Supplemental material to "Long-distance entangling gates between quantum dot spins mediated by a superconducting resonator"

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In these supplemental materials, we present a derivation of the effective N-spin Hamiltonian presented in the main text from the original N-DQD Hamiltonian. We use the $\{|+\rangle_i, |-\rangle_i\}$ orbital eigenbasis introduced into the main text, and we start with the Hamiltonian $H_N = H_0 + V$ where

$$H_0 = \hbar\omega_r a^\dagger a + \sum_{i=1}^{N}\left(\frac{1}{2}\hbar\omega_{ai}\tau_{zi} + \frac{1}{2}\hbar\omega_{zi}\sigma_{zi}\right), \tag{1}$$

and

$$\begin{aligned} V =& \sum_{i=1}^{N}(g_{xi}\sigma_{xi} + g_{zi}\sigma_{zi})(\cos(\theta_i)\tau_{zi} - \sin(\theta_i)\tau_{xi}) \\ &+ (a^\dagger + a)\sum_{i=1}^{N} g_{ACi}(1 - \cos(\theta_i)\tau_{zi} + \sin(\theta_i)\tau_{xi}). \end{aligned} \tag{2}$$

We also introduce the superoperator

$$\mathcal{L}(X) = \sum_{i,j} |i\rangle\langle i| \frac{X}{E_i - E_j} |j\rangle\langle j|, \tag{3}$$

where the sum is taken over all eigenstates $|i\rangle$ and $|j\rangle$ of H_0 and where $H_0|i\rangle = E_i|i\rangle$. Note that the action of this superoperator is only well-defined on operators which are purely off-diagonal in the H_0 eigenbasis.

We now wish to apply the Schrieffer-Wolff transformation e^S, where S is an anti-unitary operator such that $e^{-S}H_N e^S$ contains no block off-diagonal terms which couple the ground and excited orbital states (i.e. no τ_x or τ_y) to leading order [1]. We partition the perturbation V into block diagonal and block off-diagonal terms $V = V_d + V_{od}$ where

$$V_d = \sum_{i=1}^{N}\big((g_{xi}\sigma_{xi} + g_{zi}\sigma_{zi})\cos(\theta_i)\tau_{zi} + g_{ACi}(a^\dagger + a)(1 - \cos(\theta_i)\tau_{zi})\big), \tag{4}$$

and

$$V_{od} = \sum_{i=1}^{N}\sin(\theta_i)\big(g_{ACi}\big(a^\dagger + a\big) - (g_{xi}\sigma_{xi} + g_{zi}\sigma_{zi})\big)\tau_{xi}. \tag{5}$$

Noting that S will be at least first order in V, we apply the transformation to second order in V to obtain

$$\begin{aligned} e^{-S}H_N e^S \approx& H_0 + V_d + V_{od} + [S, H_0] \\ &+ [S, V_d] + [S, V_{od}] + \frac{1}{2}[S,[S,H_0]] \\ &+ \mathcal{O}(V^3). \end{aligned} \tag{6}$$

We demand that all block off-diagonal terms in this expression vanish to first order in V. This is clearly satisfied if $[S, H_0] + V_{od} = 0$, and it is easy to verify that this is the case if $S = \mathcal{L}(V_{od})$ [1]. Evaluating, we obtain

$$\begin{aligned} S = \frac{i}{\hbar}\sum_{j=1}^{N}\sin(\theta_j)(&\frac{g_{ACj}}{\omega_{aj}^2 - \omega_r^2}\left(\omega_{aj}\big(a^\dagger + a\big)\tau_{yj} - \omega_r\left(\frac{a^\dagger - a}{i}\right)\tau_{xj}\right) \\ &+ \frac{g_{xj}}{\omega_{aj}^2 - \omega_{zj}^2}(\omega_{zj}\sigma_{yj}\tau_{xj} - \omega_{aj}\sigma_{xj}\tau_{yj}) - \frac{g_{zj}}{\omega_{aj}}\sigma_{zj}\tau_{yj}). \end{aligned} \tag{7}$$

Now we define P to be the asymmetric projector onto the orbital ground subspace

$$P = \sum_{s_1,...,s_N,n} |\{s_1, ..., s_N\}, \{-, ..., -\}, n\rangle\langle\{s_1, ..., s_N\}, n| \,. \tag{8}$$

We take our transformed Hamiltonian and project onto the subspace to arrive at an effective spin-resonator Hamiltonian. Dropping additive constants,

$$\begin{aligned} H_N' &= P^\dagger e^{-S} H_N e^S P \\ &\approx P^\dagger \left(H_0 + V_d + \frac{1}{2}[S, V_{od}] \right) P \\ &= H_0' + V', \end{aligned} \tag{9}$$

where

$$\begin{aligned} H_0' = \hbar\omega_r' a^\dagger a + \sum_{j=1}^{N} (&\frac{1}{2}\hbar\omega_{zj}\sigma_{zj} - \cos(\theta_j)(g_{xj}\sigma_{xj} + g_{zj}\sigma_{zj}) \\ &+ \sin^2(\theta_j)\frac{g_{xj}\omega_{zj}}{\hbar\left(\omega_{aj}^2 - \omega_{zj}^2\right)}(g_{zj}\sigma_{xj} - g_{xj}\sigma_{zj})), \end{aligned} \tag{10}$$

and

$$\begin{aligned} V' = \sum_{j=1}^{N} (&\left(g_{xj}'\sigma_{xj} + g_{zj}'\sigma_{zj} + g_{ACj}(1 + \cos(\theta_j))\right)\left(a^\dagger + a\right) \\ &- \sin^2(\theta_j)\frac{\omega_{aj} g_{ACj}^2}{\hbar\left(\omega_{aj}^2 - \omega_r^2\right)}\left(a^{\dagger 2} + a^2\right)). \end{aligned} \tag{11}$$

We define the primed constants

$$\begin{aligned} \omega_r' &= \omega_r - 2\sum_{i=1}^{N} \frac{g_{ACi}^2}{\hbar^2}\sin^2(\theta_i)\frac{\omega_{ai}}{\omega_{ai}^2 - \omega_r^2}, \\ g_{xi}' &= g_{xi}\frac{g_{ACi}}{\hbar}\sin^2(\theta_i)\omega_{ai}\left(\frac{1}{\omega_{ai}^2 - \omega_{zi}^2} + \frac{1}{\omega_{ai}^2 - \omega_r^2}\right), \\ g_{zi}' &= g_{zi}\frac{g_{ACi}}{\hbar}\sin^2(\theta_i)\omega_{ai}\left(\frac{1}{\omega_{ai}^2} + \frac{1}{\omega_{ai}^2 - \omega_r^2}\right). \end{aligned}$$

It is convenient for the next step to work in a basis where H_0' is diagonal. This can be achieved with a y-rotation of the spin bases. This rotation simplifies the resulting expressions, but is sufficiently small that we can, to good approximation, treat the result as acting on the original spin basis. Performing the rotation yields

$$H_0' = \omega_r' a^\dagger a + \sum_{j=1}^{N} \frac{1}{2}\hbar\omega_{zj}'\sigma_{zj}', \tag{12}$$

where we define

$$\omega_{zi}' = \omega_{zi} - 2\cos(\theta_i)\frac{g_{zi}}{\hbar} + 2\omega_{zi}\left(\frac{\cos^2(\theta_i)}{\omega_{zi}^2} - \frac{\sin^2(\theta_i)}{\omega_{ai}^2 - \omega_{zi}^2}\right)\frac{g_{xi}^2}{\hbar^2},$$

and where $\vec{\sigma'}$ are the Pauli operators in this new spin basis. This rotation leaves the form of V' unchanged to second order in V.

At this point, we have derived both resonator energy corrections ω_r' as well as effective spin-resonator couplings [2]. Our goal, however, is to obtain an effective spin-spin Hamiltonian. To this end, we apply another Schrieffer-Wolff transformation $e^{S'}$, this time to eliminate the couplings between the ground and excited resonator states. We proceed as before, working this time to second order in V'. This approach is inconsistent mathematically, as we are ultimately after terms which are fourth order in V, but we have already neglected some terms which were third and fourth order in V. The ultimate justification of this step is the agreement with numerical results presented in the main text.

We once again define a superoperator

$$\mathcal{L}'(X) = \sum_{i,j} |i\rangle\langle i| \frac{X}{E'_i - E'_j} |j\rangle\langle j|, \tag{13}$$

where now the sum is taken over the eigenstates of H'_0. Noting that this time, the perturbation V' contains no block-diagonal terms, it is easy to show that the correct transformation is

$$\begin{aligned} S' =& \mathcal{L}'(V') \\ =& \frac{i}{\hbar}\sum_{j=1}^{N}\Big(\frac{g'_{xj}}{\omega_r'^2 - \omega_{zj}'^2}\left(\omega'_r\sigma'_{xj}\left(\frac{a^\dagger - a}{i}\right) - \omega'_{zj}\sigma'_{yj}\left(a^\dagger + a\right)\right) \\ &+ \frac{1}{\omega'_r}\left(\frac{a^\dagger - a}{i}\right)\left(g'_{zj}\sigma'_{zj} + g_{ACj}(1-\cos(\theta_j))\right) \\ &- \sin^2(\theta_j)\frac{\omega_{aj}g^2_{ACj}}{2\omega'_r\left(\omega^2_{aj} - \omega^2_r\right)}\left(\frac{a^{\dagger 2} - a^2}{i}\right)\Big). \end{aligned} \tag{14}$$

Now, let P' be the asymmetric projector onto the low-energy subspace

$$P' = \sum_{s_1,\dots,s_N} |\{s_1,\dots,s_N\},0\rangle\langle\{s_1,\dots,s_N\}|. \tag{15}$$

Again, we take our transformed Hamiltonian, project onto the low-energy subspace, and drop any additive constants to arrive at

$$\begin{aligned} H''_N =& P'^\dagger\left(H'_0 + \frac{1}{2}[S',V']\right)P' \\ =& \sum_{i=1}^{N}\left(\frac{1}{2}\hbar\omega'_{zi}\sigma'_{zi} + \frac{g'_{xi}\omega'_{zi}}{\hbar\left(\omega_r'^2 - \omega_{zi}'^2\right)}\left(g'_{zi}\sigma'_{xi} - g'_{xi}\sigma'_{zi}\right) - \left(\left(\frac{1}{\omega_r'^2 - \omega_{zi}'^2} + \frac{1}{\omega_r'^2}\right)\omega'_r g'_{xi}\sigma'_{xi} + \frac{2g'_{zi}}{\omega'_r}\sigma'_{zi}\right)\frac{\Sigma}{\hbar}\right) \\ &- \sum_{i\neq j}\frac{\omega'_r}{\hbar}\left(\frac{g'_{xj}}{\omega_r'^2 - \omega_{zj}'^2}\sigma'_{xj} + \frac{g'_{zj}}{\omega_r'^2}\sigma'_{zj}\right)\left(g'_{xi}\sigma'_{xi} + g'_{zi}\sigma'_{zi}\right), \end{aligned} \tag{16}$$

where we define

$$\Sigma = \sum_{j=1}^{N} g_{ACj}(1+\cos(\theta_j)).$$

This is our effective spin-spin Hamiltonian. It is once again desirable, however, to work in a basis in which the single-spin operators are diagonal. If we perform another very small spin basis rotation to diagonalize the first sum, working to second order in V', we finally obtain the desired effective N-spin Hamiltonian

$$H''_N = \sum_{i=1}^{N}\frac{1}{2}\hbar\omega''_{zi}\sigma''_{zi} - \sum_{i\neq j}\frac{\omega'_r}{\hbar}\left(\frac{g'_{xj}}{\omega_r'^2 - \omega_{zj}'^2}\sigma''_{xj} + \frac{g'_{zj}}{\omega_r'^2}\sigma''_{zj}\right)\left(g'_{xi}\sigma''_{xi} + g'_{zi}\sigma''_{zi}\right), \tag{17}$$

where $\vec{\sigma}''$ are the Pauli operators in the new spin basis, and where we have defined the dressed spin splittings

$$\omega''_{zi} = \omega'_{zi} - 2\frac{g'^2_{xi}\omega'_{zi}}{\hbar^2\left(\omega_r'^2 - \omega_{zi}'^2\right)} - 4\frac{g'_{zi}}{\hbar^2\omega'_r}\Sigma.$$

[1] S. Bravyi, D. P. DiVincenzo, and D. Loss, Annals of Physics **326**, 2793 (2011).
[2] F. Beaudoin, D. Lachance-Quirion, W. A. Coish, and M. Pioro-Ladrière, Nanotechnology **27**, 464003 (2016).