1.1 Use Gauss's theorem to prove the following:

a) Any excess charge placed on a conductor must lie entirely on its surface,

The charges inside a conductor will, almost by definition, redistribute themselves so that $\bar{E} = 0$ inside the conductor.

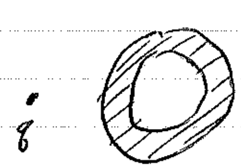
By Gauss' law, $\rho \propto \nabla \cdot \bar{E} = 0$ here,

hence $\rho = 0$ inside the conductor,
& any excess charge must be on the surface.

1.1, cont'd

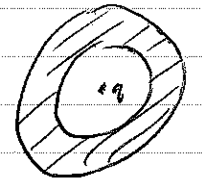
b) A closed hollow conductor shields its interior from fields due to charges outside, but does not shield its exterior from fields due to charges placed inside it.

Charges outside:



Charges in the conductor will redistribute themselves so that $\bar{E} = 0$ in the conductor and, since there are no sources in the interior, $\bar{E} = 0$ in the interior also, to match $\bar{E}$ in the conductor.

Charges inside:



Let S be a surface enclosing the hollow conductor. By Gauss's law,

$$\int_S \bar{E} \cdot \hat{n}\, da \propto q \neq 0$$

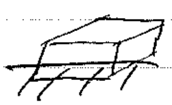
hence $\bar{E} \neq 0$ outside the conductor.

1.1, cont'd

c) The electric field at the surface of a conductor is normal to the surface and has a magnitude σ/ε_0, where σ is the charge density on the surface.

Infinitesimally close to the surface, it looks like an infinite plane, so by symmetry, $\bar{E}$ is perpendicular to the surface.

Apply Gauss's law to an infinitesimal box straddling the surface, of area Δa and ~~thickness Δt~~.



height Δt.

$$\int_S \bar{E}\cdot\hat{n}\,da = q/\varepsilon_0$$

$LHS = |E|\,\Delta a$, since $\bar{E}=0$ inside conductor & $\bar{E}\perp\hat{n}$ on the sides, so that the only contribution is from the top face.

$$RHS = \frac{1}{\varepsilon_0}(\sigma\,\Delta a)$$

$$\Rightarrow \boxed{|E| = \sigma/\varepsilon_0}$$

1.5 The time-averaged potential of a neutral hydrogen atom is given by

$$\bar{\Phi} = \frac{q}{4\pi\varepsilon_0}\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right)$$

where q is the magnitude of the electronic charge, and $\alpha^{-1} = a_0/2$, a_0 the Bohr radius.
Find the distribution of charge (both continuous & discrete) that will give this potential & interpret your result physically.

$$\nabla^2 \Phi = -\rho/\varepsilon_0, \quad \nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) \text{ for } \psi=\psi(r) \text{ only.}$$

$r \neq 0$:

$$\nabla^2\Phi = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\,\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right)\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\frac{\partial}{\partial r}\left(-\alpha e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right) + e^{-\alpha r}\left(\frac{\alpha}{2}\right)\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\left[(-\alpha)^2 e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right) + (-\alpha)e^{-\alpha r}\left(\frac{\alpha}{2}\right) - \alpha e^{-\alpha r}\left(\frac{\alpha}{2}\right)\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\left[\alpha^2 e^{-\alpha r}\left(\cancel{1}+\frac{\alpha r}{2}\right) - \cancel{\alpha^2 e^{-\alpha r}}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{\alpha^3}{2}e^{-\alpha r}$$

$$= -\rho/\varepsilon_0 \Rightarrow \rho = -\frac{q}{4\pi}\frac{\alpha^3}{2}e^{-\alpha r}$$

(cont'd)

1.5, cont'd

$\underline{r=0}$

For r close to 0,

$$\Phi \simeq \frac{q}{4\pi\varepsilon_0}\frac{1}{r}$$

$$\nabla^2\Phi = \frac{q}{4\pi\varepsilon_0}(-4\pi)\,\delta(\bar{x})$$

$$= -\rho/\varepsilon_0 \Rightarrow \rho = +q\,\delta(\bar{x})$$

so that altogether,

$$\boxed{\rho = q\,\delta(\bar{x}) - \frac{q}{4\pi}\frac{\alpha^3}{2}e^{-\alpha r}}$$

The first term is due to the proton at the center, and the second term is from the electron cloud.

1.10 Prove the mean value theorem:
For charge-free space, the value of the electrostatic potential Φ at any point is equal to the average of the potential over the surface of any sphere centered on that point.

In other words, for a sphere of radius R centered on $\bar{x}$, claim

$$\Phi(\bar{x}) = \frac{1}{4\pi R^2}\int \Phi \, R^2 d\Omega = \frac{1}{4\pi}\int \Phi \, d\Omega$$

for any R, where $\nabla^2 \Phi = 0$.

Apply Green's second identity with $\phi = \Phi$, $\psi = \frac{1}{r}$: (r = distance from $\bar{x}$)

$$\int_V d^3x' \Big(\Phi \underbrace{\nabla^2 \tfrac{1}{r}}_{= -4\pi \delta^3(\bar{x}-\bar{x}')} - \tfrac{1}{r} \underbrace{\nabla^2 \Phi}_{=0} \Big) = \int_{\partial V} \Big(\Phi \underbrace{\tfrac{\partial}{\partial r}\big(\tfrac{1}{r}\big)}_{= -\frac{1}{r^2}\big|_{r=R}} - \tfrac{1}{r}\tfrac{\partial}{\partial r}\Phi \Big) da$$

for V the sphere of radius R.

$$\Rightarrow -4\pi \, \Phi(\bar{x}) = -\frac{1}{R^2}\int_{\partial V} \Phi \, da - \int_{\partial V} \frac{1}{R}\frac{\partial \Phi}{\partial r} da$$

Now, $\displaystyle\int_{\partial V} \frac{1}{R}\frac{\partial \Phi}{\partial r} da = \frac{1}{R}\int_{\partial V} \frac{\partial \Phi}{\partial r} da \propto \int_{\partial V} E \cdot \hat{n} \, da = 0$ from Gauss' law

$$\Rightarrow \boxed{\Phi(\bar{x}) = \frac{1}{4\pi R^2}\int_{\partial V} \Phi \, da}$$

1.12 Prove Green's reciprocation theorem:

if Φ is the potential due to volume-charge density ρ in volume V
and surface-charge density σ on $S = \partial V$,
& Φ' is the potential due to volume charge density ρ', & surface charge-density σ',
where the boundary is a conductor, then

$$\int_V \rho \Phi' d^3x + \int_{\partial V} \sigma \Phi' da = \int_V \rho' \Phi d^3x + \int_{\partial V} \sigma' \Phi da$$

Apply Green's second identity with $\phi = \Phi$, $\psi = \Phi'$:

$$\int_V \left(\Phi \nabla^2 \Phi' - \Phi' \nabla^2 \Phi \right) d^3x = \int_{\partial V} \left[\Phi \partial_n \Phi' - \Phi' \partial_n \Phi \right] da$$

Now, $\nabla^2 \Phi = -\rho/\varepsilon_0$,
and since the boundary is a conductor,
from Gauss' law, ~~with~~ $E_n = -\partial_n \Phi = \sigma/\varepsilon_0$

$$\Rightarrow \int_V \left(\Phi' \rho - \Phi \rho' \right) d^3x = \int_{\partial V} \left(\Phi \sigma' - \Phi' \sigma \right) da$$

Result follows.