



# Evidence for the strongest version of the 4d $a$ -theorem via $a$ -maximization along RG flows

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## Abstract

In earlier work, we (KI and BW) gave a two line “almost proof” (for supersymmetric RG flows) of the weakest form of the conjectured 4d  $a$ -theorem, that  $a_{\text{IR}} < a_{\text{UV}}$ , using our result that the exact superconformal R-symmetry of 4d SCFTs maximizes  $a = 3 \text{Tr} R^3 - \text{Tr} R$ . The proof was incomplete because of two identified loopholes: theories with accidental symmetries, and the fact that it is only a local maximum of  $a$ . Here we discuss and extend a proposal of Kutasov (which helps close the latter loophole) in which  $a$ -maximization is generalized away from the endpoints of the RG flow, with Lagrange multipliers that are conjectured to be identified with the running coupling constants.  $a$ -maximization then yields a monotonically decreasing “ $a$ -function” along the RG flow to the IR. As we discuss, this proposal in fact suggests the strongest version of the  $a$ -theorem: that 4d RG flows are gradient flows of an  $a$ -function, with positive-definite metric. In the perturbative limit, the RG flow metric thus obtained is shown to agree precisely with that found by very different computations by Osborn and collaborators. As examples, we discuss a new class of 4d SCFTs, along with their dual descriptions and IR phases, obtained from SQCD by coupling some of the flavors to added singlets. © 2004 Published by Elsevier B.V.

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### 1. Introduction

There is an intuition that RG flows are a one-way process, with information about the UV modes lost as one coarse-grains. More precisely (since even an RG fixed point conformal field theory (CFT) has UV modes going above the cutoff), the intuition is that non-trivial RG flows should always decrease the number of *massless* degrees of freedom: relevant deformations will lift some massless degrees of freedom, and RG flow to the IR coarse-grains away these lifted modes, with no new modes becoming massless.

Let us distinguish several possibilities:

- (1) One can define a quantity,  $c$ , that properly counts the massless degrees of freedom of a CFT (e.g.,  $c > 0$  for all unitarity CFTs, and  $c = c_1 + c_2$  for two decoupled CFTs) such that the endpoints of all (unitarity) RG flows satisfy  $c_{\text{IR}} < c_{\text{UV}}$ .
- (2) A stronger claim is that  $c$  can be extended to a monotonically decreasing “ $c$ -function”  $c(g(t))$  along the entire RG flow to the IR:

$$\dot{c}(g) = -\beta^I(g) \frac{\partial c}{\partial g^I} \leq 0, \tag{1.1}$$

with  $\dot{c} = 0$  iff the theory is conformal. Here  $\dot{\phantom{x}} = \frac{d}{dt}$ , with  $t = -\log \mu$  the RG “time”, increasing towards the IR, and  $\dot{g}^I(t) = -\beta^I(g)$ , with  $g^I(t)$  the running couplings.

- (3) The strongest possibility is that RG flow is gradient flow of the  $c$ -function,

$$\beta^I(g) = G^{IJ}(g) \frac{\partial c(g)}{\partial g^J} \quad \text{and} \quad \frac{\partial c(g)}{\partial g^I} = G_{IJ}(g) \beta^J(g) \tag{1.2}$$

(here  $G^{IJ} \equiv (G_{IJ})^{-1}$ ) with  $G^{IJ}(g) > 0$  a positive-definite metric (all eigenvalues positive) on the space of coupling constants. Eq. (1.2) then implies  $\dot{c} \leq 0$ ,

$$\dot{c}(g(t)) = -\beta^I \frac{\partial c}{\partial g^I} = -G_{IJ} \beta^I \beta^J \leq 0, \tag{1.3}$$

with  $\dot{c} = 0$  iff the theory is conformal.

The possibility that RG flow is gradient flow with positive-definite metric was proposed (and verified to 3-loop order in 4d multi-component  $\lambda\phi^4$  theory) by Wallace and Zia [1]. In 2d, Zamolodchikov [2] defined a function  $c(g)$ , equal to the central charge of the Virasoro algebra for CFTs, which he proved satisfies (1.3) with  $G_{IJ}(g) > 0$  (for unitary theories).  $G_{IJ}$  is determined from the two-point functions  $\langle \mathcal{O}_I(x) \mathcal{O}_J(y) \rangle$  of the operators that  $g^I$  and  $g^J$  source. This proves version (2) above in 2d, and suggests the strongest version (3) (if the dot product with  $\beta^I$  could be eliminated from both sides of (1.3)). It was also demonstrated [2] that the strongest version (1.2) is indeed true, at least in conformal perturbation theory, in the vicinity of any 2d RG fixed point.

The apparent generality of these intuitions suggest that analogous statements should apply for RG flows in any spacetime dimension. Cardy [3] conjectured that an appropriate quantity<sup>2</sup> for counting the number of massless degrees of freedom of 4d CFTs is the

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<sup>2</sup> This candidate does not have an analog for odd spacetime dimensions, unfortunately.

conformal anomaly  $a$  on a curved spacetime:<sup>3</sup>

$$a \sim \int_{S^4} \langle T_{\mu}^{\mu} \rangle. \tag{1.4}$$

The weakest version of the 4d  $a$ -theorem conjecture is then that the conformal anomaly  $a$  satisfies  $a > 0$  for every (unitary) 4d RG fixed point, and  $a_{UV} > a_{IR}$  for the endpoints of all (unitary) 4d RG flows. Every known computable example (both non-supersymmetric and using SUSY exact results) is strikingly, and often highly non-trivially, compatible with this conjecture. It would be very interesting and powerful if this  $a$ -theorem conjecture is indeed a completely general property of all (unitary) 4d RG flows. At present, however, there is not yet a general, and generally accepted, proof of the conjectured 4d  $a$ -theorem. See, e.g., [4,6–8,30] for further discussion of the  $a$ -theorem conjecture.

Given the striking successes of the weaker version of the 4d  $a$ -theorem, it is natural to consider the 4d analogs of the stronger possibilities (2) and (3) above: perhaps  $a$  can be extended to a monotonically decreasing “ $a$ -function”  $a(g^I)$  along the entire RG flow, and perhaps the beta functions are gradients of this  $a$ -function, with positive-definite metric, as in (1.2). Osborn and collaborators [9,10] investigated this in perturbation theory for 4d QFTs (by considering renormalization with spatially dependent couplings) and indeed found a candidate  $a$ -function  $a(g)$  which satisfies a relation similar to (1.2):

$$\frac{\partial a(g)}{\partial g^I} = (G_{IJ} + \partial_I W_J - \partial_J W_I) \beta^J, \tag{1.5}$$

where  $a(g) = a_{\text{conf}}(g) + W_I(g) \beta^I(g)$ .

The candidate  $a$ -function  $a(g_I)$  coincides with the conformal anomaly<sup>4</sup>  $a_{\text{conf}}(g)$  at the endpoints of the RG flow. The possible term  $\partial_{[I} W_{J]}$  in (1.5), a possible difference from gradient flow (1.2), was found to vanish in every example, to all orders checked. Also, it is not manifest in this approach that  $G_{IJ}(g) > 0$  ( $G_{IJ}(g)$  is defined via beta functions  $\beta_{\mu\nu} \sim G_{IJ}(g) \partial_{\mu} g^I \partial_{\nu} g^J$  upon taking the couplings to be spatially dependent), but  $G_{IJ} > 0$  was verified to be true in every example, to all orders checked [9,10].

Here we will explore these ideas in supersymmetric theories, where it is possible to obtain exact results. Supersymmetry relates the stress tensor to a particular R-symmetry, which we will refer to as the superconformal R-symmetry (even when the theory is not conformal). The matter chiral superfields  $Q_i$  have superconformal  $U(1)_R$  charge

$$R(Q_i) = \frac{2}{3} \Delta(Q_i) = \frac{2}{3} \left( 1 + \frac{1}{2} \gamma_i \right), \tag{1.6}$$

<sup>3</sup> A general curved 4d spacetime background has two independent anomaly coefficients,  $\langle T_{\mu}^{\mu} \rangle = a(\text{Euler}) + c(\text{Weyl})^2$ , but  $\langle \text{Weyl} \rangle^2 = 0$  vanishes on a conformally flat background such as  $S^4$ . This is just as well, since its coefficient  $c$  (so named because it also appears in  $\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle$  in flat space) is known to not have definite monotonicity under RG flow [4,5]. So we will not discuss  $c$  further, and will replace “ $c$ ” with “ $a$ ” in the conjectured 4d analogs of the above statements.

<sup>4</sup> To avoid repeatedly writing  $3/32$ , we rescale  $a$  relative to other references,  $a_{\text{here}} = (32/3)a_{\text{usual}}$ , and write our  $a$ -function as  $a_{\text{here}}(g) = (32/3)\tilde{a}_{\text{Osborn}}(g)$ . To avoid a factor of  $4/3$  which would then show up in (1.5), we also rescale our  $G_{IJ}$  relative to [9,10]:  $G_{IJ}^{\text{here}} = \frac{4}{3}G_{IJ}^{\text{there}}$ .

related to  $Q_i$ 's anomalous dimension. The exact beta functions are related to the violations of the superconformal R-symmetry. For example, the NSVZ exact beta function [11] for the gauge coupling of gauge group  $G$ , with matter fields  $Q_i$  in representations  $r_i$ , is

$$\beta_{\text{NSVZ}}(g) = \left( \frac{3g^3/16\pi^2}{1 - \frac{g^2 T(G)}{8\pi^2}} \right) \hat{\beta}_G(R),$$

$$\hat{\beta}_G(R) \equiv - \left[ T(G) + \sum_i T(r_i)(R_i - 1) \right], \quad (1.7)$$

with  $T(G)$  the quadratic Casimir of the adjoint and  $T(r_i)$  that of representation  $r_i$ . Likewise, the exact beta function for the coupling  $h$  of a superpotential term  $W = h \prod_i Q_i^{n(W)_i}$  can be written as (using  $\Delta(h) = 3 - \Delta(W)$  to write  $h \sim \mu^{(3/2)(R(W)-2)}$ ):

$$\beta_W(h) \equiv -\dot{h} = \frac{3}{2} h \hat{\beta}_W(R),$$

$$\hat{\beta}_W(R) \equiv R(W) - 2 = \sum_i n(W)_i R(Q_i) - 2. \quad (1.8)$$

$\hat{\beta}_G(R)$  and  $\hat{\beta}_W(R)$  are simply linear combinations of the R-charges, independent of the coupling constants. They are defined to have the same sign as the full beta functions, and represent the violation of the R-symmetry by the interactions:  $\hat{\beta}_G(R)$  is the coefficient  $\text{Tr} R G^2$  of the  $U(1)_R$  current's ABJ anomaly, and  $\hat{\beta}_W(R)$  gives the violation of the R-symmetry by the superpotential.

At the superconformal endpoints of RG flow, the superconformal R-current evolves to a conserved  $U(1)_{R_*} \subset SU(2, 2|1)$ , as the interactions flow to a zero of their beta functions. The superconformal R-charges of the fields determine the exact operator dimensions of gauge-invariant chiral primary operators via  $\Delta(\mathcal{O}) = \frac{3}{2} R_*(\mathcal{O})$  (computable in terms of  $R_*(Q_i)$  since R-charges are simply additive). Moreover, as shown in [5,12], the 't Hooft anomalies of  $U(1)_{R_*}$  determine the exact central charge of the SCFT:

$$a_{\text{SCFT}} = 3 \text{Tr} R_*^3 - \text{Tr} R_*. \quad (1.9)$$

It was shown in [13] how to uniquely pick out the special  $U(1)_{R_*} \subset SU(2, 2|1)$ , from among all possible conserved R-symmetries (satisfying  $\hat{\beta}(R) = 0$ ): it is that which *maximizes* the combination of 't Hooft anomalies

$$a_{\text{trial}}(R) = 3 \text{Tr} R^3 - \text{Tr} R. \quad (1.10)$$

At the unique local maximum, the function (1.10) coincides with the conformal anomaly  $a_{\text{SCFT}}$  (1.9), hence the name “*a*-maximization”. E.g., for a free chiral superfield  $a_{\text{trial}}(R) = 3(R - 1)^3 - (R - 1)$ , as plotted in Fig. 1, with local maximum at point (A). The same qualitative picture of Fig. 1 applies for interacting theories. The function  $a_{\text{trial}}(R)$ , and its local maximum  $R_*$  and value  $a_*$ , can be exactly computed, even for strongly interacting RG fixed points, via the power of 't Hooft anomaly matching. See, e.g., [14–20] for some extensions and applications of *a*-maximization.

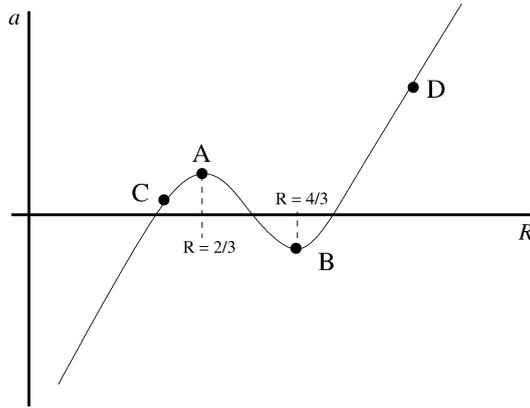


Fig. 1. The trial central charge  $a_{\text{trial}}(R)$  (with  $R_*$  values indicated for free field case).

$a$ -maximization has several immediate general corollaries. E.g., it implies [13] in complete generality, for any 4d  $\mathcal{N} = 1$  SCFT,<sup>5</sup> that the superconformal  $R_*$  charges, and hence the exact scaling dimension of chiral primary operators and the central charges  $a_*$  and  $c_*$ , are necessarily very special numbers: *quadratic irrationals*, of the general form

$$R_*, a_* \in \left\{ \frac{n + \sqrt{m}}{p} \mid n \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 0}, p \in \mathbb{Z}_{\neq 0} \right\}. \tag{1.11}$$

Quadratic irrational numbers are a measure zero subset of the reals,<sup>6</sup> with special properties (e.g., precisely they have continued fraction form that is periodic). The result (1.11) implies that the superconformal  $U(1)_R$  charges and central charge  $a_*$  cannot vary continuously; therefore, for any SCFT, they cannot depend on any continuous moduli.

As also discussed in [13],  $a$ -maximization gives a two line “almost proof” of the  $a$ -theorem for supersymmetric RG flows: relevant deformations will break some of the flavor symmetries, placing additional constraints on the IR  $R$ -symmetry as compared with the UV one,  $\mathcal{F}_{\text{IR}} \subset \mathcal{F}_{\text{UV}}$ , and maximizing a function over a subspace leads to smaller maximal value, hence  $a_{\text{IR}} < a_{\text{UV}}$ —**QED!** However, as also pointed out in [13], each of these two lines has possible exceptions. First of all, the IR SCFT can have additional accidental symmetries not present in the UV theory, in which case  $\mathcal{F}_{\text{IR}} \not\subset \mathcal{F}_{\text{UV}}$ ; the result of [13] implies that  $a_{\text{trial}}$  should be maximized over *all* flavor symmetries, including all accidental ones, so it is crucial that accidental symmetries be properly included. The two-line proof needs to be supplemented with additional physical information to apply to cases with accidental

<sup>5</sup> Theories with accidental symmetries could be exceptions to these general statements, though all known such examples, for example, those associated with singular points of  $\mathcal{N} = 2$  Seiberg–Witten curves [21,22], still satisfy the above general statements.

<sup>6</sup> Rational numbers are a subset of the quadratic irrationals. SCFTs with string dual descriptions are typically limited to this subset, though recently string geometry examples were obtained for which the  $R$ -charges are not rational [23], though they are indeed quadratic irrational, compatible with (1.11) (and the general prediction from (1.11) is that any (generally singular)  $H_5$ , such that  $AdS_5 \times H^5$  is dual to a  $\mathcal{N} = 1$  SCFT, must have quadratic irrational volumes). There are many SUSY gauge theory examples with  $R$ -charges that are quadratic irrational but not rational.

symmetries. The caveat for the second line of the proof is the fact that the maximum is only a local one. E.g., in Fig. 1, suppose that the UV theory is at local maximum (A): perturbing away from there will reduce  $a$ , but we need to rule out the possibility that the deformation might eventually drive the value of  $a$  up to a point such as (D) in the IR, with  $a_{(D)} > a_{(A)}$ , violating  $a_{\text{IR}} < a_{\text{UV}}$ .

In [19] Kutasov made a very interesting proposal, which helps close the second loophole by extending  $a$ -maximization away from the RG fixed points. Assuming that  $\mathcal{F}_{\text{IR}} \subset \mathcal{F}_{\text{UV}}$  (in Section 4, we will discuss an extension for certain accidental symmetries), the idea is to implement the additional constraints associated with  $\mathcal{F}_{\text{IR}} \subset \mathcal{F}_{\text{UV}}$  via Lagrange multipliers. We will write this generally as

$$a(R, \lambda_I) = 3 \text{Tr} R^3 - \text{Tr} R + \sum_I \lambda_I \hat{\beta}^I(R), \tag{1.12}$$

with  $\hat{\beta}^I(R)$  the linear constraints on the R-charges mentioned above, and  $\hat{\beta}^I = 0$  at the IR SCFT. Extremizing (1.12) w.r.t.  $R$ , holding the Lagrange multipliers  $\lambda_I$  fixed, yields  $R(\lambda_I)$ , and plugging back into (1.12) gives

$$a(\lambda_I) \equiv a(R(\lambda_I), \lambda_I) \quad \text{such that} \quad \frac{\partial a(\lambda)}{\partial \lambda_I} = \hat{\beta}^I(R(\lambda)), \tag{1.13}$$

using the fact that  $R(\lambda_I)$  solves  $\partial a / \partial R = 0$ . The observation now is that the function  $a(\lambda_I)$  interpolates between  $a_{\text{UV}}$  and  $a_{\text{IR}}$ , and (1.13) suggests that  $a(\lambda_I)$  is monotonic, using the physical intuition that beta functions are expected to have a definite sign along the entire RG flow: once a coupling hits a zero of the beta function, it just stops running (e.g., it does not overshoot a zero).

It was conjectured in [19] that the Lagrange multipliers  $\lambda_I$  are to be identified with the running coupling constants  $g_I^2$  in some scheme. The extremizing solution  $R(\lambda)$  of (1.12) is interpreted as the RG flow of the superconformal R-charges, and  $a(\lambda)$  (1.13) is interpreted as a monotonically decreasing  $a$ -function along the RG flow to the IR. For relevant interactions,  $\dot{\lambda}_I > 0$ , so (1.13) with  $\hat{\beta}^I < 0$  implies that  $\dot{a} < 0$ . Likewise, for irrelevant interactions,  $\dot{\lambda}_I < 0$  and (1.13), with  $\hat{\beta}^I > 0$ , again leads to  $\dot{a} < 0$ .

We will expand upon and further check the interpretation of (1.13) as defining a monotonically decreasing  $a$ -function along the RG flow. Our main point is that this proposal suggests the strongest version (3) of the  $a$ -theorem conjecture: that the exact RG flows are indeed gradient flows of the  $a$ -function (1.13), as in (1.2), with metric on the space of coupling constants given by

$$G_{IJ}(g) = f_J^K(g) \frac{\partial \lambda_K(g)}{\partial g^I}, \quad \text{where} \quad \hat{\beta}^K(R) = f_J^K(g) \beta^J(g). \tag{1.14}$$

A sufficient condition for this metric to be positive-definite is that the  $f_J^K(g)$  are positive, e.g.,  $g$  does not flow beyond the apparent pole in the denominator of  $\beta_{\text{NSVZ}}(g)$  in (1.7), and the relation (scheme change) between the  $\lambda_K$  and the  $g^J$  are monotonic.

In Sections 2.1 and 2.2, we review the RG flow of the R-symmetry in the stress tensor supermultiplet, and the  $a$ -maximization method [13] for determining the superconformal R-charge at RG fixed points, as well as the extension of [15] for cases with accidental

symmetries. In Section 2.3 we review Kutasov’s proposal for  $a$ -maximization with Lagrange multipliers [19], first for the case of gauge interactions only. In Section 2.4, we use (1.6) and the R-charges  $R(\lambda)$  obtained by extremizing (1.12) to compute the anomalous dimensions

$$\gamma_i(\lambda) = 3R(\lambda_I)Q_i - 2 = 1 - \sqrt{1 + \frac{\lambda C(r_i)}{|G|}}, \quad (1.15)$$

comparing with perturbative computations of  $\gamma_i(g)$ . This provides both a non-trivial check of  $a$ -maximization and its extensions, and also a means to determine the relation,  $\lambda_I(g)$ , of  $\lambda_I$  to the coupling constant  $g$  in a given scheme, e.g., that of the NSVZ beta function. In Section 2.4, we will check (1.15) to three loops, comparing with the computations of [24] (the one-loop check was already verified in [13], and the two-loop check was discussed and verified in [19]). In Section 2.5 we will discuss  $a$ -maximization along the RG flow for superpotential interactions, obtaining the one-loop (scheme-independent part) relation between the Lagrange multiplier and the superpotential Yukawa coupling. In Section 2.6, after reviewing  $a$ -maximization with Lagrange multipliers for  $SU(N_c)$  SQCD (which was discussed in [19]), we apply this method to its magnetic  $SU(N_f - N_c)$  Seiberg [25] dual. Analyzing the magnetic theory, we point out that the  $R(\lambda_I)$  which extremizes (1.12) is a solution of a quadratic equation and that, in the RG flow of  $R(\lambda_I)$  to the IR,  $\lambda$  can flow from increasing on one branch to decreasing  $\lambda$  on the other branch.

In Section 3, we point out that (1.13), with the Lagrange multipliers interpreted as the running coupling constants, demonstrates that RG flow is indeed gradient flow, with metric (1.14). We compute this metric for gauge (this case already appears in [19]) and Yukawa interactions. In the perturbative limit, we compare these metrics with those computed by Freedman and Osborn [10], and find perfect agreement for the leading, scheme-independent coefficients. In other words, the  $a$ -function (1.13), computed by  $a$ -maximization with Lagrange multipliers, agrees with that proposed and computed perturbatively in [9,10] (at least to leading perturbative order).

In Section 4, we propose an extension of the Lagrange multiplier method of [19] to apply for RG flows with accidental symmetries associated with gauge-invariant operators hitting their unitarity bound and becoming free. This extension leads to a monotonically decreasing  $a$ -function for such RG flows, showing in particular that  $a$ -maximization indeed ensures that  $a_{\text{IR}} < a_{\text{UV}}$  for these RG flows too. We also comment in Section 4 on the challenge of finding a natural, monotonically decreasing  $a$ -function for RG flows associated with the Higgs mechanism: there are contributions (the eaten matter fields) whose effect is to reduce  $a$  in the IR, as well as contributions (the uneaten matter fields) whose effect is to increase  $a$  in the IR, and the challenge is to find an interpolating function which makes it manifest that the former always outweighs the latter.

Finally, in Section 5, we illustrate some of these ideas with a new class of 4d SCFTs, which are simply a deformation of SQCD, where some general fraction of the flavors are coupled to added singlets. For two special cases, no flavors coupled to singlets and all flavors coupled to singlets, these theories coincide with SQCD and its magnetic Seiberg dual [25]. Our generalizations interpolate between these two special cases. As we discuss, these new SCFTs, which we generally refer to as SSQCD (for singlets + SQCD) have a dual description, obtained as a deformation of Seiberg duality [25]. Though these new SCFTs are

simply related to SQCD, they could not have been analyzed before the introduction of the  $a$ -maximization method [13]. In ordinary SQCD, mesons hitting their unitarity bound and decoupling coincides with the entire magnetic dual being IR-free [25]. In our SSQCD generalizations, on the other hand, mesons can decouple with the rest of the SCFT remaining interacting. In the magnetic dual description, the superpotential term involving that meson becomes irrelevant, but other superpotential terms and the dual gauge coupling can remain interacting.

**Note added:** The results of our Section 2.4 (including, in particular, the scheme change with  $\partial \ln F_i / \partial g \sim C(r_i)^2 g^3 + O(g^5)$ ) were subsequently independently obtained in [31].

## 2. The superconformal R-symmetry, $a$ -maximization, and Lagrange multipliers

### 2.1. The flowing R-charges

$\mathcal{N} = 1$  supersymmetry puts the stress-energy tensor  $T_{\mu\nu}$  into a current supermultiplet,  $T_{\alpha\dot{\alpha}}(x, \theta, \bar{\theta})$ , whose first component is a  $U(1)_R$  current (and other components include the supercharge currents). For superconformal theories, this R-current is conserved, and is the  $U(1)_R \subset SU(2, 2|1)$  in the superconformal algebra. For non-conformal theories, supersymmetry relates the dilatation current divergence  $T_{\mu}^{\mu}$  to that of this R-current, via

$$\bar{\nabla}^{\dot{\alpha}} T_{\alpha\dot{\alpha}} = \nabla_{\alpha} L_T, \quad (2.1)$$

with  $L_T$  the chiral superfield trace anomaly, e.g.,

$$L_T = -\frac{\hat{\beta}(R)}{64\pi^2} (W^{\alpha} W_{\alpha})_{\text{gauge}} - \frac{\tau_{IJ}}{96\pi^2} (W^I W^{\alpha J})_{\text{flavor}} + \frac{c}{24\pi^2} \mathcal{W}^2 - \frac{a}{24\pi^2} \mathcal{E}, \quad (2.2)$$

with the first term the gauge beta function, the second the contribution associated with background fields coupled to flavor currents, and the last two terms the contributions associated with a background metric and gauge field coupled to the superconformal R-current. See [12] for a discussion of the latter terms. We will refer to the  $U(1)_R$  current in  $T_{\alpha\dot{\alpha}}$  as the superconformal R-current, whether or not the theory is conformal, keeping in mind that in the non-conformal case this R-symmetry is violated.

Whether or not the theory is conformal, supersymmetry relates the superconformal R-charges to the scaling dimensions of the fields:

$$R(Q_i) = \frac{2}{3} \Delta(Q_i) = \frac{2}{3} \left( 1 + \frac{1}{2} \gamma_i \right), \quad (2.3)$$

with  $\gamma_i$  the anomalous dimension of field  $Q_i$ . Consider a RG flow, e.g., with asymptotically free gauge fields and matter in the UV, to an interacting RG fixed point in the IR. Along this RG flow we can write the superconformal R-current as

$$R^{\mu} = R_{\text{cons}}^{\mu} + X_{\text{flow}}^{\mu}, \quad (2.4)$$

with  $R_{\text{cons}}^{\mu}$  a conserved current, and  $X_{\text{flow}}^{\mu}$  not conserved. The current  $X_{\text{flow}}^{\mu}$  gets an anomalous dimension, and becomes irrelevant, flowing to zero in the IR, so the R-symmetry in the stress tensor supermultiplet flows as  $R \rightarrow R_{\text{cons}}$  in the IR.

As an example, consider SQCD:  $SU(N_c)$  gauge theory with  $N_f$  fundamental flavors  $Q_f$  and  $\tilde{Q}_{\tilde{f}}$  (taking  $N_f$  in the superconformal window [25]  $\frac{3}{2}N_c < N_f < 3N_c$ ). There is a unique conserved R-symmetry that commutes with all the flavor symmetries and charge conjugation,  $R_{\text{cons}}(Q_f) = R_{\text{cons}}(\tilde{Q}_{\tilde{f}}) = 1 - N_c/N_f$ . This R-symmetry is conserved along the entire RG flow, but it is only the R-symmetry in the stress tensor supermultiplet at the IR SCFT fixed point. Along the RG flow, the R-symmetry in the stress tensor supermultiplet is the sum of terms (2.4), with  $X_{\text{flow}}^\mu \rightarrow 0$  in the IR (see, e.g., [26]). The superconformal R-charges evolve along the RG flow, from  $R_{\text{UV}}(Q_f) = R_{\text{UV}}(\tilde{Q}_{\tilde{f}}) = R_{\text{free}} = 2/3$  (asymptotic freedom), to those of the IR SCFT,  $R_{\text{IR}}(Q_f) = R_{\text{IR}}(\tilde{Q}_{\tilde{f}}) = R_{\text{cons}} = 1 - N_c/N_f$ .

Using the result of [5,12], the conformal anomaly at the UV and IR endpoints of the RG flow are given by  $a_{\text{UV}} = 3 \text{Tr} R_{\text{UV}}^3 - \text{Tr} R_{\text{UV}}$  and  $a_{\text{IR}} = 3 \text{Tr} R_{\text{IR}}^3 - \text{Tr} R_{\text{IR}}$ . 't Hooft anomaly matching does not equate  $a_{\text{UV}}$  and  $a_{\text{IR}}$ , because the R-charges themselves are different in the UV and the IR, with the R-current in  $T_{\alpha\dot{\alpha}}$  not even conserved along the RG flow. E.g., for SQCD (with  $N_f$  in the superconformal window)

$$a_{\text{UV}} = 2(N_c^2 - 1) + 2N_c N_f \left( 3 \left( -\frac{1}{3} \right)^3 + \frac{1}{3} \right) = 2(N_c^2 - 1) + \frac{2}{9}(2N_c N_f), \quad (2.5)$$

the free-field contribution expected by asymptotic freedom ( $a_V^{\text{free}} = 2$  and  $a_Q^{\text{free}} = 2/9$  in our normalizations). At the IR endpoint of the RG flow, the conformal anomaly is

$$\begin{aligned} a_{\text{IR}} &= 2(N_c^2 - 1) + 2N_c N_f \left( 3 \left( -\frac{N_c}{N_f} \right)^3 + \frac{N_c}{N_f} \right) \\ &= 4N_c^2 - 2 - \frac{6N_c^4}{N_f^2} \equiv a_{\text{SQCD}}(N_c, N_f), \end{aligned} \quad (2.6)$$

where we used  $R_{\text{IR}} = R_{\text{cons}}$ . 't Hooft anomaly matching is used to evaluate these  $R_{\text{IR}}$  't Hooft anomalies using the weakly coupled degrees of freedom of the UV endpoint of the flow (since  $R_{\text{IR}}$ , unlike the R-symmetry in  $T_{\alpha\dot{\alpha}}$ , is here conserved along the entire RG flow). As predicted by the  $a$ -theorem conjecture,  $a_{\text{UV}} > a_{\text{IR}}$ . In the UV, the matter fields are at point (A) in Fig. 1, and in the IR they are at a lower point such as (C) in Fig. 1.

It is non-trivial that  $a_{\text{SCFT}} > 0$ , even at strongly coupled RG fixed points, as desired for a count of massless d.o.f. E.g., expression (2.6) satisfies  $a_{\text{SQCD}}(N_c, N_f) > a_{\text{SQCD}}(N_c, N_f - 1)$ , as expected by the  $a$ -theorem conjecture, since we can RG flow from the theory with  $N_f$  flavors in the UV to one with  $N_f - 1$  flavors in the IR by giving a mass to a flavor. If continued to sufficiently small  $N_f$ , (2.6) would give negative  $a$ . But  $N_f$  never gets sufficiently small to violate  $a > 0$ , because for  $N_f \leq \frac{3}{2}N_c$  something different happens, as can be seen from the fact that the mesons  $M = Q\tilde{Q}$  hit the unitarity bound  $R(M) \geq 2/3$ ; in fact, the entire magnetic dual then becomes free [25].

### 2.2. $a$ -maximization at RG fixed points

Let us briefly recall the argument of [13], that the exact superconformal R-symmetry maximizes  $a_{\text{trial}} = 3 \text{Tr} R_t^3 - \text{Tr} R_t$ . We write the general trial  $U(1)_R$  symmetry as  $R_t =$

$R_0 + \sum_I s_I F_I$ , where  $R_0$  is an arbitrary R-symmetry, the  $F_I$  are non-R flavor symmetries, and  $s_I$  are real coefficients. The superconformal R-symmetry  $U(1)_{R^*} \subset SU(2, 2|1)$  corresponds to some particular values of the  $s_{*I}$ , that we would like to determine. The result of [13] is that they are uniquely determined by the 't Hooft anomaly relations

$$9 \text{Tr} R_*^2 F_I = \text{Tr} F_I \quad \text{for all } F_I, \tag{2.7}$$

$$\text{Tr} R_* F_I F_J = -\frac{1}{3} \tau_{IJ} < 0. \tag{2.8}$$

Relation (2.7) is equivalent to the statement that the exact superconformal R-symmetry extremizes  $a_{\text{trial}} = 3 \text{Tr} R_i^3 - \text{Tr} R_i$ ; because  $a_{\text{trial}}$  is a cubic function, (2.7) is a quadratic equation for  $R$  in each variable  $s_I$ . The inequality (2.8) then implies that the correct extremum is the unique one which locally maximizes  $a_{\text{trial}}$ .

Relation (2.7) was obtained in [13] by using supersymmetry to relate the two corresponding anomaly triangle diagrams,  $\langle F_I R R \rangle$  and  $\langle F_I T T \rangle$ . A non-R flavor supercurrent  $J_I$  is at one vertex and the super-stress tensor  $T_{\alpha\dot{\alpha}}$ , containing both the superconformal  $U(1)_R$  current and the stress tensor, is at the other two vertices. Using a result of [27], the  $\langle J_I(z_1) T_{\alpha\dot{\alpha}}(z_2) T_{\beta\dot{\beta}}(z_3) \rangle$  three-point function, and hence its anomaly, is uniquely determined by the superconformal Ward identities up to an overall normalization coefficient; this implies that the anomalies on the two sides of (2.7) have fixed ratio, and the factor of 9 can then be fixed by considering the free-field case, where the fermions have  $R = -1/3$ . Another way to obtain (2.7) is to consider the anomalous violation of the flavor supercurrent  $J_I$  upon turning on a background coupled to  $T_{\alpha\dot{\alpha}}$ , i.e., a background metric and background gauge fields coupled to the superconformal R-current: (2.7) is obtained upon arguing that  $\bar{D}^2 J_I = k_I \mathcal{W}^2$ , with no contribution proportional to the chirally projected super Euler density  $\mathcal{E}$  [13].

The equality in (2.8), obtained in [5], relates the 't Hooft anomaly for  $\langle R F_I F_J \rangle$  to the coefficients  $\tau_{IJ}$  of the flavor current two-point functions  $\langle J_I^\mu(x) J_J^\nu(y) \rangle$ . The inequality in (2.8) then follows upon using unitarity to argue that the current–current two-point function coefficients are a positive-definite matrix,  $\tau_{IJ} > 0$ . The extremum condition (2.7) is a quadratic equation, and inequality (2.8) determines that the correct solution is uniquely determined to be that which locally maximizes  $a_{\text{trial}}$ .

For a general  $\mathcal{N} = 1$  SUSY gauge theory, with gauge group  $G$  and matter chiral superfields  $Q_i$  in representations  $r_i$  of  $G$ , (2.7) constrains the superconformal R-charges  $R(Q_i) \equiv R_i$  to satisfy

$$\sum_i |r_i|(F_I)_i (9(R_i - 1)^2 - 1) = 0. \tag{2.9}$$

$(F_I)_i \equiv F_I(Q_i)$  are any flavor charges of the matter fields, which must be  $G$ -anomaly-free:

$$\text{Tr} F_I G^2 = \sum_i (F_I)_i T(r_i) = 0, \tag{2.10}$$

with  $T(r_i)$  the quadratic Casimir of representation  $r_i$ . Superpotential interactions further constrain the charges  $(F_I)_i$ ; for now, consider the case of gauge interactions only. The

general solution for  $R_i$ , satisfying (2.7) for any flavor charges  $(F_I)_i$  satisfying (2.10), is

$$R_i = 1 - \frac{1}{3} \sqrt{1 + \frac{\lambda_* T(r_i)}{|r_i|}}. \tag{2.11}$$

$\lambda_*$  is a parameter that is fixed by the constraint that  $U(1)_R$  be anomaly-free:

$$\text{Tr } RG^2 = T(G) + \sum_i T(r_i)(R_i - 1) = T(G) - \frac{1}{3} \sum_i \sqrt{1 + \frac{\lambda_* T(r_i)}{|r_i|}} = 0. \tag{2.12}$$

The branch of the square-roots are determined by (2.8), which for gauge interactions has sign corresponding to negative anomalous dimensions, since (2.11) and (1.6) yield for the RG fixed point anomalous dimensions:

$$\gamma_i(g_*) = 3R_i - 2 = 1 - \sqrt{1 + \frac{\lambda_* T(r_i)}{|r_i|}} = 1 - \sqrt{1 + \frac{\lambda_* C(r_i)}{|G|}}. \tag{2.13}$$

As standard, we define group theory factors as

$$\begin{aligned} \text{Tr}_{r_i}(T^A T^B) &= T(r_i) \delta^{AB}, \\ \sum_{A=1}^{|G|} T_{r_i}^A T_{r_i}^A &= C(r_i) \mathbf{1}_{|r_i| \times |r_i|}, \quad \text{so } C(r_i) = \frac{|G| T(r_i)}{|r_i|}, \end{aligned} \tag{2.14}$$

normalizing quadratic Casimirs so that  $T(G) = N_c$  and  $T(\text{Fund}) = \frac{1}{2}$  for  $SU(N_c)$ .

As discussed in [13], a non-trivial check of  $a$ -maximization is that (2.13) indeed reproduces the correct anomalous dimensions for perturbatively accessible RG fixed points:

$$\gamma_i(g) = -\frac{g^2}{4\pi^2} C(r_i) + O(g^4). \tag{2.15}$$

Expanding the exact result (2.13) for small  $\lambda$  and comparing with (2.15) yields

$$\lambda_* = \frac{g_*^2 |G|}{2\pi^2} + O(g_*^4), \tag{2.16}$$

with both  $\lambda_*$  and  $g_*$  determined at the RG fixed point in terms of the group theory factors [13] by the condition that  $U(1)_{R_*}$  be anomaly-free (equivalently,  $\beta_{\text{NSVZ}} = 0$ ).

The above results are valid as long as there are no accidental symmetries in the IR. They require modification when IR accidental symmetries are present [15], because we must  $a$ -maximize over all flavor symmetries, including all accidental symmetries. Restricting the landscape of allowed R-charges, by not accounting for the possibility of mixing with all accidental symmetries, would lead to incorrect results. A crucial issue then becomes how one can determine what accidental symmetries might be present.

One particular type of accidental symmetry, which is under control, is that associated with gauge-invariant composite operators hitting a unitarity bound, and becoming free. To be concrete, suppose that  $\dim(M)$  operators  $M = Q\bar{Q}$  become free, with an accidental  $U(1)_M$  symmetry, under which only the composite operators  $M$  are charged; the  $U(1)_M$  charge is  $F_M$ , with  $F_M(M) = 1$  and all other fields neutral.  $a$ -maximization must include

mixing with  $U(1)_M$ :  $R_{\text{trial}} = R_{\text{trial}}^{(0)} + s_M F_M$ .  $a_{\text{trial}} = 3 \text{Tr} R_{\text{trial}}^3 - \text{Tr} R_{\text{trial}}$  can be computed using 't Hooft anomaly matching. Maximizing over  $s_M$  yields  $R_*(M) = 2/3$ , as appropriate for a free field, with  $R_*(M) \neq R_*(Q) + R_*(\tilde{Q})$  because of the mixing with  $U(1)_M$ . There is an important residual effect on the quantity to be maximized for determining  $y \equiv R(Q)$  and  $\tilde{y} \equiv R(\tilde{Q})$  [15] (see [16] for a derivation along the lines sketched here):

$$a^{(1)}(y, \tilde{y}, \dots) = a^{(0)}(y, \tilde{y}, \dots) + \dim(M) \left( \frac{2}{9} - 3(y + \tilde{y} - 1)^3 + y + \tilde{y} - 1 \right). \quad (2.17)$$

The additional term in (2.17) vanishes when  $R_0(M) \equiv y + \tilde{y} = 2/3$ , as does its first derivative. This ensures that  $a$ -maximization yields  $R_*$  charges and central charge  $a_{\text{CFT}}$  that are continuous and smooth (first derivative continuous, though higher derivatives are generally discontinuous) across a transition where the operators  $M$  become free (say as a function of parameters that can be varied, such as  $N_c/N_f$ ).

### 2.3. $a$ -maximization with Lagrange multipliers

We first review Kutasov's proposal [19] for the case of gauge interactions only. The idea is to implement the constraint that the superconformal  $U(1)_R$  be anomaly-free at the IR fixed point via a Lagrange multiplier  $\lambda$ , maximizing (1.12)

$$a(R_i, \lambda) = 2|G| + \sum_i |r_i| [3(R_i - 1)^3 - (R_i - 1)] - \lambda \left( T(G) + \sum_i T(r_i)(R_i - 1) \right). \quad (2.18)$$

Extremizing (2.18) w.r.t.  $R_i$  yields

$$R_i(\lambda) = 1 - \frac{1}{3} \sqrt{1 + \frac{\lambda T(r_i)}{|r_i|}} = 1 - \frac{1}{3} \sqrt{1 + \frac{\lambda C(r_i)}{|G|}}. \quad (2.19)$$

Plugging back into (2.18) yields

$$a(\lambda) \equiv a(R_i(\lambda), \lambda) = 2|G| - \lambda T(G) + \frac{2}{9} \sum_i |r_i| \left( 1 + \frac{\lambda T(r_i)}{|r_i|} \right)^{3/2}. \quad (2.20)$$

Because  $R_i(\lambda)$  solves  $\partial a / \partial R_i = 0$ , we have

$$\frac{d}{d\lambda} a(\lambda) = \frac{\partial}{\partial \lambda} a(R_i, \lambda) = -T(G) - \sum_i T(r_i)(R_i - 1) \equiv \hat{\beta}_G(R_i). \quad (2.21)$$

Extremizing now in  $\lambda$  has solution  $\lambda_*$ , where (2.21) vanishes, and  $R_i(\lambda_*)$  are the same as in (2.11). Also, evaluating (2.18) with both  $R_i$  and  $\lambda$  extremized yields  $a(R(\lambda_*), \lambda_*) = a_{\text{SCFT}}$ , since the additional term proportional to  $\lambda$  in (2.18) vanishes at  $\lambda = \lambda_*$ .

The proposal of [19] is to interpret (2.19) and (2.20) as the running R-charges and  $a$ -function, along the *entire RG flow*, from the UV to the IR, with the Lagrange multiplier  $\lambda$  interpreted as the running gauge coupling  $g^2$  in some scheme. The RG flow from UV to IR corresponds to  $\lambda : 0 \rightarrow \lambda_*$ . Since  $\lambda$  is increasing along the RG flow to the IR,  $\dot{\lambda} > 0$ ,

and the beta function along the RG flow is negative, (2.21) implies that this  $a$ -function is monotonically decreasing along the RG flow,  $\dot{a} \leq 0$ , with  $\dot{a} = 0$  at precisely the IR SCFT, where the beta function vanishes.

The RG flow can be pictured using Fig. 1. In the UV,  $\lambda = 0$  and the matter chiral superfields all have  $R_i = 2/3$ , at point (A). Extremizing (2.18) w.r.t.  $R_i$  implies that  $R_i$  should sit at a point where the slope of the function in Fig. 1 equals  $\lambda T(r_i)$ , giving (2.19). Increasing  $\lambda$  thus takes  $R_i$  to where the slope is positive, i.e., down the hill to the left of point (A), reducing  $a$ . Eventually the flow hits a zero of the beta function and stops, with  $R(Q_i)$  at some point (C) in Fig. 1.

2.4. Comparing with the explicit perturbative computations of Jack, Jones, and North [24]

The proposal is that (2.19) gives the exact R-charges along the entire RG flow. Hence the exact anomalous dimensions, along the entire RG flow, are given by

$$\gamma_i(\lambda) = 2(\Delta(Q_i) - 1) = 3R_i - 2 = 1 - \sqrt{1 + \frac{\lambda C(r_i)}{|G|}}. \tag{2.22}$$

In this subsection, we will compare this with explicit perturbative computations, extending the higher-loop check made in [19]. Note that the expression (2.22) is obviously compatible with the  $a$ -maximization result (2.13) for the exact anomalous dimension at RG fixed points. The check here is thus also a higher-loop extension of the check in [13] between the exact  $a$ -maximization results and explicit perturbative computations, for those RG fixed point theories which are perturbatively accessible.

Expanding (2.22) in  $\hat{\lambda} \equiv \lambda/2|G|$  yields (for uniform notation, we take  $(-1)!! \equiv 1$ )

$$\begin{aligned} \gamma_i(\lambda) &= \sum_{p=1}^{\infty} \frac{(2p-3)!!}{p!} (-\hat{\lambda})^p C(r_i)^p \\ &= -\hat{\lambda} C(r_i) + \frac{\hat{\lambda}^2}{2} C(r_i)^2 - \frac{\hat{\lambda}^3}{2} C(r_i)^3 + \frac{5\hat{\lambda}^4}{8} C(r_i)^4 + \dots \end{aligned} \tag{2.23}$$

Comparing with the 1-loop anomalous dimensions (2.15) then yields

$$\hat{\lambda} \equiv \frac{\lambda}{2|G|} = \frac{g^2}{4\pi^2} + \sum_{q=2}^{\infty} A_q g^{2q}, \tag{2.24}$$

the analog of (2.16), now interpreted as applying along the entire RG flow; (2.24) is indeed compatible with the interpretation of  $\lambda$  as corresponding to the running coupling. The undetermined coefficients  $A_{q \geq 2}$  in (2.24) reflect the standard renormalization scheme freedom to reparametrize the coupling constant. In general, if one scheme has coupling  $g$  and wavefunction renormalization factors  $Z_i(g)$ , another could have coupling  $g'(g)$  and wavefunction renormalization  $Z'_i(g') = Z_i(g) F_i(g)$ . The anomalous dimensions and beta function of the two schemes are then related by

$$\gamma'_i(g') = \gamma_i(g) - \beta(g) \frac{\partial \ln F_i(g)}{\partial g} \quad \text{and} \quad \beta'(g') = \frac{\partial g'(g)}{\partial g} \beta(g). \tag{2.25}$$

We will compare the prediction (2.23) with the explicit higher-loop computations of [24], assuming initially that the only scheme difference is a change of coupling constant  $\lambda = \lambda(g)$ , as in (2.24), assuming initially that  $F_i(g) = \text{const}$  in (2.25).

Keeping arbitrary  $A_q$  in (2.24), (2.23) yields

$$\gamma_i(g) = \sum_{p=1}^{\infty} \frac{(2p-3)!!}{p!} \left( -\frac{g^2 C(r_i)}{4\pi^2} - \sum_{q=2}^{\infty} A_q C(r_i) g^{2q} \right)^p. \quad (2.26)$$

Expanding yields predicted expressions for the  $p$ -loop anomalous dimensions:

$$\begin{aligned} \gamma_i^{(1)} &= -\frac{C(r_i)}{4\pi^2} g^2, & \gamma_i^{(2)} &= \left( \frac{C(r_i)^2}{32\pi^4} - A_2 C(r_i) \right) g^4, \\ \gamma_i^{(3)} &= \left( -\frac{C(r_i)^3}{128\pi^6} + A_2 \frac{C(r_i)^2}{4\pi^2} - A_3 C(r_i) \right) g^6, \\ \gamma_i^{(4)} &= \left( \frac{5}{8} \frac{C(r_i)^4}{(4\pi^2)^4} - \frac{3}{2} A_2 \frac{C(r_i)^3}{(4\pi^2)^2} + \frac{1}{2} \left( 2 \frac{A_3}{4\pi^2} + A_2^2 \right) C(r_i)^2 - A_4 C(r_i) \right) g^8, \quad \text{etc.} \end{aligned} \quad (2.27)$$

The prediction, for general  $p$ -loops, is that the highest power of  $C(r_i)$  is  $C(r_i)^p$ . The coefficient of this highest power term is hence scheme-independent, and predicted to be:

$$\begin{aligned} \gamma_i^{(p)}(g) &= \left( \frac{(2p-3)!!}{p!} \left( -\frac{C(r_i)}{4\pi^2} \right)^p \right. \\ &\quad \left. + \sum_{\ell=1}^{p-1} (\text{scheme-dependent coeffs.}) C(r_i)^\ell \right) g^{2p}. \end{aligned} \quad (2.28)$$

Moreover, for each  $p$ , the scheme-dependent coefficients of  $C(r_i)^\ell$  in (2.28) are fixed in terms of those of lower orders of perturbation theory for  $2 \leq \ell < p$  (only the coefficient of the  $\ell = 1$  term is not already determined by the results from lower orders in perturbation theory). The structure of the scheme-dependent coefficients is predicted to be such that there exists a particular scheme, corresponding to setting all  $A_{q>2} = 0$ , in (2.24) in which the  $p$ -loop anomalous dimension has only the  $C(r_i)^p$  term in (2.28).

As discussed in [19], the predicted  $\gamma^{(2)}$  in (2.27) indeed agrees with that obtained from explicit computation of the Feynman diagrams: the scheme-independent  $C(r_i)^2$  term indeed has the same coefficient,<sup>7</sup> and matching the coefficient of the  $C(r_i)$  term fixes the coefficient  $A_2$  in the expression (2.24) for  $\lambda$  in the particular scheme adopted in [24]:

$$A_2 = \frac{b_1}{64\pi^4}, \quad \text{with } b_1 \equiv 3T(G) - \sum_i T(r_i), \quad \text{in the particular scheme of [24].} \quad (2.29)$$

<sup>7</sup> In comparing with [24], note that we define anomalous dimensions as  $\Delta(Q_i) = 1 + \frac{1}{2}\gamma_i$ , whereas the definition in [24] would not have the  $\frac{1}{2}$ , so  $\gamma_{\text{here}} = 2\gamma_{\text{there}}$ .

We can now go to *three loops*, comparing the prediction (2.27) with the perturbative results of [24]. We indeed find precise agreement for the scheme-independent coefficient of the  $g^6 C(r_i)^3$  term! However, using (2.29) in (2.27), our prediction for the (scheme-dependent) coefficient of the  $g^6 C(r_i)^2$  term in  $\gamma_i^{(3)}$  is twice that obtained in [24]. Fortunately, this difference (as in (2.29)) is proportional to (the leading term of)  $\beta(g)$ . Thus (2.27) can be salvaged by including a further scheme difference (2.25), between that of the Lagrange multiplier method and that of [24], coming from a non-trivial difference in the wavefunction renormalization starting at two loops:  $\partial \ln F_i / \partial g \sim C(r_i)^2 g^3$ .

### 2.5. Including superpotential interactions

Let us now consider the case of both gauge interactions and those associated with a superpotential term  $W = h \prod_i Q_i^{n(W)_i}$ . If this  $W$  is relevant, the IR SCFT has the added constraint that the superpotential<sup>8</sup> has total R-charge 2, which can again be implemented with a Lagrange multiplier. The prescription is then to modify (2.18) by adding a term  $\lambda_W (R(W) - 2)$ , with  $R(W) = \sum_i R_i n(W)_i$ . Extremizing  $a(R_i, \lambda_G, \lambda_W)$  w.r.t. the  $R_i$ , holding  $\lambda_G$  and  $\lambda_W$  fixed, then modifies (2.19) to

$$R_i(\lambda_G, \lambda_W) = 1 - \frac{1}{3} \sqrt{1 + \frac{\lambda_G T(r_i)}{|r_i|} - \frac{n(W)_i \lambda_W}{|r_i|}}. \tag{2.30}$$

Plugging  $R_i(\lambda_G, \lambda_W)$  back into  $a(R_i, \lambda_G, \lambda_W)$  yields the  $a$ -function

$$a(\lambda_G, \lambda_W) = 2|G| - \lambda_G T(G) + \lambda_W (n(W) - 2) + \frac{2}{9} \sum_i |r_i| \left( 1 + \frac{\lambda_G T(r_i)}{|r_i|} - \frac{n(W)_i \lambda_W}{|r_i|} \right)^{3/2}, \tag{2.31}$$

with  $n_W = \sum_i n(W)_i$  the degree of the superpotential. This  $a$ -function satisfies

$$\frac{\partial a}{\partial \lambda_G} = \hat{\beta}_G \quad \text{and} \quad \frac{\partial a}{\partial \lambda_W} = \hat{\beta}_W, \tag{2.32}$$

proportional to the exact gauge and Yukawa beta functions, as defined in (1.7) and (1.8).

The conjecture is again that  $\lambda_W$  can be interpreted as the running superpotential Yukawa coupling  $h^2$ , in some appropriate scheme. Using (2.19) for the exact R-charges yields exact anomalous dimensions

$$\gamma_i = 3R_i - 2 = 1 - \sqrt{1 + \frac{\lambda_G T(r_i)}{|r_i|} - \frac{\lambda_W n(W)_i}{|r_i|}}. \tag{2.33}$$

We can again write this exact expression for the anomalous dimensions as

$$\gamma_i = 1 - \sqrt{1 - 2\gamma_i^{(1)}}, \tag{2.34}$$

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<sup>8</sup> We use the fact that the form of the superpotential is not renormalized along the RG flow: the only renormalization is that of the overall coupling  $h$  (coming from the renormalization of the kinetic terms). Non-perturbative corrections to the superpotential are avoided if there is sufficient matter, so that  $\sum_i T(r_i) \geq T(G)$ .

with

$$\gamma_i^{(1)} = -\frac{\lambda_G T(r_i)}{2|r_i|} + \frac{n(W)_i \lambda_W}{2|r_i|}, \tag{2.35}$$

to be identified with the one-loop anomalous dimension. Comparing with explicit perturbative computations allows us to check this result, e.g., verifying the  $1/|r_i|$  dependence in (2.33) and (2.35), and to find the leading relation between  $\lambda_W$  and  $h^2$ .

To fix the normalization, let us first compare (2.35) with perturbation theory for a single chiral superfield  $Q$ , with cubic superpotential  $W = \frac{1}{6}hQ^3$  (so  $n(W) = 3$  in (2.35)):

$$\gamma_Q^{(1)} = \frac{|h|^2}{16\pi^2} = \frac{3\lambda_W}{2}, \quad \text{hence } \lambda_W = \frac{|h|^2}{24\pi^2} + \mathcal{O}(h^4). \tag{2.36}$$

With many chiral superfields  $Q_i$  and superpotential  $W = \frac{1}{6}h^{ijk}Q_iQ_jQ_k$ , the one-loop anomalous dimension matrix is

$$\gamma^{(1)i}_j = \frac{h^{ikl}h^*_{jkl}}{16\pi^2}. \tag{2.37}$$

Suppose that the matter fields form distinct irreps of a group, with  $h^{ijk} = hT^{r_i r_j r_k}$ , with  $T^{r_i r_j r_k}$  an invariant tensor to contract the group indices of those irreps. Schur’s lemma then ensures that the anomalous dimension matrix (2.37) is diagonal and proportional to the identity matrix for each irrep, and taking the trace fixes the coefficient to be

$$\gamma^{(1)i}_j = \delta^i_j \frac{h^{klm}h^*_{klm}}{16\pi^2|r_i|} \quad (\text{with } h^{klm}h^*_{klm} = |h|^2 T^{r_i r_j r_k} T^*_{r_i r_j r_k} \equiv |h|^2 |T|^2), \tag{2.38}$$

giving  $\gamma^{(1)} \sim 1/|r_i|$ , as predicted from (2.33). Comparing (2.33) and (2.38) yields,

$$\lambda_W = \frac{|h|^2 |T|^2}{24\pi^2} + \text{higher-loop (scheme-dependent) corrections}. \tag{2.39}$$

As in the previous subsection, one can do higher-loop comparisons with the results of [24], where the anomalous dimensions were computed to three loops, including the contributions from Yukawa couplings. But there is significant scheme freedom in redefining the Yukawa couplings, including their tensor structure, so we will not here explicitly discuss the higher order dictionary (2.39) between  $\lambda_W$  and the Yukawa couplings in the scheme of [24].

*2.6. An example: electric and magnetic SQCD*

For  $SU(N_c)$  SQCD, with  $N_f$  fundamental flavors  $Q_f, \tilde{Q}_{\tilde{f}}$ , (2.19) gives [19]

$$R_Q(\lambda) = R_{\tilde{Q}}(\lambda) = 1 - \frac{1}{3}\sqrt{1 + \frac{\lambda_G}{2N_c}}, \tag{2.40}$$

and thus the  $a$ -function along the flow is [19]

$$a(\lambda) = 2(N_c^2 - 1) - \lambda_G N_c + \frac{4}{9}N_c N_f \left(1 + \frac{\lambda_G}{2N_c}\right)^{3/2}. \tag{2.41}$$

The asymptotically free UV theory corresponds to  $\lambda = 0$ , and the RG flow to the IR corresponds to  $\lambda : 0 \rightarrow \lambda^*$ , where

$$\frac{\lambda_{G^*}}{2N_c} = \left(\frac{3N_c}{N_f}\right)^2 - 1 \tag{2.42}$$

is where the R-charges (2.40) are anomaly-free, and hence (2.41) is critical and  $\beta_{\text{NSVZ}} = 0$ .

The magnetic dual [25] is  $\tilde{G} = SU(\tilde{N}_c) \equiv SU(N_f - N_c)$  SCQD, with  $N_f$  dual quarks  $q^f, \tilde{q}^f$ , and  $N_f^2$  added singlets  $M_{f\tilde{g}}$ , with superpotential

$$W = hM_{f\tilde{g}}q_f\tilde{q}^{\tilde{g}}. \tag{2.43}$$

The quantity to maximize for the RG flow of the dual theory is

$$a = 2(\tilde{N}_c^2 - 1) + 2\tilde{N}_cN_f(3(R_q - 1)^3 - (R_q - 1)) + N_f^2(3(R_M - 1)^3 - (R_M - 1)) - \lambda_{\tilde{G}}(\tilde{N}_c + N_f(R_q - 1)) + \lambda_h(2R_q + R_M - 2). \tag{2.44}$$

Extremizing in  $R_q$  and  $R_M$ , holding  $\lambda_{\tilde{G}}$  and  $\lambda_h$  fixed yields

$$R(q) = R(\tilde{q}) = 1 - \frac{1}{3}\sqrt{1 + \frac{\lambda_{\tilde{G}}}{2\tilde{N}_c} - \frac{\lambda_h}{\tilde{N}_cN_f}},$$

$$R(M) = 1 - \epsilon\frac{1}{3}\sqrt{1 - \frac{\lambda_h}{N_f^2}}. \tag{2.45}$$

Increasing  $\lambda_{\tilde{G}}$ , and hence the magnetic gauge group coupling  $\tilde{g}^2$ , lowers  $R(q)$ , whereas increasing  $\lambda_h$  increases  $R(q)$  and  $R(M)$ . Plugging back into (2.44) yields  $a$ -function

$$a(\lambda_{\tilde{G}}, \lambda_h) = 2(\tilde{N}_c^2 - 1) - \lambda_{\tilde{G}}\tilde{N}_c + \frac{4}{9}\tilde{N}_cN_f\left(1 + \frac{\lambda_{\tilde{G}}}{2\tilde{N}_c}\right)^{3/2} + \epsilon\frac{2}{9}N_f^2\left(1 - \frac{\lambda_h}{N_f^2}\right)^{3/2}, \tag{2.46}$$

whose  $\lambda$  gradients give  $\hat{\beta}_{\tilde{G}}$  and  $\hat{\beta}_W$ .

The  $\epsilon = \pm$  in (2.45) corresponds to the choice of branch sign in the square root, and is a main point of this subsection. Taking  $N_f > \frac{3}{2}N_c$ , the magnetic theory is asymptotically free, and the UV limit has the free-field R-charges  $R(q) \rightarrow 2/3$  and  $R(M) \rightarrow 2/3$ , and hence  $\lambda_{\tilde{G}} \rightarrow 0$  and  $\lambda_h \rightarrow 0$ , with  $\epsilon = +1$  in (2.45). As the magnetic theory RG flows to the IR,  $\lambda_h$  increases, and hence  $R(M)$  moves to  $R(M) > 2/3$  (unitarity requires  $R(M) \geq 2/3$ , with equality iff it is a free field). In Fig. 1,  $R(M)$  flows from point (A) towards point (B). If the IR fixed point is sufficiently strong coupling,  $R(M)$  can increase past  $R(M) = 1$ , in which case  $\lambda_h$  must first increase to  $N_f^2$  on the  $\epsilon = +1$  branch of (2.45), and then we must switch to the  $\epsilon = -1$  branch, after which  $\lambda_h$  must decrease as we flow farther in the IR.

As an extreme example, for  $N_f \approx 3N_c$  (just below) the electric theory is barely asymptotically free and hence weakly coupled in the IR, whereas the magnetic dual is very strongly coupled in the IR. At the RG fixed point, we know from the electric side that  $R_{\text{IR}}(Q) \approx 2/3$ , and thus  $R_{\text{IR}}(M) \approx 4/3$ , i.e.,  $R(M)$  in the magnetic theory flows from  $R_{\text{UV}}(M) = 2/3$  to  $R_{\text{IR}}(M) \approx 4/3$ . Using (2.45), the flow starts in the UV with  $\epsilon = +1$  and

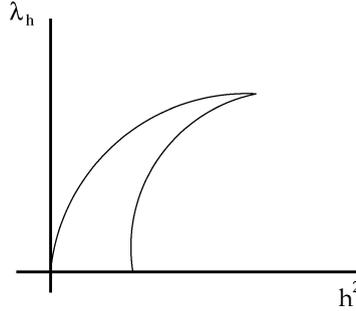


Fig. 2. Hypothetical plot of  $\lambda_h(h^2)$ , with  $\epsilon = +1$  on the top part and  $\epsilon = -1$  on the bottom.

$\lambda_h$  increasing from zero to its maximal value  $\lambda_h = N_f^2$ , after which the continued flow to the IR is on the  $\epsilon = -1$  branch, with  $\lambda_h$  decreasing, with  $\lambda_h \rightarrow 0$  at the IR fixed point. Though  $\lambda_h \approx 0$  at the IR fixed point, the magnetic dual is certainly strongly coupled, and we expect that  $h_*^2$  is not small. As we will discuss in the next section, in order to have positive-definite metric  $G_{IJ}$  and monotonically decreasing  $a$ -function, we expect that the Jacobian  $\partial\lambda_K/\partial g^I$  should be positive (positive eigenvalues); assuming the off-diagonal terms to be negligible, this requires  $d\lambda_h/dh^2 > 0$ , suggesting the “shark fin” shape of Fig. 2.

The slope of the beta function at a RG fixed point,  $\beta'(\alpha_*)$ , is a scheme-independent quantity, which gives the anomalous dimension of the leading irrelevant operator along which we flow into a RG fixed point (i.e.,  $F_{\mu\nu}F^{\mu\nu}$  for gauge interactions). For SUSY gauge theories,  $\beta'(\alpha_*)$  was argued to be related to the anomalous dimension of the Konishi current at the RG fixed point [28]. Using a claimed map of this current to that of the magnetic dual it was argued that  $\beta'(g_*^2)_{\text{elec}} = \beta'_{\text{min}}(g_*^2, h_*^2)_{\text{mag}}$  [28]. For  $N_f/N_c = \frac{3}{2} + \delta$ , with  $\delta \ll 1$ , the magnetic RG fixed point is weakly coupled and  $\beta'_{\text{min}}(g_*^2, h_*^2)_{\text{mag}}$  can be perturbatively computed; doing so, the claim of [28] leads to a prediction for  $\beta'(\alpha_*)$  in the corresponding, strongly coupled electric theory [28],  $\beta'(\alpha_*) = (28/3)\delta^2$ . We do not, however, find this qualitative behavior, of having  $\beta'(\alpha_*) \rightarrow 0$  as  $\delta \rightarrow 0$ , in  $(d\hat{\beta}/d\lambda)_{\lambda_*} = (N_f/6N_c)^2$ , as computed using (2.41) and (2.42). The factor from  $\beta_{\text{NSVZ}}/\hat{\beta}$  in (1.7) does not help (if anything, it is large in this limit); the only apparent way to get  $\beta' \rightarrow 0$  would be if  $(d\lambda/d\alpha)|_{\alpha_*} \rightarrow 0$  as  $\delta \rightarrow 0$ . We do not know whether or not this is the case.

### 3. RG flow = gradient flow: evidence for the strongest version of the $a$ -theorem

Writing the general  $a$ -function again as  $a(\lambda) = a(R(\lambda), \lambda)$  with

$$a(R, \lambda_I) = 3 \text{Tr} R^3 - \text{Tr} R + \sum_I \lambda_I \hat{\beta}^I(R), \tag{3.1}$$

and  $R(\lambda)$  obtained by extremizing in  $R$ , the  $\lambda_K$  gradients of this function give

$$\frac{\partial a(\lambda)}{\partial \lambda_K} = \hat{\beta}^K(R(\lambda)). \tag{3.2}$$

The  $\hat{\beta}^K(R)$  are proportional to the exact beta functions, which we will write as

$$\hat{\beta}^K(R) = f_J^K(g)\beta^J(g). \tag{3.3}$$

Thus (3.2) demonstrates that the exact RG flow is indeed gradient flow! Writing the  $\lambda_I$  as functions of the couplings  $g^J$  in a general scheme, we have

$$\frac{\partial a}{\partial g^I} = \frac{\partial a}{\partial \lambda_K} \frac{\partial \lambda_K}{\partial g^I} = f_J^K(g) \frac{\partial \lambda_K}{\partial g^I} \beta^J(g) \equiv G_{IJ}(g)\beta^J(g). \tag{3.4}$$

This gives the beta-functions as gradients of the  $a$ -function, as in (1.2), with metric for the space of  $g^I$  coupling constants

$$G_{IJ}(g) = f_J^K(g) \frac{\partial \lambda_K}{\partial g^I}. \tag{3.5}$$

A sufficient condition for  $G_{IJ}(g) > 0$  and the strongest version of the  $a$ -theorem is  $f_J^K(g) > 0$  (e.g., we do not continue past the apparent pole associated with the denominator of  $\beta_{\text{NSVZ}}$ ) and the coupling constant reparametrization  $\lambda_K(g)$  is monotonic,  $\partial \lambda_K / \partial g^I > 0$ .

Using (3.5) and (1.7), the exact metric for gauge couplings is (this case appears already in [19])

$$G_{gg} = \frac{\hat{\beta}}{\beta} \frac{d\lambda_G}{dg} = \frac{16\pi^2}{3g^3} \left( 1 - \frac{g^2 T(G)}{8\pi^2} \right) \frac{d\lambda_G}{dg}, \tag{3.6}$$

with  $\lambda_G(g)$  that for the NSVZ  $g$  scheme. As long as  $g^2 T(G) < 8\pi^2$  and  $\lambda_G(g)$  is monotonic, (3.6) satisfies  $G_{gg} > 0$ . Using (2.24) and (2.29), for weak coupling we approximate:

$$\begin{aligned} G_{gg} &= \frac{16\pi^2}{3g^3} \left( 1 - \frac{g^2 T(G)}{8\pi^2} \right) \left( \frac{g|G|}{\pi^2} + \frac{|G|g^3 b_1}{8\pi^4} + \dots \right) \\ &\approx \frac{16|G|}{3g^2} \left( 1 + \frac{g^2}{8\pi^2} (b_1 - T(G)) \right). \end{aligned} \tag{3.7}$$

Likewise, for Yukawa couplings, using (3.5) and (1.8), the exact metric is

$$G_{hh} = \frac{\hat{\beta}}{\beta} \frac{d\lambda_h}{dh} = \frac{4}{3} \frac{d\lambda_h}{d(h^2)}, \tag{3.8}$$

which satisfies  $G_{hh} > 0$  as long as  $\lambda_h(h)$  is monotonic. Using (2.39), we can approximate for weak coupling

$$G_{hh} = \frac{4}{3} \frac{d\lambda_h}{d(h^2)} \approx \frac{4}{3} \left( \frac{1}{24\pi^2} + \mathcal{O}(h^2) \right). \tag{3.9}$$

Consider, e.g., the magnetic dual of SQCD, with gauge group  $SU(\tilde{N}_c)$ , with gauge coupling  $\tilde{g}$ , and superpotential (2.43), with Yukawa coupling  $h$ . The  $a$ -function (2.44) gives the beta-functions as gradient flow:

$$\begin{pmatrix} \frac{\partial a}{\partial \tilde{g}} \\ \frac{\partial a}{\partial h} \end{pmatrix} = \frac{4}{3} \begin{pmatrix} \frac{\partial \lambda_{\tilde{G}}}{\partial \tilde{g}} & \frac{\partial \lambda_h}{\partial \tilde{g}} \\ \frac{\partial \lambda_{\tilde{G}}}{\partial h} & \frac{\partial \lambda_h}{\partial h} \end{pmatrix} \begin{pmatrix} 4\pi^2 g^{-3} (1 - \frac{\tilde{g}^2 T(\tilde{G})}{8\pi^2}) & 0 \\ 0 & (2h)^{-1} \end{pmatrix} \begin{pmatrix} \beta_{\text{NSVZ}}(\tilde{g}) \\ \beta_W(h) \end{pmatrix}. \tag{3.10}$$

A sufficient condition for positive metric in (3.10) is positivity of the Jacobian  $d\lambda_K/dg^I$  and  $\tilde{g}^2 T(G) < 8\pi^2$ . Assuming that the off-diagonal components of the metric are not appreciable (they are zero in perturbation theory), positivity of the Jacobian requires  $d\lambda_h/dh^2 > 0$ , which motivated the shark-fin shape of Fig. 2, for the case of  $N_f \approx 3N_c$ .

As we discussed in the introduction, we can compare metrics  $G_{IJ}$ , as computed above, with those computed by Osborn and collaborators in the context of 4d field theories on curved spacetime, with spatially dependent couplings. The supersymmetric case was considered by Freedman and Osborn in [10]. To compare expressions, we need to account for our rescalings mentioned in footnote 4,  $a_{\text{here}}(g) = (32/3)\tilde{a}_{\text{there}}(g)$ , and  $G_{IJ}^{\text{here}}(g) = \frac{4}{3}G_{IJ}^{\text{there}}(g)$ . We then find that the leading, scheme-independent, term in both the metric  $G_{gg}$  (3.7), and also the Yukawa coupling metric (3.9), agree precisely with those found by Freedman and Osborn [10]! (The coefficient of the subleading, scheme-dependent term in (3.7), however, does not agree with that obtained in [10]: rather than  $b_1 - T(G)$  of (3.7), the coefficient obtained in [10] was  $\frac{5}{2}b_1 - T(G)$ . The apparent difference,  $\sim b_1$ , could be completed at higher orders into a difference  $\sim \beta(g)$ , which would at least vanish at the endpoints of the RG flow. More work is needed to verify if this is a real difference in the metric and  $a$ -function, or perhaps associated with a scheme discrepancy.)

The method of Osborn was to consider renormalization for spatially dependent coupling constants, e.g., with  $G_{IJ}$  coming from beta functions  $\beta_{\mu\nu} \sim G_{IJ}(g)\partial_\mu g^I \partial_\nu g^J$ . This is very reminiscent of the AdS/CFT correspondence, where coupling constants correspond to fields in the bulk, with  $G_{IJ}$  naturally associated with the sigma model metric  $G_{IJ}^{\text{bulk}}$  of these bulk fields. Indeed, in [29] it was argued that the AdS holographic RG flow leads to  $\dot{c} = -G_{IJ}\beta^I\beta^J$ , with metric  $G_{IJ} = 2cG_{IJ}^{\text{bulk}}$ . This again suggests that RG flow is gradient flow, with positive-definite metric, though it is important to emphasize that the AdS/CFT correspondence seems limited to a very restricted subset of all possible CFTs. In any case,  $G_{IJ} = 2cG_{IJ}^{\text{bulk}}$  gives a nice insight into the result for the leading perturbative metric,  $G_{gg} \sim |G|/g^2$  (3.7): it matches with the ( $SL(2, Z)$ -invariant) dilaton kinetic terms in the bulk:  $\mathcal{L}_{\text{bulk}} = -\frac{1}{2}(\tau_2)^{-2}\partial_\mu\tau\partial^\mu\bar{\tau}$  (here  $\tau = \frac{\theta}{2\pi} + 4\pi i g^{-2}$ , so  $\frac{1}{4}(d(\log \tau_2))^2 = (d(\log g))^2$ ).

#### 4. $a$ -maximization along RG flows with accidental symmetries, and comments about Higgsing

The Lagrange multiplier method needs to be extended in order to apply to RG flows with accidental symmetries, or those associated with Higgsing [19]. In this section, we will discuss an extension of the proposal of [19] for the case of accidental symmetries associated with gauge-invariant operators hitting the unitarity bound and becoming free. This extension defines a monotonically decreasing  $a$ -function along such RG flows. This shows, in particular, that  $a$ -maximization indeed ensures that  $a_{\text{UV}} > a_{\text{IR}}$  is automatically satisfied for such RG flows. We will next discuss Higgsing RG flows, where we do not yet have a good candidate  $a$ -function, or general argument for  $a_{\text{UV}} > a_{\text{IR}}$ .

##### 4.1. Accidental symmetries

Accidental symmetries, present in the IR but not in the UV, challenge the  $a$ -theorem conjecture. Additional symmetries broaden the landscape over which we are maximizing

$a_{\text{trial}}$ , increasing the value of  $a_{\text{IR}}$ . To avoid violating  $a_{\text{IR}} < a_{\text{UV}}$  thus requires that the IR theory must not have too much accidental symmetry; at present, however, we do not know of a general way to prove that the possible accidental symmetries are always sufficiently bounded so as to be compatible with  $a_{\text{IR}} < a_{\text{UV}}$ . Here we will limit our discussion to a particular type of accidental symmetry, that of a gauge-invariant operator hitting its unitarity bound and becoming free (without additional free fields, such as free magnetic quarks and gluons, whose existence would have been hard to predict from the spectrum of gauge-invariant operators of the UV theory).

Near the UV start of the RG flow, we will use for the  $a$ -function, following [19],

$$a^{(0)}(R, \lambda_I) = 3 \text{Tr} R^3 - \text{Tr} R + \sum_I \lambda_I \hat{\beta}_I(R). \tag{4.1}$$

Extremizing this in the  $R_i$  has solution  $R_i^{(0)}(\lambda_I)$ , and plugging back in gives  $a$ -function  $a^{(0)}(\lambda_I) = a^{(0)}(R_i(\lambda_I), \lambda_I)$ . We propose that these  $R^{(0)}(\lambda_I)$  and  $a^{(0)}(\lambda_I)$  give the R-charges and the  $a$ -function initially along the RG flow, up until the point where the accidental symmetry arises: until the flow hits a value of the Lagrange multiplier/coupling constants  $\lambda_I^{(0)}$  where a gauge-invariant composite operator  $M$  hits  $R(M) = 2/3$ . At that point on the RG flow, including the effect of the accidental  $U(1)_M$  means patching onto another  $a$ -function, with the correction term of [15] added to (4.1):

$$a^{(1)}(R_i, \lambda_I) = a^{(0)}(R, \lambda_I) + \text{dim}(M) \left( \frac{2}{9} - 3(R_M - 1)^3 + R_M - 1 \right), \tag{4.2}$$

with  $R_M = \sum_i R_i m_i$  for  $M = \prod_i Q_i^{m_i}$ . Now (4.2) is extremized to find  $R_i^{(1)}(\lambda_I)$ , and plugging these back into (4.2) gives  $a$ -function  $a^{(1)}(\lambda_I) = a^{(1)}(R_i^{(1)}(\lambda_I), \lambda_I)$ . If other operators  $M'$  hit  $R(M') = 2/3$  further down the RG flow, we had similarly patch onto the  $a$ -function  $a^{(2)}$  obtained by adding the analogous correction term to (4.2).

So the running R-charges  $R_i(\lambda_I)$  and  $a$ -function  $a(\lambda_I)$  along the entire RG flow are proposed to be given by this patching procedure, with the patches occurring at every place along the RG flow where some gauge-invariant operator hits the unitarity bound. The important point is that, despite the patching together, the  $R_i(\lambda_I)$  and  $a(\lambda_I)$  thus obtained are continuous along the entire RG flow, as presumably are  $\hat{R}_i(\lambda_I)$  and  $\hat{a}(\lambda_I)$ , because the added term in (4.2) vanishes at the patching location, where  $R_M = 2/3$ , as does its first derivative w.r.t.  $R_M$ . Moreover, the patched together  $a$ -function still satisfies

$$\frac{\partial a(\lambda_I)}{\partial \lambda_I} = \hat{\beta}_I(R),$$

with  $\hat{\beta}_I(R)$  the same linear combinations of the (patched together) R-charge  $R_i$ , proportional to the exact beta functions, as in (1.7) and (1.8). Thus the patched-together  $a$ -function continues to satisfy  $\hat{a}(\lambda_I) < 0$ . In particular, for the endpoints of the RG flow, this demonstrates that  $a$ -maximization automatically ensures that the accidental symmetries of the above type never violate  $a_{\text{IR}} < a_{\text{UV}}$ .

Here is a suggestive way to obtain this same patching-together prescription. Consider coupling the  $N_f^2$  composite, gauge-invariant meson operators  $Q_f \hat{Q}_{\tilde{g}}$  to the same number

of added sources,  $L^{f\tilde{g}}$ , and also introduce into the theory the same number of added gauge-invariant fields  $M_{f\tilde{g}}$ , with added superpotential

$$W = L^{f\tilde{g}} Q_f \tilde{Q}_{\tilde{g}} + h L^{f\tilde{g}} M_{f\tilde{g}}. \tag{4.3}$$

We think of the second term, with coupling  $h$ , as a perturbation. Starting at  $h = 0$ , we have  $R(M) = 2/3$  and  $R(L) = 2 - R(Q\tilde{Q})$ , so the  $h$  perturbation is relevant if  $R(Q\tilde{Q}) > 2/3$ . In this case, the effect of the two terms in (4.3) is that  $L$  and  $M$  are both massive, and hence should be integrated out. The  $L$  e.o.m. sets  $M_{f\tilde{g}} = Q_f \tilde{Q}_{\tilde{g}}$ , the  $M$  e.o.m. sets  $L^{f\tilde{g}} = 0$ , and the upshot is that we are back to where we would have been had we not included the  $2N_f^2$  additional fields  $L^{f\tilde{g}}$  and  $M_{f\tilde{g}}$ . In particular, these massive fields make cancelling contributions to 't Hooft anomalies and hence to the  $a$ -function  $a = 3 \text{Tr } R^3 - \text{Tr } R$ .

On the other hand, if  $R(Q\tilde{Q}) < 2/3$ , the second term in (4.3) is irrelevant, and the  $N_f^2$  fields  $M_{f\tilde{g}}$  are then decoupled free fields, with  $R(M) = 2/3$ . This gives the  $2/9$  term in (4.2), and the remaining additional terms in (4.2) are the contribution of the fields  $L^{f\tilde{g}}$  (whose R-charge is fixed by the first term in (4.3) to be  $R(L) = 2 - R(Q\tilde{Q})$ ). The  $a$ -function computed with these added fields and superpotential interactions involves additional Lagrange multipliers, associated with the added superpotential terms, but should be equivalent to the patched-together prescription described above.

#### 4.2. Higgsing

Giving a chiral superfield an expectation value breaks the gauge group  $G \rightarrow H$ . There is then a Higgsing RG flow, from the unbroken  $G$  theory in the UV (as the vevs then negligible), to the  $H$  theory in the IR, with the massive  $G/H$  fields decoupled. We do not have a candidate  $a$ -function, or a general argument that  $a_{\text{IR}} < a_{\text{UV}}$ , for Higgsing RG flows. We will simply illustrate the challenge here, taking  $W_{\text{tree}} = 0$  for simplicity.

When  $G \rightarrow H$ , the  $G$  matter fields  $Q_i$  decompose into  $H$  representations as  $Q_i \rightarrow \sum_{\mu} Q_{i\mu}$ , some of which are eaten. As with other RG flows, we can compute  $\Delta a \equiv a_{\text{IR}} - a_{\text{UV}}$  from the IR vs. UV R-charges of the chiral superfields, with the gauge field contribution unchanged and canceling in  $\Delta a$ . The fact that the low energy group does change, from  $G$  to  $H$ , is accounted for by the contribution to  $\Delta a$  of the  $|G| - |H|$  matter fields eaten by the Higgs mechanism. At the IR fixed point, these eaten matter fields will have  $R_{\text{IR}}(Q_{\text{eaten}}) = 0$ , as seen by the fact that their fermionic components pair up to get a mass with the  $G/H$  gauginos; their contribution to  $\Delta a$  then correctly accounts for  $G \rightarrow H$ . We will write the total  $\Delta a$  as  $\Delta a = \Delta a_{\text{eaten}} + \Delta a_{\text{uneaten}}$ . The  $a$ -theorem conjecture predicts  $\Delta a < 0$ . The eaten contribution satisfies  $\Delta a_{\text{eaten}} < 0$  if  $R_{\text{UV}}(Q_{\text{eaten}}) > 0$ , e.g., at point (C) in Fig. 3, which is the case for RG fixed points with  $W_{\text{tree}} = 0$  and sufficient matter to avoid generating  $W_{\text{dyn}}$ . (Theories with  $W_{\text{tree}}$  can have matter with negative R-charge, as seen, e.g., in [15] for the theory with  $W_{\text{tree}} = \text{Tr } X^{k+1}$ .)

Very generally, however,  $\Delta a_{\text{uneaten}} > 0$ , because Higgsing leads to an IR theory that is less asymptotically free than the UV theory. The uneaten matter fields move up the hill of Fig. 3 (which is a blown-up portion of Fig. 1), from point (C) in the UV, to a larger value in the IR. Those that are  $H$ -charged move partially up the hill, and those that are  $H$ -singlets are IR-free, and hence move all the way up to point (A) in the IR. The  $a$ -theorem

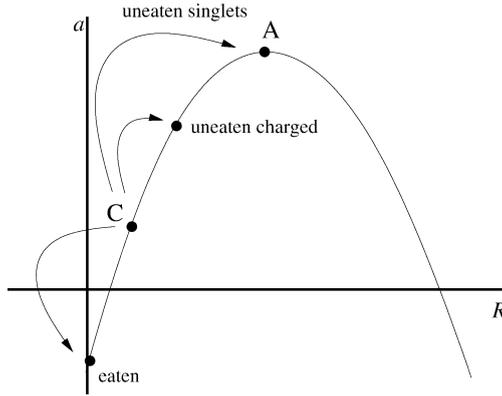


Fig. 3. Eaten and uneaten matter fields contribute oppositely to  $\Delta a$ .

prediction that  $\Delta a < 0$  thus requires that  $\Delta a_{\text{eaten}}$  be sufficiently negative, to compensate for  $\Delta a_{\text{uneaten}} > 0$ .

To illustrate all this, consider  $SU(N_c)$  SQCD with  $N_f$  flavors in the superconformal window range  $\frac{3}{2}N_c < N_f < 3N_c$ . As reviewed in Section 2, this theory has

$$a_{\text{SCFT}} = a_{\text{SQCD}}(N_c, N_f) \equiv 2(N_c^2 - 1) + 2N_c N_f \left( \frac{N_c}{N_f} - 3 \frac{N_c^3}{N_f^3} \right). \tag{4.4}$$

Giving an expectation value to one of the flavors yields a  $SU(N_c) \rightarrow SU(N_c - 1)$  Higgsing RG flow, with  $N_f \rightarrow N_f - 1$ , and  $a$ -theorem prediction

$$a_{\text{SQCD}}(N_c, N_f) > a_{\text{SQCD}}(N_c - 1, N_f - 1) + \frac{2}{9}(2N_f - 1), \tag{4.5}$$

with the last term from the  $2N_f - 1$  uneaten singlets (decomposing  $(\mathbf{N}_c) \rightarrow (\mathbf{N}_c - \mathbf{1}) + (\mathbf{1})$ ). This inequality can be thought of as a statement about the contributions of the  $2N_c N_f$  matter fields to  $\Delta a \equiv a_{\text{IR}} - a_{\text{UV}}$ . In the UV limit of the Higgsing flow, all of these fields start at point (C) in Fig. 3, with  $R_{\text{UV}} = 1 - (N_c/N_f)$ . In the IR limit, the  $2(N_c - 1)(N_f - 1)$  uneaten charged matter fields move slightly up the hill of Fig. 3 (to  $R_{\text{IR}} = 1 - (N_c - 1/N_f - 1)$ ), contributing to an increase in  $a$ . The  $2N_f - 1$  uneaten singlets also contribute positively to  $\Delta a$ , moving up the hill in Fig. 3 from point (C) to point (A), with  $R = 2/3$ . Only the  $|G| - |H| = 2N_c - 1$  eaten matter fields contribute to a decreased value of  $a_{\text{IR}}$ , moving down the hill of Fig. 3 from point (C) to  $R_{\text{IR}}(Q_{\text{eaten}}) = 0$ .

Since  $\Delta a_{\text{uneaten}} > 0$ , it is non-trivial to prove that the eaten matter field contribution is sufficient to ensure that  $\Delta a < 0$ . Indeed, (4.5) would be violated for  $N_f$  sufficiently small, if we did not account for the effect of accidental symmetries for  $N_f \leq \frac{3}{2}N_c$ . Upon taking into account these accidental symmetries,  $\Delta a < 0$  is satisfied. Proving that Higgsing RG flows always satisfy  $\Delta a < 0$  thus generally requires accounting for accidental symmetries. Perhaps it is possible to prove that  $a_{\text{IR}} < a_{\text{UV}}$  is satisfied whenever the unitarity bound condition is satisfied by all gauge-invariant operators, with accidental symmetries giving

$R = 2/3$  for any gauge-invariant operators appearing to violate the unitarity bound, but we have not found an effective way to implement this.

An attempt to generalize the proposal of [19] for defining a flowing  $a$ -function for Higgsing RG flows would be to introduce several Lagrange multipliers, to interpolate along each of the three flows depicted in Fig. 3,  $\lambda_e$  for the eaten matter fields,  $\lambda_{u.c.}$  for the uneaten charged matter, and  $\lambda_{u.s.}$  for uneaten singlet matter fields. The Higgsing RG flow would then correspond to some path  $\lambda_e(t), \lambda_{u.c.}(t), \lambda_{u.s.}(t)$ , along which we would like to find a monotonically decreasing  $a$ -function. Some clever choice of path would be required, since only the flow associated with  $\lambda_e$  has the needed sign of decreasing  $a$ .

### 5. New SCFTs from SQCD with singlets: SSQCD

In this section, we illustrate some of the points discussed in the previous sections with a new set of examples. Consider  $SU(N_c)$  SQCD with  $N_f$  fundamental flavors  $Q_i$  and  $\tilde{Q}_{\tilde{i}}$  (with  $i = 1, \dots, N_f$ ), and  $N'_f$  additional flavors  $Q'_{i'}$  and  $\tilde{Q}'_{\tilde{i}'}$  (with  $i' = 1, \dots, N'_f$ ), with the  $N'_f$  flavors coupled to  $N'^2_f$  singlets  $S^{i'\tilde{j}'}$  by a superpotential term

$$W = h S^{i'\tilde{j}'} Q'_{i'} \tilde{Q}'_{\tilde{j}'}. \tag{5.1}$$

For  $h = 0$ , the theory is just SQCD, with  $N_f + N'_f$  flavors, which flows to an interacting SCFT in the superconformal window  $\frac{3}{2}N_c < N_f + N'_f < 3N_c$ . The superpotential (5.1) is a relevant deformation of these SCFTs,  $h : 0 \rightarrow h_* \neq 0$ , driving a RG flow to a new family of SCFTs in the IR, labeled by  $(N_c, N_f, N'_f)$ . The usual SQCD RG fixed points are the special case  $N'_f = 0$  (electric description) or  $N_f = 0$  (dual, magnetic description).

The  $SU(N_f + N'_f - N_c)$  Seiberg dual [25] of the theory with  $h = 0$  can be deformed by the superpotential (5.1), whose effect in the dual is simply a mass term that pairs up the  $N'^2_f$  added singlets  $S$  with the  $N'^2_f$  mesons  $M'$  (which  $Q' \tilde{Q}'$  map to). The dual description of the new RG fixed points associated with (5.1) is thus simply a deformation of Seiberg duality, where we integrate out the massive gauge singlets  $S'$  and  $M'$ . What is left is an  $SU(\tilde{N}_c)$  gauge theory, with  $\tilde{N}_c \equiv N_f + N'_f - N_c$ , with  $N_f$  flavors of dual quarks,  $q'$ , and  $\tilde{q}'$  (if  $Q \in \mathbf{N}_f$  of  $SU(N_f)_L$ , then  $q' \in \tilde{\mathbf{N}}_f$ ), and  $N'_f$  flavors  $q$ , and  $\tilde{q}$  (if  $Q' \in \mathbf{N}'_f$  of  $SU(N'_f)$ , then  $q \in \tilde{\mathbf{N}}'_f$ ), and  $N^2_f$  gauge singlets  $M_{i\tilde{j}}$ , and  $2N_f N'_f$  singlets  $P_{i\tilde{j}'}$ , and  $P'_{\tilde{i}j'}$ , with superpotential (suppressing flavor and color indices)

$$W = M q' \tilde{q}' + P q' \tilde{q} + P' \tilde{q}' q. \tag{5.2}$$

The first term in (5.2) is similar to the superpotential (5.1) of the electric theory, with an exchange  $N_f \leftrightarrow N'_f$  in the number of flavors coupled to singlets. But the additional  $P$  and  $P'$  terms in (5.2) distinguish the magnetic duals from the original electric theory (5.1), so the duality does *not* simply equate the SCFT, obtained from the electric theory with  $(N_c, N_f, N'_f)$ , to that obtained from the electric theory with  $(N_f + N'_f - N_c, N'_f, N_f)$ . Duality equates these two SCFTs only for the special case of SQCD,  $N_f N'_f = 0$ ; for  $N_f N'_f \neq 0$ , the electric  $(N_c, N_f, N'_f)$  and  $(N_f + N'_f - N_c, N'_f, N_f)$  theories are distinct (each with their own, distinct, magnetic dual). The duality map for mesons, singlets, and

baryonic operators is

$$\begin{aligned}
 Q\tilde{Q} &\rightarrow M, & S &\rightarrow -q\tilde{q}, & Q\tilde{Q}' &\rightarrow P, & Q'\tilde{Q} &\rightarrow P', \\
 Q^r Q'^{N_c-r} &\longleftrightarrow q'^{N_f-r} q^{N_f-N_c+r}
 \end{aligned}
 \tag{5.3}$$

(with  $r$  an arbitrary integer).

Both the electric and magnetic theories have an  $SU(N_f)_L \times SU(N_f)_R \times SU(N'_f)_L \times SU(N'_f)_R \times U(1)_B \times U(1)_{B'} \times U(1)_F \times U(1)_{R_0}$  flavor symmetry. E.g., taking  $h \neq 0$  in (5.1) breaks the axial  $SU(N_f + N'_f)$  to  $SU(N_f) \times SU(N'_f) \times U(1)_F$ , so the  $U(1)_F$  charges are  $F(Q) = F(\tilde{Q}) = N'_f / (N_f + N'_f)$  and  $F(Q') = F(\tilde{Q}') = -N_f / (N_f + N'_f)$ . It is straightforward to list all of the flavor charges in the electric and magnetic duals, and to verify that they are compatible with the mappings (5.3), and also to verify that all of their 't Hooft anomalies match. All of these checks are guaranteed to work, because they worked for the original Seiberg duality [25], and the above new SCFTs and duality are obtained from those via a relevant deformation and its map to the dual description.

Despite the fact that these new SCFTs are such a simple deformation of those associated with SQCD, they could not have been quantitatively analyzed prior to the introduction [13] of the  $a$ -maximization method for determining the superconformal R-charges. The reason is that there are three independent R-charges,  $R(Q) = R(\tilde{Q}) \equiv y$ ,  $R(Q') = R(\tilde{Q}') \equiv y'$ , and  $R(S) \equiv z$ , but only two constraints among them, anomaly freedom and the constraint that the superpotential (5.1) respect the R-symmetry:

$$N_c + N_f(R(Q) - 1) + N'_f(R(Q') - 1) = 0 \quad \text{and} \quad R(S) + 2R(Q') = 2. \tag{5.4}$$

This is because the R-symmetry can mix with the  $U(1)_F$  flavor symmetry, whose effect is to allow  $R(Q)$  and  $R(Q')$  to differ. We will first discuss  $a$ -maximization at the RG fixed points, imposing (5.4) at the outset, and then next  $a$ -maximization along the RG flow, with (5.4) imposed along the lines of [19], with Lagrange multipliers.

### 5.1. $a$ -maximization at the RG fixed point

Before getting started, it is worth noting that the superconformal R-charges, obtained via  $a$ -maximization in the above electric and magnetic dual theories, will be compatible with the duality maps (5.3), which require

$$2R_*(Q) = R_*(M), \quad R_*(S) = 2R_*(q), \quad R_*(Q) + R_*(Q') = R_*(P). \tag{5.5}$$

The two duals have the same flavor symmetries and 't Hooft anomalies, so we are maximizing the same function  $a_{\text{trial}}(s)$  in both descriptions. The result is that the superconformal R-charges of the electric and magnetic theories are related by

$$R_*(q') = 1 - R_*(Q), \quad R_*(q) = 1 - R_*(Q'), \tag{5.6}$$

which imply (5.5).

In the electric theory we have  $R(Q) = R(\tilde{Q}) \equiv y$ ,  $R(Q') = R(\tilde{Q}') \equiv y'$ , and  $R(S) \equiv z$ , which are subject to the constraints (5.4) at the RG fixed point. We use these to eliminate  $y'$  and  $z$  in favor of  $y$ , and we then obtain  $y$  at the RG fixed point by maximizing

$a_{\text{trial}} = 3 \text{Tr} R^3 - \text{Tr} R$ , which we write as (taking  $N_c$ ,  $N_f$ , and  $N'_f$  all large, to simplify the expressions, holding fixed  $x \equiv N_c/N_f$  and  $n \equiv N'_f/N_f$ ):

$$\begin{aligned} & \frac{a}{2N_f N'_f}(x, n, y) \\ &= \frac{x}{n} [3(y-1)^3 - y + 1] + x \left[ 3 \left( \frac{1-y-x}{n} \right)^3 - \frac{1-y-x}{n} \right] \\ & \quad + \frac{n}{2} \left[ 3 \left( 2 \left( \frac{x+y-1}{n} \right) - 1 \right)^3 - \left( 2 \left( \frac{x+y-1}{n} \right) - 1 \right) \right] + \frac{x^2}{n}. \end{aligned} \quad (5.7)$$

Maximizing this with respect to  $y$  determines the superconformal R-charge to be

$$\begin{aligned} y = & \frac{1}{3x - 3n(4 + nx)} \left\{ -3[2n(2+n) + (n(n-4) - 1)x + x^2] \right. \\ & \left. + \sqrt{n^2(9x^2(x-2n)^2 + 8n(1-n^2)x + 4n^2)} \right\}. \end{aligned} \quad (5.8)$$

The result (5.8) is only valid over a range of  $x$  and  $n$  for which no gauge-invariant operator violates the unitarity bound. The first operator to hit the unitarity bound is the meson  $M = Q\tilde{Q}$ , which hits the unitarity bound when  $R(Q) = 1/3$ ; solving (5.8) for the value  $x_M(n)$  such that  $y(x_M(n)) = 1/3$ , the unitarity bound is hit at  $x_M(n) = \frac{1}{3}(1 + 5n - \sqrt{1 - 14n + 13n^2})$ . So (5.8) is valid for  $x < x_M(n)$ , and needs correction to account for the accidental symmetry associated with the free-fields  $M$  when  $x \geq x_M(n)$ .

We also know that, when  $N_f + N'_f \leq \frac{3}{2}N_c$ , i.e., when  $x \geq x_{\text{FM}}(n) \equiv \frac{2}{3}(1+n)$ , the theory is in a free magnetic phase, with IR-free quarks,  $SU(N_f + N'_f - N_c)$  gluons, and singlets  $M$ ,  $P$ ,  $P'$ . The phases are as in Fig. 4: for  $n = N'_f/N_f < 2$  (e.g., for the usual SQCD, where  $n = 0$ ) the theory goes directly from having no accidental symmetries to free magnetic phase, where the entire magnetic theory is IR-free. On the other hand, for  $n \geq 2$ , there is a wedge in the  $(x, n)$  parameter space where the field  $Q\tilde{Q}$  hits its unitarity bound, while the dual is still asymptotically free. In this wedge, the IR theory remains an interacting SCFT, with only the field  $M$  becoming free and decoupled.

In the wedge  $x_M < x < x_{\text{FM}}$ , where  $M = Q\tilde{Q}$  hits the unitarity bound, but the theory is not in the free magnetic phase, the effect of the accidental  $U(1)_M$  symmetry is, as in [15], simply to replace the  $M$  field contributions with those of free fields: we instead maximize the quantity

$$a^{(1)} = a^{(0)} + \left( \frac{2}{9} - 3(2y-1)^3 + (2y-1) \right) N_f^2. \quad (5.9)$$

The maximizing solution for the superconformal R-charges, and the maximal value  $a$  for the central charge, are pasted-together with the solution (5.8) at  $x = x_M(n)$ . Because the added quantity in (5.9) has a second order zero at  $y = 2/3$ , these pasted together quantities are continuous and smooth (first derivatives match) at  $x = x_M(n)$ .

The magnetic description of the decoupling of  $M$  in the wedge  $x_M(n) < x < x_{\text{FM}}(n)$  is very simple, the term  $Mq'\tilde{q}'$  in the dual superpotential (5.2) is then irrelevant: when its coefficient is small,  $R(Mq'\tilde{q}') > 2$ , because  $R(M) \approx 2/3$  and  $R(q') > 2/3$  for  $x > x_M(n)$ .

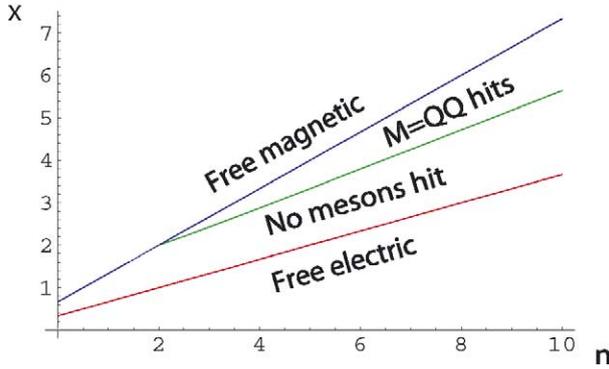


Fig. 4. Phases of SSQCD.

In the IR, this irrelevant term goes away, and the dual superpotential becomes

$$W_{\text{mag}} = Pq'\tilde{q} + P'\tilde{q}'q. \tag{5.10}$$

When we now compute  $\tilde{a}_{\text{trial}}$  in the magnetic theory, with superpotential (5.10), we obtain the same result as on the electric side, reproducing the correction term in (5.9).

5.2. *a*-function, via *a*-maximization with Lagrange multipliers

For the electric theory, *a*-maximization along the RG flow, imposing (5.4) with Lagrange multipliers, yields

$$\begin{aligned} R(Q) &= 1 - \frac{1}{3}\sqrt{1 + \frac{\lambda_G}{2N_c}}, & R(Q') &= 1 - \frac{1}{3}\sqrt{1 + \frac{\lambda_G}{2N_c} - \frac{\lambda_S}{N_c N_f'}}, \\ R(S) &= 1 - \frac{1}{3}\epsilon\sqrt{1 - \frac{\lambda_S}{N_f'^2}}, \end{aligned} \tag{5.11}$$

with both branches  $\epsilon = \pm 1$  generally needed, as we discussed in Section 2.6. Plugging these back into  $a(R_i, \lambda_I)$  yields  $a(\lambda_G, \lambda_S)$ ,

$$\begin{aligned} a &= \frac{4}{9}N_c N_f \left(1 + \frac{\lambda_G}{2N_c}\right)^{3/2} + \frac{4}{9}N_c N_f' \left(1 + \frac{\lambda_G}{2N_c} - \frac{\lambda_S}{N_c N_f'}\right)^{3/2} \\ &\quad + \frac{2}{9}N_f'^2 \epsilon \left(1 - \frac{\lambda_S}{N_f'^2}\right)^{3/2} + 2N_c^2 - \lambda_G N_c + \lambda_S. \end{aligned} \tag{5.12}$$

It would be interesting to determine the RG flow path of the gauge coupling and superpotential coupling Lagrange multipliers,  $\lambda_G(t)$  and  $\lambda_S(t)$  to their eventual IR values, where (5.12) is critical. It is gradient flow, as discussed in Section 3, but to actually determine the full trajectory requires knowing the full  $\lambda_I(g)$ .

Similarly,  $a$ -maximization along the RG flow, with Lagrange multipliers, in the magnetic dual yields

$$\begin{aligned}
 R(q) &= 1 - \frac{1}{3} \sqrt{1 + \frac{\tilde{\lambda}_G}{\tilde{N}_c} - \frac{\tilde{\lambda}_P}{2\tilde{N}_c N'_f}}, & R(M) &= 1 - \frac{1}{3} \epsilon_M \sqrt{1 - \frac{\tilde{\lambda}_M}{N_f^2}}, \\
 R(q') &= 1 - \frac{1}{3} \sqrt{1 + \frac{\tilde{\lambda}_G}{\tilde{N}_c} - \frac{\tilde{\lambda}_M}{\tilde{N}_c N_f} - \frac{\tilde{\lambda}_P}{2\tilde{N}_c N_f}}, \\
 R(P) &= 1 - \frac{1}{3} \epsilon_P \sqrt{1 - \frac{\tilde{\lambda}_P}{2N_f N'_f}}.
 \end{aligned} \tag{5.13}$$

In the wedge  $x_M(n) < x < x_{FM}(n)$ , where  $M$  decouples but the theory is otherwise interacting, the RG fixed point has  $\tilde{\lambda}_M^* = 0$ . This happens when  $R(q') > 2/3$ , hence  $\tilde{\lambda}_P/2N_f > \tilde{\lambda}_G$  in (5.13).

### 5.3. Predictions and checks of the $a$ -theorem

Having obtained the superconformal R-charge  $R_*$  via  $a$ -maximization, as discussed above, we can compute  $a(N_c, N_f, N'_f) = 3 \text{Tr } R_*^3 - \text{Tr } R_*$  for our new SCFTs. There are many RG flows associated with these theories, and in this subsection we will discuss and check some of the  $a_{UV} > a_{IR}$  predictions.

First, there is the RG flow associated with superpotential (5.1). In the UV limit of this flow,  $h \rightarrow 0$ , and the theory is the SCFT associated with ordinary SQCD with  $N_f + N'_f$  flavors plus the  $N_f'^2$  decoupled singlets, so  $a_{UV} = a_{\text{SQCD}}(N_c, N_f + N'_f) + \frac{2}{9} N_f'^2$ . The IR limit is our new SSQCD superconformal field theory, with  $a_{IR} = a(N_c, N_f, N'_f)$ , so  $a_{UV} > a_{IR}$  means

$$\begin{aligned}
 &2N_c^2 + 2N_c(N_f + N'_f) \left( 3 \left( -\frac{N_c}{N_f + N'_f} \right)^3 - \left( -\frac{N_c}{N_f + N'_f} \right) \right) + \frac{2}{9} N_f'^2 \\
 &> a(N_c, N_f, N'_f).
 \end{aligned} \tag{5.14}$$

For simplicity, we again consider the limit of large  $N_c$ ,  $N_f$ , and  $N'_f$ , holding fixed  $x \equiv N_c/N_f$  and  $n \equiv N'_f/N_f$ . Defining  $\hat{a}(x, n) \equiv a(N_c, N_f, N'_f)/2N_f N'_f$ , (5.14) becomes

$$\frac{x^2}{n} + x \left( 1 + \frac{1}{n} \right) \left( -3 \left( \frac{x}{1+n} \right)^3 + \frac{x}{1+n} \right) + \frac{n}{9} > \hat{a}(x, n). \tag{5.15}$$

We have verified numerically that this prediction is indeed satisfied.

Another RG flow is to start at our SSQCD fixed point and deform by giving a  $Q$  flavor a mass. The IR theory is again SSQCD, but with  $N_f \rightarrow N_f - 1$ , and  $a_{UV} > a_{IR}$  becomes

$$a(N_c, N_f, N'_f) > a(N_c, N_f - 1, N'_f). \tag{5.16}$$

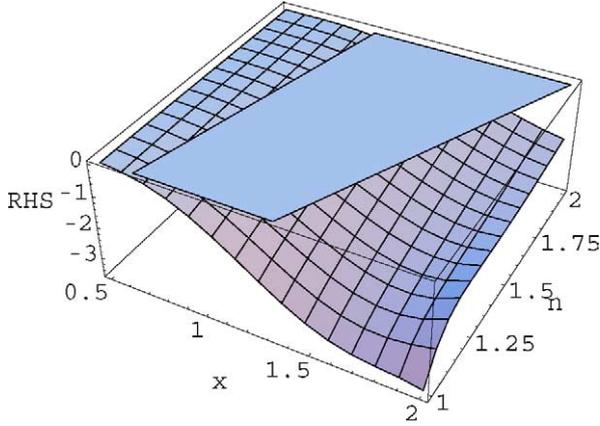


Fig. 5.  $Q$  mass RG flow, checking  $a_{IR} < a_{UV}$ , i.e.,  $0 > (x \frac{\partial}{\partial x} + n \frac{\partial}{\partial n} - 1)\hat{a}$  in the conformal window.

In the limit discussed above, this becomes

$$\hat{a}(x, n) > (1 - \epsilon)\hat{a}(x(1 + \epsilon), n(1 + \epsilon)) \tag{5.17}$$

with  $\epsilon \equiv 1/N_f > 0$ . The order  $\epsilon$  term then gives

$$0 > \left( x \frac{\partial}{\partial x} + n \frac{\partial}{\partial n} - 1 \right) \hat{a}(x, n), \tag{5.18}$$

which must hold for all  $x$  and  $n$  in the conformal window,  $3x > 1 + n > \frac{3}{2}x$ . In Fig. 5, we have plotted the function on the right-hand side of (5.18). The plane at the top of the graph indicates both the conformal window as well as where the right-hand side of (5.18) would equal 0, so  $a_{IR} < a_{UV}$  is indeed always satisfied in the conformal window.

Now consider giving a mass to one of the  $q'$  flavors, which is equivalent to giving, say  $S_{N'_f N'_f}$  a non-zero expectation value. This drives the theory in the IR to a similar RG fixed point, with  $N_c \rightarrow N_c$ ,  $N_f \rightarrow N_f$ , and  $N'_f \rightarrow N'_f - 1$ . In addition, the IR fixed point has  $2N'_f - 1$  decoupled free singlets, coming from the  $S_i N'_f$ . The  $a$ -theorem thus requires

$$a(N_c, N_f, N'_f) > a(N_c, N_f, N'_f - 1) + \frac{2}{9}(2N'_f - 1). \tag{5.19}$$

As above, we divide both sides by  $2N_f N'_f$  and take the term proportional to  $\epsilon \equiv 1/N_f > 0$  to write this inequality as

$$\hat{a} + \frac{\partial \hat{a}}{\partial n} > \frac{2}{9}n. \tag{5.20}$$

Once again, we find numerically that (5.20) is satisfied.

Now consider giving  $Q_{N_f} \tilde{Q}_{N_f}$  a non-zero expectation value. This leads to

$$a(N_c, N_f, N'_f) > a(N_c - 1, N_f - 1, N'_f) + \frac{2}{9}(2N_f + 2N'_f - 1), \tag{5.21}$$

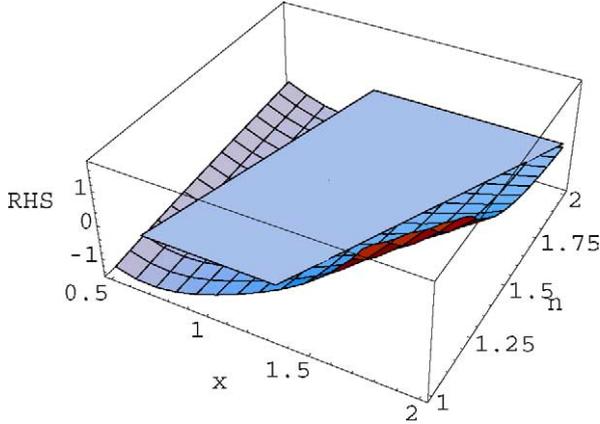


Fig. 6.  $Q$  vev Higgsing satisfies  $a_{\text{IR}} < a_{\text{UV}}$  in the conformal window.

with the last term from the uneaten  $SU(N_c - 1)$  singlets, which are IR-free. We can write (5.21) as

$$\hat{a}(x, n) > (1 - \epsilon)\hat{a}((x - \epsilon)(1 + \epsilon), n(1 + \epsilon)) + \frac{2}{9}\left(1 + \frac{1}{n}\right)\epsilon, \tag{5.22}$$

so, taking the  $\epsilon$  term,

$$0 > -\left(1 + (1 - x)\frac{\partial}{\partial x} - n\frac{\partial}{\partial n}\right)\hat{a} + \frac{2}{9}\left(1 + \frac{1}{n}\right). \tag{5.23}$$

This inequality is shown in Fig. 6, where there appears to be a region where it is violated. But within the conformal window, the inequality is indeed satisfied. (Outside of the conformal window, additional contributions of free fields come to the rescue.)

There is a similar Higgsing RG flow upon giving  $Q_f \tilde{Q}'_{N_f}$  an expectation value (i.e.,  $P$  in the dual), and  $a_{\text{UV}} > a_{\text{IR}}$  is

$$a(N_c, N_f, N'_f) > a(N_c - 1, N_f, N'_f - 1) + \frac{2}{9}(2N_f), \tag{5.24}$$

where there are fewer singlets than in (5.21) because some pair up with the  $S_{i'N'_f}$  to get a mass. We write (5.24) as

$$\hat{a}(x, n) > \left(1 - \frac{1}{n}\epsilon\right)\hat{a}(x - \epsilon, n - \epsilon) + \frac{2}{9n}\epsilon, \tag{5.25}$$

and hence

$$\left(\frac{1}{n} + \frac{\partial}{\partial x} + \frac{\partial}{\partial n}\right)\hat{a} > \frac{2}{9n}. \tag{5.26}$$

Once again, we numerically verified that this inequality is true.

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