The exact superconformal $R$-symmetry minimizes $\tau_{RR}$

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Abstract

We present a new, general constraint which, in principle, determines the superconformal $U(1)_R$ symmetry of 4d $\mathcal{N} = 1$ SCFTs, and also 3d $\mathcal{N} = 2$ SCFTs. Among all possibilities, the superconformal $U(1)_R$ is that which minimizes the coefficient, $\tau_{RR}$, of its two-point function. Equivalently, the superconformal $U(1)_R$ is the unique one with vanishing two-point function with every non-$R$ flavor symmetry. For 4d $\mathcal{N} = 1$ SCFTs, $\tau_{RR}$ minimization gives an alternative to $a$-maximization. $\tau_{RR}$ minimization also applies in 3d, where no condition for determining the superconformal $U(1)_R$ had been previously known. Unfortunately, this constraint seems impractical to implement for interacting field theories. But it can be readily implemented in the AdS geometry for SCFTs with AdS duals.

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1. Introduction

Our interest here will be in the coefficients $\tau_{IJ}$ of two-point functions of globally conserved currents $J^\mu_I$ ($I$ labels the various currents) in $d$-dimensional CFTs:

$$\langle J^\mu_I(x)J^\nu_J(y)\rangle = \frac{\tau_{IJ}}{(2\pi)^d} \left( \delta^2 \delta^{\mu\nu} - \partial^\mu \partial^\nu \right) \frac{1}{(x-y)^{2(d-2)}}. \tag{1.1}$$

The general form (1.1) of the correlator is completely fixed by conformal invariance, with the specific dynamics of the theory entering only in the coefficients $\tau_{IJ}$. Unitarity restricts $\tau_{IJ}$ to be a positive matrix (positive eigenvalues). For 4d CFTs, $\tau_{IJ}$ give [1,2] the violation of scale

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invariance, \( \langle T_{\mu} \rangle = \frac{1}{4} \tau_{IJ} (F^I)_{\mu\nu} (F^J)^{\mu\nu} \), when the global currents are coupled to background gauge fields.

We will here consider field theories with four supercharges: \( N = 1 \) in 4d, and \( N = 2 \) in 3d (one could also consider \( N = (2, 2) \) in 2d), and their renormalization group fixed point SCFTs (where there are an additional four superconformal supercharges). The stress tensor of these theories lives in a supermultiplet \( T_{\alpha \dot{\beta}}(x, \theta, \bar{\theta}) \) (in 4d Lorentz spinor notation; for \( d < 4 \) the dot on \( \dot{\beta} \) is unnecessary), which also contains a \( U(1)_R \) current—this is “the superconformal \( U(1)_R \) symmetry”. Supersymmetry relates this current and its divergence to the dilatation current and its divergence. The scaling dimension of chiral operators are related to their superconformal \( U(1)_R \) charge by

\[
\Delta = \frac{d - 1}{2} R. \tag{1.2}
\]

For a chiral superfield, writing \( \Delta = \frac{1}{2} d - 1 + \frac{1}{2} \gamma \), with \( \gamma \) the anomalous dimension, (1.2) yields

\[
R = \frac{d - 2}{d - 1} + \frac{1}{d - 1} \gamma. \tag{1.3}
\]

There are often additional non-\( R \) flavor currents, whose charges we will write as \( F_i \), with \( i \)-labeling the flavor symmetries. In superspace, these currents reside in a different kind of supermultiplet, which we will write as \( J_i(x, \theta, \bar{\theta}) \). When there are such additional flavor symmetries, the superconformal \( U(1)_R \) of RG fixed point SCFTs cannot be determined by the symmetries alone, as the \( R \)-symmetry can mix with the flavor symmetries. Some additional dynamical information is then needed to determine precisely which, among all possible \( R \)-symmetries, is the superconformal one, in the \( T_{\alpha \dot{\beta}} \) supermultiplet.

We will here present a new condition that, in principle, completely determines which is the superconformal \( U(1)_R \). We write the most general possible trial \( R \)-symmetry as

\[
R_t = R_0 + \sum_i s_i F_i, \tag{1.4}
\]

where \( R_0 \) is any initial \( R \)-symmetry, and \( F_i \) are the non-\( R \) flavor symmetries. The subscript “\( t \)” is for “trial”, with the \( s_i \) arbitrary real parameters. The superconformal \( R \)-symmetry, which we will write as \( R \) without the subscript, corresponds to some special values \( s_i^* \) of the coefficients in (1.4), that we would like to determine, \( R = R_t |_{s_j = s_j^*} \).

As we will discuss, the fact that the superconformal \( R \)-symmetry and the non-\( R \) flavor symmetries reside in different kinds of supermultiplets, implies that their current–current two-point function necessarily vanishes, \( \langle J_R^\mu(x) J_F^\nu(y) \rangle = 0 \), i.e.

\[
\tau_{Ri} = 0 \quad \text{for all non-\( R \) symmetries} \ F_i. \tag{1.5}
\]

This condition uniquely characterizes the superconformal \( R \)-symmetry among all possibilities (1.4). To see this, use (1.4) to write (1.5) as

\[
0 = \tau_{Ri} = \tau_{R0i} |_{s_j = s_j^*} = \tau_{R0i} + \sum_j s_j^* \tau_{ij} \quad \text{for all} \ i. \tag{1.6}
\]

Here \( \tau_{R0i} \) is the coefficient of the \( \langle J_R^\mu(x) J_F^\nu(y) \rangle \) current–current two-point function of the currents for \( R_0 \) and \( F_i \), and \( \tau_{ij} \) is the coefficient of the \( \langle J_{F_i}^\mu(x) J_{F_j}^\nu(y) \rangle \) of the current–current two-point function for the non-\( R \) flavor symmetries \( F_i \) and \( F_j \). The conditions (1.6) is a set of
linear equations which uniquely determines the $s_j^*$, if the coefficients $\tau_{R_0 i}$ and $\tau_{ij}$ are known. Unitarity implies that the matrix $\tau_{ij}$ is necessarily positive, with non-zero eigenvalues, so it can be inverted, and the solution of (1.6) is

$$s_j^* = -\sum_i (\tau^{-1})_{ij} \tau_{R_0 i}. \quad (1.7)$$

The conditions (1.6) can be phrased as a minimization principle: the exact superconformal $R$-symmetry is that which minimizes the coefficient $\tau_{R_i R_i}$ of its two-point function among all trial possibilities (1.4). Using (1.4), the coefficient of the trial $R$-current $R_i$ two-point function is a quadratic function of the parameters $s_j$:

$$\tau_{R_i R_i}(s) = \tau_{R_0 R_0} + 2 \sum_i s_i \tau_{R_0 i} + \sum_{ij} s_i s_j \tau_{ij}. \quad (1.8)$$

Our result (1.5) implies that the exact superconformal $R$-symmetry extremizes this function,

$$\frac{\partial}{\partial s_i} \tau_{R_i R_i}(s) \bigg|_{s_j = s_j^*} = 2\tau_{R_i} = 0. \quad (1.9)$$

The unique solution of (1.9) is a global minimum of the function (1.8) since

$$\frac{\partial^2}{\partial s_i \partial s_j} \tau(s) = 2\tau_{ij} > 0, \quad (1.10)$$

with the last inequality following from unitarity.

The value of $\tau_{R_i R_i}$ at its unique minimum is the coefficient $\tau_{RR}$ of the superconformal $R$-current two-point function. As is well known, supersymmetry relates this to the coefficient, “$c$”, of the stress tensor two-point function, $\tau_{RR} \propto c$; as we will discuss, the proportionality factor is

$$\tau_{RR} = \frac{(2\pi)^d}{d(d^2 - 1)(d^2 - 2)(d - 1)(d - 2)} C_T \quad \text{or, for } d = 4, \quad \tau_{RR} = \frac{16}{3} c. \quad (1.11)$$

$\tau_{RR}$ minimization immediately implies some expected results. For non-Abelian flavor symmetry, (1.5) is automatically satisfied for all flavor currents with traceless generators, if the superconformal $R$-symmetry is taken to commute with these generators. This shows, as expected, that the superconformal $R$-symmetry does not mix with such non-Abelian flavor symmetries. Similarly, (1.5) is automatically satisfied by any baryonic flavor currents which are odd under a charge conjugation symmetry, taking the superconformal $U(1)_R$ to be even under charge conjugation. So, as expected, the superconformal $U(1)_R$ does not mix with baryonic symmetries which are odd under a charge conjugation symmetry.

As a simple example of $\tau_{RR}$ minimization, consider a single, free, chiral superfield $\Phi$ in $d$-spacetime dimensions. The $R$-symmetry can mix with a non-$R$ $U(1)_F$ flavor current, under which $\Phi$ has charge 1 (the “Konishi current”). Write the general trial $R$-charges for the scalar and fermion components as $R(\phi) = R_i$, $R(\psi) = R_i - 1$. As we will review, the free field two-point function of this $R$-current is

$$\tau_{R_i R_i} = \frac{\Gamma(\frac{d}{2})^2 2^{d-2}}{(d-1)(d-2)} \left( \frac{1}{d-2} R_i^2 + (R_i - 1)^2 \right) \quad (1.12)$$
with the two terms the scalar and fermion contributions. Taking the derivative w.r.t. $R_t$,
\[ \tau_{R,F} = \frac{1}{2} \frac{d}{dR_t} \tau_{R,R_t} = \frac{\Gamma(d)^2 2^{d-2}}{(d-1)(d-2)} \left( \frac{R_t}{d-2} + R_t - 1 \right). \]  
(1.13)

Requiring $\tau_{RF} = 0$ then gives the correct result (1.3), with anomalous dimension $\gamma = 0$, for a free chiral superfield in $d$-spacetime dimensions.

The above considerations all apply independent of spacetime dimension; they are equally applicable for 4d $\mathcal{N} = 1$ SCFTs as with 3d $\mathcal{N} = 2$ SCFTs. For 4d $\mathcal{N} = 1$ SCFTs, there is already a known method for determining the superconformal $R$-symmetry: $a$-maximization [3]. It was shown in [3] that the $s_i^*$ can be determined by $a$-maximization, maximizing w.r.t. the $s_i$ in (1.4) the combination of ’t Hooft anomalies
\[ a_{\text{trial}}(R_t) = \frac{3}{32} (3 \text{Tr} R_t^3 - \text{Tr} R_t) \]  
(1.14)

(where we decided here to include the conventional normalization prefactor). For example, for a free 4d chiral superfield we locally maximize the function
\[ a_{\text{trial}}(R_t) = \frac{3}{32} (3(R_t - 1)^3 - (R_t - 1)). \]  
(1.15)

The local maximum of (1.15) is at $R = 2/3$, which indeed coincides with the global minimum of (1.12), but it is illustrative to see how the functions themselves differ.

$a$-maximization in 4d is much more powerful than $\tau_{R,R_t}$ minimization, because one can use the power of ’t Hooft anomaly matching to exactly compute $a_{\text{trial}}(R_t)$ (1.14), whereas the current two-point functions $\tau_{R,R_t}$ and $\tau_{ij}$ needed for $\tau_{R,R_t}$ minimization receive quantum corrections. Actually, once the exact superconformal $R$-symmetry is known, there is a nice way to evaluate $\tau_{ij}$ in terms of ’t Hooft anomalies [4]:
\[ \tau_{ij} = -3 \text{Tr} R F_i F_j, \]  
(1.16)
as we will review in what follows. (The result (1.16) generally cannot be turned around, and used as a way to determine the superconformal $U(1)_R$, because plugging (1.4) in (1.16) cannot always be inverted to solve for the $s^*$.)

In the context of the AdS/CFT correspondence, the criterion (1.6) for determining the superconformal $U(1)_R$ becomes more useful and tractable, because the AdS duality gives a weakly coupled dual description of $\tau_{R,R_t}$ and $\tau_{ij}$; these quantities become the coefficients of gauge field kinetic terms in the AdS bulk [5]. As we will discuss in a separate paper [6], these coefficients are computable by reducing SUGRA on the corresponding Sasaki–Einstein space. We will show in [6] that the conditions (1.6) are in fact equivalent to the “geometric dual of $a$-maximization” of Martelli, Sparks, and Yau [7].

There is no known analog of $a$-maximization for 3d $\mathcal{N} = 1$ SCFTs, and in 3d there is no useful analog of ’t Hooft anomalies and matching (aside from a $Z_2$ parity anomaly matching [8]). $\tau_{R,R_t}$ minimization gives an alternative to $a$-maximization in 4d, which applies equally well to 3d $\mathcal{N} = 2$ SCFTs.

$a$-maximization in 4d ties the problem of finding the superconformal $U(1)_R$ together with Cardy’s conjecture [9], that the conformal anomaly $a$ counts the degrees of freedom of a quantum field theory, with $a_{UV} > a_{IR}$ and $a_{CFT} > 0$. The result that $a$ is maximized over its possibilities implies that relevant deformations decrease $a$ [3], in agreement with Cardy’s conjecture. Unfortunately, we have not gained any new insights here into general RG inequalities from our $\tau_{RR}$
minimization result. Indeed, $\tau_{RR}$ is related to the conformal anomaly $c$ in 4d, which is known to not have any general behavior, neither generally increasing nor generally decreasing, in RG flows to the IR. And there is no analogous argument to that of [3], to conclude that $\tau_{RR}$ generally increases in RG flows in the IR, from the fact that $\tau_{RR}$ is minimized among all possibilities: the quantum corrections to $\tau_{RR}$, coming from the relevant interactions, can generally have either sign. (The difference is that the argument of [3] was based on 't Hooft anomalies, which do not get any quantum corrections for conserved currents.)

Our $\tau_{RR}$ minimization result applies for SCFTs at their RG fixed point. It would be interesting to extend $\tau_{RR}$ minimization to study RG flows away from the RG fixed point. Perhaps this can be done by using Lagrange multipliers, as in [10], to impose the constraint that one minimize only over currents that are conserved by the relevant interactions.

2. Current two point functions; free fields and normalization conventions

Two-point functions of currents and stress tensors for free bosons and fermions in $d$-spacetime dimensions were worked out, e.g. in [11]. To compare with [11], rewrite (1.1) as

$$\langle J_\mu^I(x) J_\nu^J(y) \rangle = \tau_{IJ} \frac{2(d-1)(d-2)}{(2\pi)^d} \frac{I_{\mu\nu}(x-y)}{(x-y)^{2(d-1)}}.$$  

(2.1)

with $I_{\mu\nu}(x) \equiv \delta_{\mu\nu} - 2x_\mu x_\nu(x^2)^{-1}$. The normalization conventions of [11] is

$$\langle J_\mu(x) J_\nu(0) \rangle = C_V I_{\mu\nu}(x), \quad \langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle = C_T I_{\mu\nu,\rho\sigma}(x),$$  

(2.2)

with $I_{\mu\nu,\rho\sigma}(x) = \frac{1}{2}(I_{\mu\rho}(x) I_{\nu\sigma}(x) + I_{\mu\sigma}(x) I_{\nu\rho}(x)) - d^{-1} \delta_{\mu\nu} \delta_{\rho\sigma}$. Thus $C_V = 2 \tau (d-1)(d-2)/(2\pi)^d$. With these normalizations, the coefficients (2.2) for a single complex scalar are

$$C_V = \frac{2}{d-2} \frac{1}{S_d}, \quad C_T = \frac{2d}{d-1} \frac{1}{S_d^2},$$  

(2.3)

where $S_d \equiv 2\pi^{\frac{1}{2}d}/\Gamma(\frac{1}{2}d)$ and the current was normalized to give $\phi$ and $\phi^*$ charges $\pm 1$. The coefficients for a free fermion having the same number of components as a 4d complex chiral fermion (half the components of a Dirac fermion) the coefficients are

$$C_V = 2 \frac{1}{S_d^2}, \quad C_T = \frac{d}{S_d^2}$$  

(2.4)

(4d) because we were here considering a fermion with the same number of components as the dimensional reduction of a 4d chiral fermion for all $d$).

More generally, let current $J_I(x)$ give charges $q_{I,b}$ to the complex bosons and charges $q_{I,f}$ to the chiral fermions. Using (2.3) and (2.4), we have

$$\tau^\text{free field}_{IJ} = \frac{\Gamma\left(\frac{d}{2}\right)^2 2^{d-2}}{(d-1)(d-2)} \left( \frac{1}{d-2} \sum_{\text{bosons } b} q_{I,b} q_{J,b} + \sum_{\text{fermions } f} q_{I,f} q_{J,f} \right).$$  

(2.5)

In particular, for a $U(1)_R$ symmetry, this gives (1.12). For $d = 4$, $\Gamma(d/2)^2 2^{d-2}/(d-1)(d-2) = 2/3$, so e.g. a 4d $U(1)_F$ non-$R$ symmetry which assigns charge $q$ to a single chiral superfield has $\tau^\text{free field}_{FF} = q^2$. 
3. Supersymmetric field theories

Supersymmetry relates the superconformal $R$-symmetry to the stress tensor: both reside in the supercurrent supermultiplet

$$ T_{\alpha\dot{\alpha}}(x,\theta,\bar{\theta}) \sim J_{R,\alpha\dot{\alpha}}(x) + S_{\alpha\dot{\alpha}\beta}(x)\theta^{\beta} + \bar{S}_{\alpha\dot{\alpha}\dot{\beta}}(x)\bar{\theta}^{\dot{\beta}} + T_{\alpha\dot{\alpha}\beta\dot{\beta}}(x)\theta^{\beta}\bar{\theta}^{\dot{\beta}} + \cdots, \quad (3.1) $$

whose first component is the superconformal $U(1)_R$ current and whose $\theta\bar{\theta}$ component is the stress energy tensor (we were omitting numerical coefficients here). Our notation is for the 4d case; similar results hold for 3d $\mathcal{N}=2$ theories, with $\bar{\theta}^{\dot{\alpha}}$ replaced with a second flavor of $\theta^{\alpha}$.

For superconformal theories, the stress tensor is traceless, and the superconformal $R$-current is conserved. As discussed in [12], the supercurrent two-point function is then of a completely determined form, with the only dependence on the theory contained in a single overall coefficient $C$:

$$ \langle T_{\alpha\dot{\alpha}}(z_1)T_{\beta\dot{\beta}}(z_2) = C \frac{(x_{z_1\bar{z}_2})(\alpha\dot{\beta})}{(x_{z_1\bar{z}_2})^{d/2}}, \quad (3.2) $$

see [12] for an explanation of the superspace notation in (3.2).

Expanding out (3.2) in superspace, the LHS includes both the $R$-current two-point function and the stress tensor two-point function. So (3.2) shows that the coefficient $C \propto \tau_{RR}$, and also $C \propto \tau_{CT}$, and so it follows that $\tau_{RR} \propto \tau_{CT}$. We could determine the precise coefficients in these relations by being careful with the coefficients in (3.1) and in expanding both sides of (3.2); instead we will fix these universal proportionality factors by considering the particular example of a free chiral superfield. Using (2.3) and (2.4) to get $\tau_{CT}$, and comparing with the free field value of $\tau_{RR}$ computed from (2.5), gives the general proportionality factor that we quoted in (1.11); e.g. for $d=3$ it is $\tau_{RR} = \pi^3 C_T/3$. In 4d, $C_T \propto c$, one of the conformal anomaly coefficients, and the proportionality can again be fixed by considering the case of a free 4d $\mathcal{N}=1$ chiral superfield, for which $c = 1/24$ and (2.5) gives $\tau_{RR} = 2/9$ (or a free 4d $\mathcal{N}=1$ vector superfield, for which $c = 1/8$ and (2.5) gives $\tau_{RR} = 2/3$); this gives the relation quoted in (1.11).

The non-$R$ global flavor currents $J^i(x)$ are the $\theta^{\alpha}\bar{\theta}^{\dot{\alpha}}$ components of superfields $J_i(x,\theta,\bar{\theta})$, whose first component is a scalar. We can write their two-point functions in superspace [12], with the coefficients given by that of the flavor current correlators, $\tau_{ij}$:

$$ \langle J_i(z_1)J_j(z_2) = \frac{\tau_{ij}}{(2\pi)^d(x_{z_1\bar{z}_2}^2)^{(d-2)/2}}, \quad (3.3) $$

In general $d$-dimensional CFTs, two-point functions of primary operators vanish unless the operators have conjugate Lorentz spin and the same operator dimension. Noting that the first component of the supermultiplet (3.1) has dimension $\Delta (T_{\alpha\dot{\alpha}}) = d-1$, and the first component of the current $J_i$ has dimension $\Delta (J_i) = d-2$ (since the $\theta^{\alpha}\bar{\theta}^{\dot{\alpha}}$ component is the current, with dimension $d-1$), the two-point function of the first components of these two different supermultiplets must vanish. Because there is no non-trivial nilpotent invariant for two-point functions [12], this implies that two-point function of the entire supermultiplets must vanish:

$$ \langle T_{\alpha\dot{\alpha}}(z_1)J_i(z_2) = 0. \quad (3.4) $$

I.e. the two-point function of any operator in the $T_{\alpha\dot{\alpha}}$ supermultiplet and any operator in the $J_i$ supermultiplet vanishes; in particular, this implies that the two-point function of the superconformal $U(1)_R$ current and all non-$R$ flavor currents necessarily vanish, $\tau_{RF} = 0$. We thus have
the general result (1.5), and this same argument applies equally for \( d = 4 \), \( \mathcal{N} = 1 \) as well as lower-dimensional SCFTs with the same number of supersymmetries.

3.1. 4d \( \mathcal{N} = 1 \) SCFTs: relating current correlators to ’t Hooft anomalies

The superspace version of an anomaly in the dilatation current is

\[
\hat{\nabla}^{\hat{\alpha}} T_{\alpha\dot{\alpha}} = \nabla_{\alpha} L_T ,
\]

(3.5)

with \( L_T \) the trace anomaly, which is the variation of the effective action with respect to the chiral compensator chiral superfield [13].

On a curved spacetime, there is the conformal anomaly

\[
\langle T_{\mu} \rangle = \frac{1}{120} \left( \frac{c(Weyl)^2}{4\pi^2} - \frac{a}{4}(Euler) \right)
\]

(3.6)

(there can also be an \( a^\prime \partial^2 R \) term, whose coefficient \( a^\prime \) is ambiguous, which was discussed in detail in [14]). The coefficient “\( c \)” is that of the stress tensor two-point function in flat space, whereas the coefficient “\( a \)” can be related to a stress tensor 3-point function in flat space. The superspace version of this anomaly, including also background gauge fields coupled to the superconformal \( R \)-current, is as in (3.5), with \( L_T = (cW^2 - a\Xi_c)/24\pi^2 \) [4]. Taking components of this superspace anomaly equation relates the conformal anomaly coefficients \( a \) and \( c \) to the \( \text{'t Hooft} \) anomalies of the superconformal \( U(1)_R \) symmetry [4]:

\[
a = \frac{3}{32} \left( 3 \text{Tr} \, R^3 - \text{Tr} \, R \right), \quad c = \frac{1}{32} \left( 9 \text{Tr} \, R^3 - 5 \text{Tr} \, R \right).
\]

(3.7)

An alternate derivation [12] of these relations follows from the fact that, in flat space, the 3-point function \( \langle T_{\alpha\dot{\alpha}}(z_1)T_{\beta\dot{\beta}}(z_2)T_{\gamma\dot{\gamma}}(z_3) \rangle \) is of a form that is completely determined by the symmetries and Ward identities, up to two overall normalization coefficients, with one linear combination of these coefficients proportional to the coefficient (3.2) of the \( T_{\alpha\beta} \) two-point function. In components, this relates the stress tensor three-point functions, and hence \( a \) and \( c \), and to the \( R \) current 3-point functions, and hence the \( \text{Tr} \, U(1)_R \) and \( \text{Tr} \, U(1)_R^3 \) \( \text{'t Hooft} \) anomalies, to these two coefficients. It follows that \( a \) and \( c \) can be expressed as linear combinations of \( \text{Tr} \, U(1)_R \) and \( \text{Tr} \, U(1)_R^3 \), and the coefficients in (3.7) can easily be determined by considering the special cases of free chiral and vector superfields.

Combining (1.11) and (3.7), we have

\[
\tau_{RR} = \frac{3}{2} \text{Tr} \, R^3 - \frac{5}{6} \text{Tr} \, R.
\]

(3.8)

It was also argued in [4] that the two-point functions \( \tau_{ij} \) of non-\( R \) flavor currents are related to \( \text{'t Hooft} \) anomalies, as

\[
\tau_{ij} = -3 \text{Tr} \, RF_i F_j.
\]

(3.9)

Again, this can be argued for either by turning on background fields, or by considering correlation functions in flat space. In the former method, one uses the fact that coupling background field strengths to the non-\( R \) currents leads to \( \Delta L_T = C_{ij} W_{\alpha i} W^{\alpha j} \), in (3.5), for some coefficients \( C_{ij} \). In components, (3.5) then gives \( \delta \langle T_{\mu}^{ij} \rangle \sim C_{ij} F_{\mu\nu i} F_{j\nu}^{\mu} \) and \( \delta (\partial_{\mu} J_{\mu}^R) \sim C_{ij} F_{\mu\nu i} F_{\mu j\nu}^R \). The former gives \( C_{ij} \sim \tau_{ij} \) and the latter gives \( C_{ij} \sim \text{Tr} \, RF_i F_j \), so \( \tau_{ij} \propto \text{Tr} \, RF_i F_j \). The coefficient in (3.9) is again easily determined by considering the special case of free field theory.
The alternate derivation would be to consider the flat space 3-point function of the stress tensor and two flavor currents, \( \langle T_{\alpha\beta}(z_1) J_i(z_2) J_j(z_3) \rangle \). It was shown in [1] that such 3-point functions are completely determined by the symmetries and Ward identities, up to two overall coefficients, and that one linear combination of these coefficients is proportional to the current–current two-point functions, and hence \( \tau_{ij} \). In our supersymmetric context, that same linear combination should be related by supersymmetry to \( \langle \partial_\mu J^\mu_R (x_1) J^\rho_F (x_2) J^\sigma_{F_j} (x_3) \rangle \), and hence to the Tr \( RF_i F_j \) \( \prime \)t Hooft anomaly.

The \( a \)-maximization [3] constraint on the superconformal \( R \)-symmetry follows from the fact that supersymmetry relates the Tr \( R^2 F_i \) and Tr \( F_i \) \( \prime \)t Hooft anomalies:

\[
9 \text{Tr} \ R^2 F_i - \text{Tr} \ F_i = 0, \tag{3.10}
\]

which again can be argued for either by considering again an anomaly with background fields, or by considering current correlation functions in flat space [3]. In the former method, one considers the anomaly of the flavor current coming from a curved background metric and background gauge field coupled to the superconformal \( R \)-current, \( \bar{\nabla}^2 J \propto \mathcal{W}^2 \). With the latter method, one uses the result of [12] that the flat space 3-point function \( \langle T_{\alpha\beta}(z_1) T_{\beta\gamma}(z_2) J_i(z_3) \rangle \) is completely determined by the symmetries and superconformal Ward identities, up to a single overall normalization constant.

We note that supersymmetry does not relate \( \tau_{Ri} \) to the \( \prime \)t Hooft anomaly Tr \( R^2 F_i \). Naively, one might have expected some such relation, in analogy with the above arguments, for example by trying to use (3.5) to relate a term \( \delta \langle T_{\mu}^\mu \rangle \sim \tau_{Ri} F_{R,\mu\nu} F_{\mu\nu}^R \) to a term \( \delta \langle \partial_\mu J^\mu_R \rangle \sim (\text{Tr} \ R^2 F_i) F_{R,\mu\nu} F_{\mu\nu}^R \), when background fields are coupled to both \( U(1)_R \) and \( U(1)_{F_i} \) currents. But there is actually no way to write such combined contributions of the \( U(1)_R \) and \( U(1)_{F_i} \) background fields to (3.5), because the former resides in the spin 3/2 chiral superfield strength \( \mathcal{W}_{a\beta\gamma} \), and the latter resides in the spin 1/2 chiral superfield strength \( W_{a\beta} \), and there is no way to combine the two of them into the spin zero chiral object \( L_T \). Likewise, in flat space, a relation between \( \tau_{Ri} \) and Tr \( R^2 F_i \) would occur if the 3-point function \( \langle T_{\alpha\beta}(z_1) T_{\beta\gamma}(z_2) J_i(z_3) \rangle \) includes a term proportional to Tr \( R^2 F_i \), were related to the two-point function \( \langle T_{\beta\gamma}(z_2) J_i(z_3) \rangle \), which is proportional to \( \tau_{Ri} \) (and, as we have argued above, vanishes). It sometimes happens that 3-point functions with a stress tensor are simply proportional to the 2-point function without the stress tensor, e.g. this is the case when the other two operators are chiral and anti-chiral primary [12]. But the \( \langle T_{\alpha\beta}(z_1) T_{\beta\gamma}(z_2) J_i(z_3) \rangle \) 3-point function in [12] is not related to the \( \langle T_{\beta\gamma}(z_2) J_i(z_3) \rangle \) two-point function. Indeed, the free field example discussed in the introduction illustrates that Tr \( R^2 F_i \) and \( \tau_{Ri} \) are not related by supersymmetry, as Tr \( R^2 F_i \neq 0 \) for this example but, as always, \( \tau_{Ri} = 0 \).

3.2. Using \( \tau_{Ri} = 0 \) to determine the superconformal \( R \)-symmetry

As discussed in the introduction, using (1.4), we have for a general trial \( R \)-symmetry

\[
\tau_{R;i} = \tau_{R;0i} + \sum_j s_j \tau_{ij}. \tag{3.11}
\]

Imposing \( \tau_{R;i} = 0 \) gives a set of linear equations, which determines the particular values \( s_j^* \) of the parameters for which the trial \( R \)-symmetry is the superconformal \( R \)-symmetry. As discussed in the introduction, this can equivalently be expressed as “the exact superconformal \( R \)-symmetry minimizes its two-point function coefficient \( \tau_{R,R}(s) \),” which is given by (1.8), and which we can
re-write using $\tau_{R_i} = 0$ for the superconformal $R$-symmetry as

$$\tau_{R_i}R_i(s) = \tau_{RR} + \sum_{ij} (s_i - s_i^*)(s_j - s_j^*)\tau_{ij},$$

(3.12)

making it manifest that $\tau_{R_i}R_i$ has a unique global minimum, when the $s_j$ are set to the particular value $s_j^*$. At $s_j = s_j^*$, the general $R$-symmetry $R_i$ in (1.4) becomes the superconformal $R$-symmetry, in the supermultiplet stress tensor $T_{a\bar{a}}$.

The function $\tau_{R_i}R_i(s)$ to minimize and the function $a_{\text{trial}}(s)$ to locally maximize in 4d are different. Let us compare the values of them and their derivatives at the extremal point $s_i = s_i^*$. For (3.11), we have:

$$\tau_{R_i}R_i|_{s^*} = \tau_{RR} = \frac{16}{3}c = \frac{3}{2} \text{Tr } R^3 - \frac{5}{6} \text{Tr } R,$$

$$\frac{\partial}{\partial s_i} \tau_{R_i}R_i|_{s^*} = 0,$$

$$\frac{\partial^2}{\partial s_i \partial s_j} \tau_{R_i}R_i = 2\tau_{ij},$$

(3.13)

whereas for $\frac{16}{3}a_{\text{trial}}(R_i) = \frac{1}{2}(3 \text{Tr } R_i^3 - \text{Tr } R_i)$ we have:

$$\frac{16}{3}a_{\text{trial}}(R_i)|_{s^*} = \frac{16}{3}a = \frac{3}{2} \text{Tr } R^3 - \frac{1}{2} \text{Tr } R,$$

$$\frac{\partial}{\partial s_i} \frac{16}{3}a_{\text{trial}}(R_i)|_{s^*} = \frac{9}{2} \text{Tr } R^2 F_i - \frac{1}{2} \text{Tr } F_i = 0,$$

$$\frac{\partial^2}{\partial s_i \partial s_j} \frac{16}{3}a_{\text{trial}}(R_i)|_{s^*} = 9 \text{Tr } R F_i F_j = -3\tau_{ij}.$$

(3.14)

The derivatives of both functions of $s$ vanish at the same values $s^*$. The values of the two functions in (3.13) and (3.14) differ, except for SCFTs with $a = c$, i.e. $\text{Tr } R = 0$, as is the case for SCFTs with AdS duals. The second derivatives of the functions in (3.13) and (3.14) are proportional, though with opposite sign, reflecting the fact that the exact superconformal $R$-symmetry minimizes $\tau_{R_i}R_i$ and maximizes $a_{\text{trial}}(R_i)$.

For the sake of comparison, let us also consider the function $\frac{16}{3}c_{\text{trial}}(R_i) = \frac{3}{2} R_i^3 - \frac{5}{6} R_i$; the value of this function and its first two derivatives at $R_i = R$, i.e. $s_i = s_i^*$, are

$$\frac{16}{3}c_{\text{trial}}(R_i)|_{s^*} = \frac{16}{3}c = \frac{3}{2} \text{Tr } R^3 - \frac{5}{6} \text{Tr } R,$$

$$\frac{\partial}{\partial s_i} \frac{16}{3}c_{\text{trial}}(R_i)|_{s^*} = \frac{9}{2} \text{Tr } R^2 F_i - \frac{5}{6} \text{Tr } F_i = -\frac{1}{3} \text{Tr } F_i,$$

$$\frac{\partial^2}{\partial s_i \partial s_j} \frac{16}{3}c_{\text{trial}}(R_i)|_{s^*} = 9 \text{Tr } R F_i F_j = -3\tau_{ij}.$$

(3.15)

The value of $\tau_{R_i}R_i$ and $c_{\text{trial}}(R_i)$ coincide at $R_i = R$. The value of their first derivatives differ for any flavor symmetries with $\text{Tr } F_i \neq 0$. General SCFTs can have flavor symmetries with $\text{Tr } F_i = 0$, but SCFTs with AdS duals always have $\text{Tr } F_i = 0$, and $\text{Tr } F_i = 0$ for general superconformal

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1 Quite generally, quiver gauge theories with only bi-fundamental matter have $\text{Tr } R = 0$, and hence $a = c$ [15,16].
quivers with only bi-fundamental matter [15,16]. The second derivatives in (3.15) differ from those of (3.13) by a factor of $-3/2$, coinciding with those of (3.14).

As a further comparison of $a$-maximization in 4d with $\tau_{RR}$ minimization, let us consider the equations for the case where the superconformal $U(1)_R$ can mix with a single non-$R$ flavor symmetry, $R_t = R_0 + s F$. $a$-maximization gives the value $s^*$ for the superconformal $U(1)_R$ as a solution of the quadratic equation

$$s^2 \text{Tr} F^3 + 2s \text{Tr} R_0 F^2 + \text{Tr} R_0^2 F - \frac{1}{9} \text{Tr} F = 0.$$  \hspace{1cm} (3.16)

$\tau_{RR}$ minimization gives $s^*$ as (1.7)

$$s^* = -\frac{\tau_0 F}{\tau_{FF}}.$$  \hspace{1cm} (3.17)

If $\text{Tr} F^3$ is non-zero, $s^*$ can also be obtained from (1.16), which here gives

$$s^* = -\frac{\text{Tr} R_0 F^2 + \frac{1}{3} \tau_{FF} \text{Tr} F^3}{\text{Tr} F^3}.$$  \hspace{1cm} (3.18)

For any given choice of $R_0$ and $F$, the value of $s^*$ obtained in these three different ways must agree. It would be nice to have a direct proof of the relations that this implies. E.g. comparing (3.18) with (3.17) gives the identity $\tau_0 F \text{Tr} F^3 = \tau_{FF} (\frac{1}{3} \tau_{FF} + \text{Tr} R_0 F^2)$ which, evidently, must hold for any choice of the $R$-symmetry $R_0$ (taking $R_0$ to equal the superconformal $U(1)_R$, both sides vanish).

4. SQCD Example

4d $\mathcal{N} = 1$ SCQD, with gauge group $SU(N_c)$ and $N_f$ fundamental and anti-fundamental flavors, $Q$ and $\tilde{Q}$, has been argued to flow to a SCFT in the IR for the flavor range $\frac{3}{2} N_c < N_f < 3 N_c$ [17]. Taking the superconformal $U(1)_R$ to be the anomaly free $R$-symmetry, the superconformal $R$-charges are $R(Q) = R(\tilde{Q}) = 1 - (N_c/N_f)$. Let us also consider the baryonic $U(1)_B$ symmetry, with $B(Q) = -B(\tilde{Q}) = 1/N_c$. Using the ‘t Hooft anomaly relations, we have

$$\tau_{RR} = \frac{3}{2} \text{Tr} R^3 - \frac{5}{6} \text{Tr} R = \frac{3}{2} \left[ N_c^2 - 1 - 2 \frac{N_c^4}{N_f^2} \right] + \frac{5}{6} \left[ N_c^2 + 1 \right],$$  \hspace{1cm} (4.1)

$$\tau_{BB} = -3 \text{Tr} RBB = 6.$$  \hspace{1cm} (4.2)

For $N_f \approx 3N_c$, where the RG fixed point is at weak coupling as in [18,19], these expressions reduce to the free field values.

There is a unique, anomaly free $U(1)_R$ symmetry that commutes with charge conjugation and the $SU(N_f)$ global symmetries. Our $\tau_{RR}$ minimization condition immediately leads to the same conclusion. $\tau_{RR}$ is minimized by having $\tau_{RR} = 0$ and $\tau_{RF} = 0$ for the $U(1)_B$ and $SU(N_f)$ global symmetries. Taking the $U(1)_R$ to be even under charge conjugation ensures that $\tau_{RR} = 0$, because the $U(1)_B$ current is odd, so charge conjugation symmetry gives $\tau_{RB} = -\tau_{BB}$. Likewise $\tau_{RF} = 0$ for the $SU(N_f)$ flavor currents, simply by the tracelessness of the generators, if $U(1)_R$ is taken to commute with $SU(N_f)$. 
5. Perturbative analysis

Consider a general 4d $\mathcal{N} = 1$ SCFT with gauge group $G$ and matter chiral superfields $Q_f$ in representations $r_f$ (of dimension $|r_f|$) of $G$, with no superpotential, $W = 0$. If the theory is just barely asymptotically free, there can be a RG fixed point at weak gauge coupling, where perturbative results can be valid. We will verify that the leading order perturbative expression for the anomalous dimension for fields,

$$\gamma_f(g) = -\frac{g^2}{4\pi^2}C(r_f) + O(g^4), \quad \text{i.e.} \quad R_f = \frac{2}{3} - \frac{g^2}{12\pi^2}C(r_f) + O(g^4)$$

agrees with $\tau_{RR}$ minimization. As standard, we define group theory factors as

$$\text{Tr}_{r_f}(T^AT^B) = T(r_f)\delta^{AB}, \quad \sum_{A=1}^{[G]} T_{r_f}^A T_{r_f}^A = C(r_f)1_{|r_f|\times|r_f|}, \quad \text{so} \quad C(r_f) = \frac{|G|T(r_f)}{|r_f|}.$$ 

The RG fixed point value $g_*$ of the coupling is determined by the constraint that the $R$-symmetry be anomaly free, $T(G) + \sum_f T(r_f)(R_f - 1) = 0$.

For the free UV theory, we minimize $\tau_{RR}$ over all possible $R$-charges $R_f$ of the matter chiral superfields, which are unconstrained for $g = 0$. As we discussed in the introduction, this gives the free field term $R_f^{(0)} = 2/3$. For $g \neq 0$, we write $R_f = R_f^{(0)} + R_f^{(1)} + \cdots$, with $R_f^{(1)}$ the $O(g^2)$ term that we would like to find via $\tau_{RR}$ minimization. For $g \neq 0$, $\tau_{RR}$ should be minimized subject to the constraint that the symmetries be anomaly free, i.e. we impose $\tau_{Ri} = 0$ over all anomaly free $U(1)_R$ and $U(1)_{F_i}$ symmetries, with $R$-charges $R_f$, and flavor $F_i$ charges $q_i(r_f)$ constrained to satisfy

$$T(G) + \sum_f T(r_f)(R_f - 1) = 0, \quad \text{and} \quad \sum_f T(r_f)q_i(r_f) = 0.$$ 

The $U(1)_R$ current assigns charges $R_f$ to the squark and $R_f - 1$ to the quarks components of $Q_f$. The $U(1)_{F_i}$ non-$R$ current assigns charges $q_i(r_f)$ to both the quark and squark components of $Q_f$. To compute $\tau_{RF_i}$, we consider the diagrams for the two-point function $\langle J_R(x_1)J_{F_i}(x_2) \rangle$. Because we take the currents to be conserved, they have vanishing anomalous dimension, so we anticipate that the various diagrams sum such that all apparent divergences cancel, and we were left with only finite contributions to $\tau_{RF_i}$. The $O(g^2)$ contributions can be written as

$$\tau_{RF_i}^{(1)} = \sum_f q_i(r_f) \left[ \frac{1}{3} R_f^{(1)} + \frac{2}{3} R_f^{(1)} \right] |r_f| + R_f^{(0)} \left( A_f^{(1)} + C_f^{(1)} \right) + (R_f^{(0)} - 1) \left( B_f^{(1)} + C_f^{(1)} \right).$$

The first two terms come from the leading diagrams, without interactions, exactly as in the free field result (1.13), but weighted by the $O(g^2)$ $R$-charges $R_f^{(1)}$. The first term is from connecting the currents at $x_1$ and $x_2$, with squark $\phi_f$ propagators, and the second from connecting them with quark $\psi_f$ propagators. The remaining contributions in (5.4) are $O(g^2)$ because they involve $O(g^2)$ interaction diagrams, and the $R$-charge weighting is thus taken as $R_f^{(0)} = 2/3$. Here $A_f^{(1)}$ is the contribution of all $O(g^2)$ 1PI diagrams connecting squark $\phi_f$, at $x_1$, to squark $\phi_f$ at $x_2$.
is similarly the contribution from all $O(g^2)$ diagrams connecting quark $\psi_f$ at $x_1$ to quark $\psi_f$ at $x_2$. We note that the group theory factors in all of these diagrams with $O(g^2)$ interactions is the same: $\text{Tr}_{rf} \sum_{A=1}^{G} T_{rf}^A T_{rf}^A = |r_f| C(r_f) = |G| T(r_f)$, i.e. $A_f^{(1)} = |G| T(r_f) A_f^{(1)}$, $B_f^{(1)} = |G| T(r_f) B_f^{(1)}$, and $C_f^{(1)} = |G| T(r_f) C_f^{(1)}$, where $A_f^{(1)}$, $B_f^{(1)}$, and $C_f^{(1)}$ are independent of the gauge group and representation, e.g. they could be computed in $U(1)$ SQED.

Using the second constraint in (5.3), $\sum_f T(r_f) q_i(r_f) = 0$, it immediately follows, without even having to compute $A_f^{(1)}$, $B_f^{(1)}$, and $C_f^{(1)}$, that their contributions to $\tau_{R_i}^{(1)}$ in (5.4) all vanish, for all anomaly free flavor symmetries $F_i$. The only contributions remaining in (5.4) are the $R_f^{(1)}$ ones, $\tau_{R_i}^{(1)} = \sum_f q_i(r_f) R_f^{(1)} |r_f|$. Our $\tau_{RR}$ minimization result implies that this must vanish, for any $q_i(r_f)$ satisfying the anomaly free constraint in (5.3). This implies that $R_f^{(1)} = \alpha C(r_f)$ for some constant $\alpha$ that is independent of the rep. $r_f$.

We have thus used $\tau_{R,R_i}$ minimization to re-derive the group theory dependence of the $O(g^2)$ term in the anomalous dimension (5.1). The coefficient is also fixed to agree with (5.1), at the fixed point $g_*$, by using the condition in (5.3) that the $R$-symmetry be anomaly free to solve for $\alpha$ (which is appropriately small when the matter content is such that the theory is barely asymptotically free). This reproduces the $O(g^2)$ contribution to the $R$-charges in (5.1) at the RG fixed point.

In principle, one could extend this analysis, and use $\tau_{RR}$ minimization to compute the anomalous dimensions to all orders. Using $a$-maximization [3] (assuming that the RG fixed point has no accidental symmetries), the general result can be written as [10]

$$ R_f = \frac{2}{3} \left( 1 + \frac{1}{2} \gamma_f(g_*) \right) = 1 - \frac{1}{3} \sqrt{1 + \frac{\lambda_* T(r_f) |r_f|}{|G|}} = 1 - \frac{1}{3} \sqrt{1 + \frac{\lambda_* C(r_f)}{|G|}}, \quad (5.5) $$

where $\lambda_*$ is a Lagrange multiplier [10], which is determined by the constraint that the $R$-symmetry be anomaly free, $T(G) + \sum_f T(r_f) (R_f - 1) = 0$. The result (5.5) was successfully compared [20,21] with the results for the anomalous dimensions to 3-loops of [22]. But, because current two-point functions get quantum corrections, $\tau_{RR}$ minimization does not seem to be a very efficient way to compute anomalous dimensions. Indeed, the higher-order quantum corrections to $\tau_{R_i}$ include diagrams where the currents at $x_1$ and $x_2$ are connected by renormalized propagators, with all quantum corrections from the interactions, and computing such $\tau_{R_i}$ contributions is already tantamount to directly computing the anomalous dimensions $\gamma_f(g)$.

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