Example 3-2: The Lorentz Force, part I (Ch.22, #10, pg. 757 in Serway and Jewett, Ed. 4)

Say we have a particle with a charge q that is moving along the $\hat{\imath}$ direction, and there is a magnetic field of magnitude $B = 15 * 10^{-3}T$ and in the $\hat{j}$ direction. What electric field would one have to apply to the charge in order to ensure that the Lorentz force vanishes.

Answer: The Lorentz force is $\overrightarrow{F} = 0 = q(\overrightarrow{v} \times \overrightarrow{B} + \overrightarrow{E}) \Rightarrow -\overrightarrow{v} \times \overrightarrow{B} = \overrightarrow{E}$. The left hand side is

$$\overrightarrow{v} \times \overrightarrow{B} = Bv\hat{\imath} \times \hat{j} = -vB\hat{k}$$

Now,

$$\overrightarrow{E} = vB\hat{k}$$

This is the electric field we need to apply to ensure that the Lorentz force is zero.

Magnetic Force On a Current

We can, with some skill, apply the magnetic force on a charge to a current. A current, of course, is just a string of moving charges. Like our previous problems, we will take an arbitrary charge in a current and write down the magnetic force on it from a magnetic field. Let's use the language of calculus:

$$\overrightarrow{dF} = dq\overrightarrow{v} \times \overrightarrow{B}$$

Now, the velocity of the electrons is the speed and direction of the current. We can write this as

$$\overrightarrow{v} = \frac{\overrightarrow{dl}}{dt}$$

where $\overrightarrow{dl}$ is a differential element of length in the direction of the current. Now, $dq\overrightarrow{v}$ is $dq\frac{\overrightarrow{dl}}{dt}$. We can rewrite this as

$$dq\overrightarrow{v} = \frac{dq}{dt}\overrightarrow{dl} = I\overrightarrow{dl}$$

Now, our initial equation is

$$\overrightarrow{dF} = I\overrightarrow{dl} \times \overrightarrow{B}$$

So what if we want to find the force on all the charges due to $\overrightarrow{B}$? Well we sum over all the charges by integrating:

$$\int \overrightarrow{dF} = \overrightarrow{F_M} = \int_{current} I\overrightarrow{dl} \times \overrightarrow{B}$$

which we integrate over the whole current. The 'M' subscript denotes that it's a magnetic force. Another, simpler way of writing this is

$$\overrightarrow{F_M} = \int_{current} \overrightarrow{I} \times \overrightarrow{B}$$

where, of course, $\overrightarrow{I} = I\overrightarrow{dl}$.

Example 3-3: Force on a Semi-Circular Wire (Ch.22, Example 22-4, pg. 740 in Serway and Jewett, Ed. 4)

What is the total magnetic force, due to a magnetic field $\overrightarrow{B} = B\hat{j}$, on a semiccircular wire with a current I that is traveling counterclockwise? The radius of the semicircle is R.

Answer: The equation we need is

$$\overrightarrow{F_M} = \int_{current} I\overrightarrow{dl} \times \overrightarrow{B}$$

Let's do this in a piecemeal fashion. We'll start with the straight wire. Here, $\overrightarrow{dl} = \hat{i}dx$. Taking the cross product,

$$\overrightarrow{dl} \times \overrightarrow{B} = \hat{i}dx \times \overrightarrow{B} = Bdx\hat{i} \times \hat{j} = Bdx\hat{k}$$

This implies that

$$\overrightarrow{F_s} = \int_s IBdx\hat{k}$$

$$= \int_{-R}^{R} IBdx\hat{k} = 2IBR\hat{k} = \overrightarrow{F_s}$$

The subscript 's' denotes 'straight', as in the straight portion of the wire. The semicircular part is trickier. Let's say θ is the angle between $\overrightarrow{I}$ and $\overrightarrow{B}$. We can then use the definition of the cross product:

$$\overrightarrow{I} \times \overrightarrow{B} = IBdl(-\hat{k})\sin\theta$$

Since we're dealing with a semicircle, $dl = Rd\theta$; replacing this we get

$$= \overrightarrow{I} \times \overrightarrow{B} = -IBRd\theta\hat{k}\sin\theta$$

Then we integrate, and voila:

$$\overrightarrow{F_c} = -\int_0^{\pi} IBRd\theta\hat{k}\sin\theta$$

$$= -|_0^{\pi}(-IBR\hat{k}\cos\theta) = IBR\hat{k}(-1-1) = -2IBR\hat{k} = \overrightarrow{F_c}$$

The 'c' here denotes 'circle'. We conclude, therefore, that the total force is zero:

$$\overrightarrow{F_T} = \overrightarrow{F_s} + \overrightarrow{F_c} = 0$$

Example 3-4: Force on a Circular Wire (Ch.22, #17, pg. 740 in Serway and Jewett, Ed. 4)

What is the magnetic force on a clockwise (from non-magnet side) current that is just above the positive pole of a magnet?

The magnetic field is bent around the wire, so at the wire its projected outward at angle θ from the vertical. The radius of the circular wire is R.

Answer: We can write the magnetic field as

$$\overrightarrow{B} = B(\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta)$$

Now we need to find $\overrightarrow{dl}$, a differential element of length in the direction of the current. This can be tricky business, so let's use a symmetry argument to make this considerably easier. Let's take the rightmost point on the wire. What is $\overrightarrow{dl}$ here? Well the current is clockwise, so $\overrightarrow{dl} = dl\hat{k}$. The little arc length dl is $Rd\phi$. Thus, $\overrightarrow{dl} = Rd\phi\hat{k}$ at this point. So at this point, the cross product is

$$\overrightarrow{I} \times \overrightarrow{B} = I\overrightarrow{dl} \times B((\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta)$$

$$= IRd\phi\hat{k} \times B(\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta)$$

$$= IBRd\phi(\hat{k} \times \hat{\imath}\sin\theta + \hat{k} \times \hat{\jmath}\cos\theta)$$

$$= IBRd\phi(\hat{\jmath}\sin\theta + -\hat{\imath}\cos\theta)$$

If we take the leftmost point, then $\overrightarrow{dl} = -Rd\phi\hat{k}$ and $B(-\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta)$. Then,

$$\overrightarrow{I} \times \overrightarrow{B} = -IRd\phi\hat{k} \times B(-\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta)$$

$$= -IRBd\phi(-\hat{k} \times \hat{\imath}\sin\theta + \hat{k} \times \hat{\jmath}\cos\theta)$$

$$= -IRBd\phi(-\hat{\jmath}\sin\theta + (-\hat{\imath})\cos\theta)$$

$$= IRBd\phi(\hat{\jmath}\sin\theta + \hat{\imath}\cos\theta)$$

We can see that the $\hat{i}$ components cancel. With more work, we could do the same calculation for many different points, and we would see that for each point on the loop there is another opposing point that cancels the $\hat{i}$ component of $\overrightarrow{I} \times \overrightarrow{B}$. This is not very rigorous, but we can conclude that the only nonzero component of $\overrightarrow{I} \times \overrightarrow{B}$ after integration will be

$$\overrightarrow{I} \times \overrightarrow{B}|_{nonzero} = IRBd\phi\hat{j}\sin\theta$$

Therefore,

$$\overrightarrow{F_M} = \int_{current} I\overrightarrow{dl} \times \overrightarrow{B} = \int_0^{2\pi} IRBd\phi\hat{j}\sin\theta = IRB\hat{j}\sin\theta \int_0^{2\pi} d\phi = 2\pi IRB\hat{j}\sin\theta$$

Up to now we have not discussed how to actually calculate the magnetic field. I have mentioned that magnetic fields are created by magnets, though I haven't written down an equation describing these fields.

I also haven't mentioned that there is an intimate connection between magnetism and electricity. This is not at all intuitive, considering the marked differences between magnetic and electric fields that I've mentioned. In fact, I'm holding out on you a bit–the electric and magnetic fields are actually different parts of the same field. This realization, however, did not come until several suggestive experiments were performed.

We start with Hans Christian Orsted, a Danish physicist who, on April 21, 1820, noticed during a lecture that he was able to deflect a compass needle by switching on and off a current. Many years later, it was discovered how to quantify this: the Biot-Savart law. We turn to this now.

Section III-2: Calculating the Magnetic Field, Part 1: Biot-Savart Law

The Biot-Savart law tells us how a current produces a magnetic field. Say we have a current where at each point the differential element of length along it is $\overrightarrow{dl}$. Now we wish to find the magnetic field at a point a distance r from the point where $\overrightarrow{dl}$ is. Obviously r is a variable and will take a different value for each point on the current. You can guess what we're going to do here–we'll write down an equation for the magnetic field produced by this little piece of current and then integrate over the whole current. The magnetic field produced by a tiny piece of current of length $\overrightarrow{dl}$, at a distance r from it, is

$$\overrightarrow{dB} = \frac{\mu_0}{4\pi}\frac{I\overrightarrow{dl} \times \hat{r}}{r^2}$$

where $\hat{r}$ is a unit vector pointing from the tiny piece of current $\overrightarrow{dl}$ to the point at which one is finding the magnetic field. Notice of course that we have a new constant here: μ_0, which is called the vacuum permeability and in SI units is $4\pi * 10^{-7}$. Finally we must integrate over the entire current:

$$\int \overrightarrow{dB} = \int_{current} \frac{\mu_0}{4\pi}\frac{I\overrightarrow{dl} \times \hat{r}}{r^2} = \overrightarrow{B}$$

As always we must apply this to problems in order to understand how to use it.

Example 3-4: Magnetic Field due to a Circular Wire (Ch.22, Example 22-6, pg. 744 in Serway and Jewett, Ed. 4)

Find the magnetic field along the axis that goes through the center of a circular wire with a counterclockwise (from the top) current.

FINISH.

Example 3-5: Magnetic Field due to a straight lightning bolt (Ch.22, #24, pg. 759 in Serway and Jewett, Ed. 4)

We have a straight lightning bolt traveling upwards. Lightning is, of course, electric current; it therefore produces a magnetic field. The current is $I = 10^4 A$. Lightning bolts are not straight, as you know, but we will approximate it as such. Anyway, what is the magnetic field at a point 100 meters away? Assume the lightning bolt is infinitely long.

Answer: Let's say the bolt is along the y axis. Therefore $\overrightarrow{dl} = \hat{j}dy$. The vector $\hat{r}$ points from an arbitrary point on the bolt to the point 100 meters away. Let's call the distance along the y axis from this point 'y'; then, if we take the point at which we're finding the magnetic field to be a point 100 meters from the bolt along the x axis,

$$\hat{r} = \frac{100\hat{i} + y\hat{j}}{\sqrt{100^2 + y^2}}$$

The square of the distance from the arbitrary point on the current to the point at which we're finding the electric field, is

$$r^2 = 100^2 + y^2$$

We have

$$\overrightarrow{B} = \int_{bolt} \frac{\mu_0}{4\pi} \frac{I\overrightarrow{dl} \times \hat{r}}{r^2} = \int_{bolt} \frac{\mu_0}{4\pi} \frac{I\hat{j}dy \times \frac{100\hat{i}+y\hat{j}}{\sqrt{100^2+y^2}}}{100^2 + y^2}$$

$$= \int_{bolt} \frac{\mu_0}{4\pi} \frac{Idy\hat{j} \times (100\hat{i} + y\hat{j})}{(100^2 + y^2)^{3/2}}$$

$$= \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} \frac{100Idy\hat{j} \times \hat{i}}{(100^2 + y^2)^{3/2}}$$

since $\hat{j} \times \hat{j} = 0$. Finally,

$$\overrightarrow{B} = \frac{100\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{dy(-\hat{k})}{(100^2 + y^2)^{3/2}}$$

This is a tricky integral that requires more advanced calculus. We can just use integral tables or mathematica to get

$$\frac{1}{5000} = \int_{-\infty}^{\infty} \frac{dy}{(100^2 + y^2)^{3/2}}$$

Thus,

$$\overrightarrow{B} = \frac{-100\mu_0 I\hat{k}}{4\pi * 5000} = \frac{-\mu_0 I\hat{k}}{200\pi} = \frac{-4\pi * 10^{-7} * 10^4\hat{k}}{200\pi}$$

$$= \frac{-10^{-3}\hat{k}}{50} = -0.2 * \frac{10^{-3}}{10}\hat{k} = -2 * 10^{-1} * 10^{-4}\hat{k}T$$

$$= -2 * 10^{-5}\hat{k}T$$

We could ask an interesting question at this point: what if we had two straight wires, each with a current, next to each other? Each of them creates a magnetic field, and we know that magnetic fields induce forces on currents. So the left current creates a magnetic field that induces a force on the right current, and vice versa. Let's see how to calculate these forces.

Example 3-6: Magnetic Force between Currents

Say we have two straight currents directed upward, I_1 and I_2, at a distance d away from one another. What is the force on one current from the other? Take I_1 to the left of I_2.

Answer: First let's note that due to Newton's Third Law, the force on 1 from 2 is equal and opposite to the force on 2 from 1. So we can just pick one and the other force we'll know via Newton.

Let's find the force on I_1 from I_2. The current I_2 produces a magnetic field which then exerts a force on I_1. So the magnetic field from I_2 is

$$\overrightarrow{B} = 2 * \frac{-\mu_0 I_1 d}{4\pi}\hat{k} \int_0^L \frac{dy}{(y^2 + d^2)^{3/2}}$$

This comes directly from the previous example. The magnetic field is from a section of I_2 of length $2L$–this is why I multiplied by 2. I could just integrate from $-L$ to L instead of from 0 to L, but the integral would be more complex in that case. And since the current is straight and the same all along the wire, I can just multiply by two to get twice the magnetic field.

I want to make a subtle point here. This is the magnetic field from a relatively section of a long wire. But what happens to the current when it gets to the end of the wire? Well it'll have to be looped around eventually to the begining to the wire; but in doing this, we're changing the magnetic field because this section of wire produces a more complicated field. However, we will for the moment ignore this, and deal with it later.

The integral is

$$\int_0^L \frac{dy}{(y^2 + d^2)^{3/2}} = \frac{L}{d^2\sqrt{d^2 + L^2}}$$

Therefore the magnetic field is

$$\overrightarrow{B} = \frac{-\mu_0 I_1 d}{2\pi} \frac{L\hat{k}}{d^2\sqrt{d^2+L^2}} = \frac{-\mu_0 I_1}{2\pi} \frac{L\hat{k}}{d\sqrt{d^2+L^2}}$$

Notice that this is into the page. We have come across another and very useful form of the right-hand rule: if you place your thumb in the direction of the current, and the curl your four fingers in the direction of your palm, then the direction of your four fingers is the direction of the magnetic field. Since the current is upwards, the magnetic field curls around the current in circles, going into the page on the right of the current and out of the page on the left of the current.

Now we can use the equation for the force on a current from a magnetic field:

$$\overrightarrow{F_{onI_2}} = \int_{current} I_2 \overrightarrow{dl} \times \overrightarrow{B}$$

The vector $\overrightarrow{dl}$ is $\overrightarrow{dl} = \hat{\jmath} dy$; therefore

$$\overrightarrow{F_{onI_2}} = \int_{current} I_2 dy \hat{\jmath} \times \left(\frac{-\mu_0 I_1}{4\pi} \frac{L\hat{k}}{d\sqrt{d^2+L^2}}\right)$$

$$= -I_2 \frac{\mu_0 I_1}{2\pi} \frac{L}{d\sqrt{d^2+L^2}} \int_0^L dy \hat{\jmath} \times \hat{k}$$

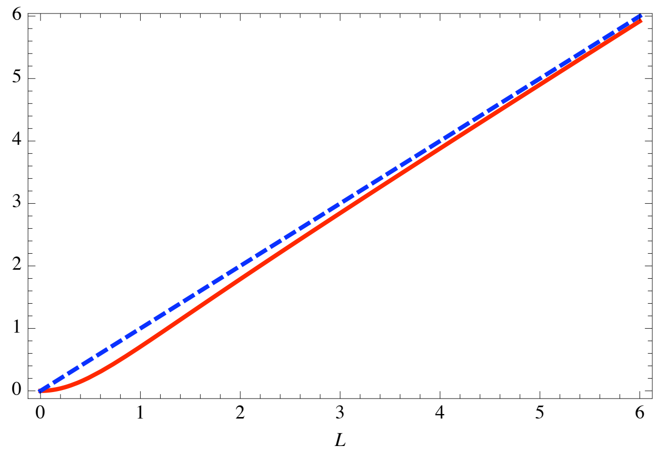
$$= -I_2 \frac{\mu_0 I_1}{2\pi} \frac{L\hat{\imath}}{d\sqrt{d^2+L^2}} \int_0^L dy$$

$$= -\frac{\mu_0 I_1 I_2}{2\pi} \frac{L^2\hat{\imath}}{d\sqrt{d^2+L^2}}$$

The force on I_2 from I_1 is to the left–in other words, it's an attractive force. Remember the subtle point I made before–that eventually the current will have to loop around, and this will change our equation for $\overrightarrow{B}$. So we will have to make the approximation that the wire is really, really long and that these trouble-making portions of the wire produce negligible magnetic fields. So the approximation is $L \gg d$ (in other words, the wires are close together compared to their length). In that case,

$$\frac{L^2}{\sqrt{d^2+L^2}} \rightarrow \frac{L^2}{\sqrt{L^2}} = L$$

We can make this more rigorous by graphing $\frac{L^2}{\sqrt{d^2+L^2}}$ and L:



The red solid graph is $\frac{L^2}{\sqrt{1+L^2}}$, and the blue dashed graph is L. I've set $d = 1$ for technical purposes–it's easier to graph that function than that with an arbitrary constant. But the graph would be qualitatively the same with any other value of the constant. You can see that the graphs begin to coalesce as L becomes large; or more precisely, as $L \gg 1$.

Thus, when $L \gg d$,

$$\overrightarrow{F_{onI_2}} = -\frac{\mu_0 L I_1 I_2}{2\pi d}\hat{\imath}$$

So when two wires have currents in the same direction, the forces between them are *attractive*; when two wires have currents in the opposite direction (as you might guess), the forces between them are *repulsive*. The magnetic field by one of these wires is, making this approximation,

$$\overrightarrow{B} = \frac{-\mu_0 I_1}{2\pi d}\hat{k}$$

since

$$\frac{L}{\sqrt{d^2 + L^2}} \to \frac{L}{\sqrt{L^2}} = 1$$

when $L \gg d$.

The Biot-Savart law is one way of calculating the magnetic field from a current. Remember when we were calculating the electric field from charges, there were two methods: a brute integration method, where we wrote down the electric field due to a tiny bit of charge and then integrated over all the charges; and then another method based on a mathematical theorem, and we discussed how it was only useful if we have symmetry in the charges. The same applies to calculating the magnetic field. The brute force method is the Biot-Savart law; the symmetry-based way is called Ampere's law, which we shall discuss now.

Section III-3: Calculating the Magnetic Field, Part 2: Ampere's Law.

Ampere's law is intimately related to a mathematical theorem called Stoke's Theorem, whose general form won't concern us here[5]. Ampere's law states that the line integral of the magnetic field around a closed loop is proportional to the current passing through the loop. It states

$$b * I = \oint \overrightarrow{B} \cdot \overrightarrow{ds}$$

where $\overrightarrow{ds}$ is a little piece of length tangent to a closed loop, and b is the proportionality constant[6], and I is the current passing through the loop. The word 'loop' is misleading, because the integral just has to be a closed contour–it doesn't have to be circular. Now let's apply this to a problem for which we already have the answer, so that we can determine the constant c.

Let's recap the equation for the magnetic field around a wire. From the previous section we get

$$\overrightarrow{B} = \frac{-\mu_0 I_1}{2\pi d}\hat{k}$$

for the magnetic field to the right of an infinite current that is oriented upwards. Let us use Ampere's law to calculate the magnetic field around a wire. First, we must decide what loop to use. Like Gauss' Law, we need to fulfill some conditions in order to make Ampere's law useful. If we're solving for $\overrightarrow{B}$, which we are, then we need to bring it out of the integral; if we can't do this, we'll be integrating over a function we don't know. We ran across this when we studied Gauss' Law, and we concluded that, while still valid, Gauss' Law is not useful unless the electric field is constant along the entirety of the surface. Analogously, Ampere's law is not useful unless the magnetic field $\overrightarrow{B}$ is constant all along the loop. And since we have a dot product, we would really prefer if $\overrightarrow{B} \cdot \overrightarrow{ds} = Bds$; in other words, if $\overrightarrow{B}$ is parallel to $\overrightarrow{ds}$. So what loop should we choose? Well we've already stated that the magnetic field around a straight current wraps around it in a circular fashion. So we should choose a circle as our loop, of radius d. This is in general, by the way, called an *Amperian Loop*.

Ok so we know that $\overrightarrow{B} \cdot \overrightarrow{ds} = Bds$, and that $\overrightarrow{B}$ is constant along the loop since it depends only on the distance one is from the wire and the magnitude of the current. Therefore,

$$\oint \overrightarrow{B} \cdot \overrightarrow{ds} = B \oint ds$$

What is the integral of ds around the whole loop? It is the circumference of the circle, which is

$$\oint ds = 2\pi d$$

Hence,

$$\oint \overrightarrow{B} \cdot \overrightarrow{ds} = B \oint ds = 2\pi d B = b * I_1$$

Thus,

$$B = \frac{bI_1}{2\pi d} = \frac{\mu_0 I_1}{2\pi d} \Rightarrow b = \mu_0$$

[5]Stoke's Theorem states that $\int_{area} (\overrightarrow{\nabla} \times \overrightarrow{B}) \cdot \overrightarrow{dA} = \oint \overrightarrow{B} \cdot \overrightarrow{dA}$

[6]This constant depends on the units used; here I derive it simply for SI units.

Amperes law is, in all its glory,

$$\oint \overrightarrow{B} \cdot \overrightarrow{ds} = \mu_0 I$$

Keep in mind the I is the current enclosed by the closed loop, and the integral is around this loop.

What if the wire were to have a radius? Well we could apply Ampere's law in the same way, with the caveat that the Amperian loop must be outside the wire. What if we wanted to find the magnetic field inside the wire? Let's see how to do that.

Magnetic Field Inside a Wire

First we must determine what Amperian loop to use. The same considerations as before apply, so we should choose a circle inside the wire. The current outside this circle contributes nothing to the integral $\oint \overrightarrow{B} \cdot \overrightarrow{ds}$. let's take the current of the whole wire to be I; what is the current inside the loop of radius r, if the radius of the wire is a? Once again let's use our secret weapon–density. The cross-section current density is

$$\sigma = \frac{I}{\pi a^2}$$

Which we will assume is constant. Then, the current in a cross-section of radius r is

$$I_{enclosed} = \pi r^2 \sigma = \frac{\pi r^2}{\pi a^2} I = \frac{r^2}{a^2} I$$

Writing down Ampere's law,

$$\oint \overrightarrow{B} \cdot \overrightarrow{ds} = \mu_0 I_{enclosed}$$

$$= \mu_0 \frac{r^2}{a^2} I$$

And now, we must integrate the magnetic field around this circle of radius r:

$$\oint \overrightarrow{B} \cdot \overrightarrow{ds} = \oint B ds = B \oint ds = B * 2\pi r$$

Therefore,

$$\mu_0 \frac{r^2}{a^2} I = B * 2\pi r \Rightarrow B = \mu_0 \frac{r^2}{2\pi a^2 r} I = \mu_0 I \frac{r}{2\pi a^2} = B$$

Which is directed around the enclosed current in a circular fashion. The unit vector in this direction is $\hat{\phi}$. So we can write

$$\overrightarrow{B_{inside}} = \mu_0 I \frac{r\hat{\phi}}{2\pi a^2}$$

We can check our work by looking at what this equation gives for the magnetic field on the surface of the wire, which should be the same as that we'd get from the equation for the magnetic field outside the wire. In other words, at the boundary they should give the same result. So at $r = a$,

$$B_{outside}(a) = \frac{\mu_0 I}{2\pi a}$$

and

$$B_{inside}(a) = \mu_0 I \frac{a}{2\pi a^2} = \frac{\mu_0 I}{2\pi a} = B_{outside}(a)$$

They give the same result, which is good!

I talked before about how the electric field obeys the Superposition Principle–which means that if we have several electric fields in a region of space, then the total electric field is just the sum of the individual electric fields. The same principle applies to magnetic fields. We shall now turn to an example of this.

Example 3-5: Magnetic Field at the Center of 4 straight wires

We have 4 straight wires at the corners of a square; the two left currents are going into the page, and the two right currents are coming out of the page. The sides of the square are of length 0.2 meter. Each of the currents is $I = 5A$. What is the magnetic field at the center of the square?

Answer: We will call the current at the top left I_a. Then the current at the top right is I_c. Directly below this is I_d, and the current at the bottom left is I_b. Let's find the magnetic field at the center from the a current (top left). The distance from the center to I_a is $r = \frac{1}{2}\sqrt{0.2^2 + 0.2^2} = 0.1\sqrt{2} = \frac{\sqrt{2}}{10}$. Using our equation we've already derived for the magnetic field outside of a (very long, straight) wire,

$$B_a = \frac{\mu_0 I_a}{2\pi r} = \frac{4\pi * 10^{-7} * 5}{2\pi \frac{\sqrt{2}}{10}} = 7.1 * 10^{-6} T$$

It is directed tangent to a circle around I_a (counterclockwise direction). We need to pick axes, so let's choose the diagonal line from b to d to be the y axis, the positive end directed toward d (top right). Thus,

$$\overrightarrow{B_a} = -B_a \hat{j}$$

We now do the same thing for I_b. We get

$$B_b = \frac{\mu_0 I_b}{2\pi r} = \frac{4\pi * 10^{-7} * 5}{2\pi \frac{\sqrt{2}}{10}} = 7.1 * 10^{-6} T$$

The magnitude of the magnetic fields are all the same, actually, since they are all the same distance from the center of the square. The direction of B_b is down and to the right, along the line connecting a and d. The positive x axis is along this line, toward a (top left). Hence

$$\overrightarrow{B_b} = -B_b \hat{i}$$

$\overrightarrow{B_c}$ is in the same direction:

$$\overrightarrow{B_c} = -B_c\hat{\imath}$$

and the direction of $\overrightarrow{B_d}$ is the same as $\overrightarrow{B_a}$:

$$\overrightarrow{B_d} = -B_d\hat{\jmath}$$

The total field, then, is

$$\overrightarrow{B} = \overrightarrow{B_a} + \overrightarrow{B_b} + \overrightarrow{B_c} + \overrightarrow{B_d}$$

$$= 7.1 * 10^{-6}(-2\hat{\imath} - 2\hat{\jmath})$$

$$= -14.2 * 10^{-6}(\hat{\imath} + \hat{\jmath})$$

which is directly downwards.

Magnetic Field Inside a Soleneid

FINISH

Section III-4: Faraday's Law and Lenz's Law