1.1 Use Gauss's theorem to prove the following:

d) Any excess charge placed on a conductor must lie entirely on its surface.

The charges inside a conductor will, almost by definition, redistribute themselves so that \( \vec{E} = 0 \) inside the conductor.

By Gauss' law, \( \rho \propto \nabla \cdot \vec{E} = 0 \) here,

hence \( \rho = 0 \) inside the conductor,

and any excess charge must lie on the surface.
6) A closed hollow conductor shields its interior from fields due to charges outside, but does not shield its exterior from fields due to charges placed inside it.

**Charges outside:**

Changes in the conductor will redistribute themselves so that $E = 0$ in the conductor and, since there are no sources in the interior, $E = 0$ in the interior also, to match $E$ in the conductor.

**Charges inside:**

Let $S$ be a surface enclosing the hollow conductor. By Gauss's law,

$$ \int_S E \cdot \nabla \, dA = \phi \neq 0 $$

Hence $E \neq 0$ outside the conductor.
1.1, cont'd

d) The electric field at the surface of a conductor is normal to the surface and has a magnitude $\sigma \varepsilon_0$, where $\sigma$ is the charge density on the surface.

In infinitesimally close to the surface, it looks like an infinite plane, so by symmetry, $E$ is perpendicular to the surface.

Apply Gauss's Law to an infinitesimal box straddling the surface, of area $\Delta A$ and height $\Delta t$.

$$\int_E \varepsilon_0 \hat{n} \, dA = \frac{\sigma}{\varepsilon_0}$$

LHS = $|E| \Delta A$, since $E=0$ inside conductor & $E \hat{n}$ on the sides, so that the only contribution is from the top face.

RHS = $\frac{1}{\varepsilon_0} (\sigma \Delta A)$

$\Rightarrow |E| = \sigma \varepsilon_0$
The time-averaged potential of a neutral hydrogen atom is given by

\[ \Phi = \frac{q}{4\pi\varepsilon_0} \frac{e^{-\alpha r}}{r} \left(1 + \frac{\alpha r}{2}\right) \]

where \( q \) is the magnitude of the electronic charge, and \( \alpha^{-1} = a_0/2 \), \( a_0 \) the Bohr radius.

Find the distribution of charge (both continuous & discrete) that will give this potential & interpret your result physically.

\[ \nabla^2 \Phi = -\rho/\varepsilon_0, \quad \nabla^2 \Phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r}\right) \text{ for } \Phi = \Phi(r) \text{ only.} \]

\[ r \neq 0: \]

\[ \nabla^2 \Phi = \frac{q}{4\pi\varepsilon_0} \frac{1}{r} \frac{\partial}{\partial r} \left(r e^{-\alpha r} \left(1 + \frac{\alpha r}{2}\right)\right) \]

\[ = \frac{q}{4\pi\varepsilon_0} \frac{1}{r} \frac{\partial}{\partial r} \left(-\alpha e^{-\alpha r} \left(1 + \frac{\alpha r}{2}\right) + e^{-\alpha r} \left(\frac{\alpha}{2}\right)\right) \]

\[ = \frac{q}{4\pi\varepsilon_0} \frac{1}{r} \left[ -\alpha^2 e^{-\alpha r} \left(1 + \frac{\alpha r}{2}\right) + (-\alpha) e^{-\alpha r} \left(\frac{\alpha}{2}\right) - \alpha e^{-\alpha r} \left(\frac{\alpha}{2}\right) \right] \]

\[ = \frac{q}{4\pi\varepsilon_0} \frac{1}{r} \left[ \alpha^2 e^{-\alpha r} \left(1 + \frac{\alpha r}{2}\right) - \alpha^2 e^{-\alpha r}\right] \]

\[ = \frac{q}{4\pi\varepsilon_0} \frac{\alpha^3}{2} e^{-\alpha r} \]

\[ = -\rho/\varepsilon_0 \implies \rho = -\frac{q}{4\pi} \frac{\alpha^3}{2} e^{-\alpha r} \]

(cont'd)
r = 0

For r close to 0,

\[ \Phi = -\frac{q}{4\pi \varepsilon_0} \frac{1}{r} \]

\[ \nabla^2 \Phi = \frac{q}{4\pi \varepsilon_0} (-4\pi) \delta(r) \]

\[ = -p(\varepsilon_0) \Rightarrow p = +q \delta(r) \]

so that altogether,

\[ p = q \delta(r) - \frac{q}{4\pi} \frac{x^3}{z^2} e^{-\alpha r} \]

The first term is due to the proton at the center, and the second term is from the electron cloud.
Prove the mean value theorem:

For charge-free space, the value of the electrostatic potential $\Phi$ at any point is equal to the average of the potential over the surface of any sphere centered on that point.

In the words, for a sphere of radius $R$ centered on $\hat{x}$, we claim

$$\Phi(\hat{x}) = \frac{1}{4\pi R^2} \int \Phi R^2 dS = \frac{1}{4\pi} \int \Phi dS$$

for any $R$, where $\nabla^2 \Phi = 0$.

Apply Green's second identity with $\Phi = \Phi$, $\Psi = \frac{1}{r}$: ($r$ = distance from $\hat{x}$)

$$\int_V \nabla' \cdot (\nabla \Phi - \frac{1}{r} \nabla \Phi) = \int_{\partial V} \left( \Phi \frac{\partial}{\partial n}(\frac{1}{r}) - \frac{1}{r} \frac{\partial}{\partial n} \Phi \right) dS$$

$$-4\pi \hat{k} \cdot (\hat{r} - \hat{x}) \int_0^R \Phi(\hat{r}) \hat{r} d\hat{r} - \frac{1}{R^2} \int_{\partial V} \Phi dS$$

For $V$ the sphere of radius $R$,

$$-4\pi \Phi(\hat{x}) = -\frac{1}{R^2} \int_{\partial V} \Phi dS - \int_{\partial V} \frac{1}{R} \frac{\partial}{\partial n} \Phi dS$$

Now, $\int_{\partial V} \frac{1}{R} \frac{\partial}{\partial n} \Phi dS = \frac{1}{R} \int_{\partial V} \frac{2\Phi}{R} dS = \oint_{\partial V} E \cdot d\hat{a} = 0$ from Gauss' law

$$\Rightarrow \Phi(\hat{x}) = \frac{1}{4\pi R^2} \int_{\partial V} \Phi dS$$
1.12 Prove Green's reciprocity theorem:

If \( \Phi \) is the potential due to volume charge density \( \rho \) in volume \( V \) and surface charge density \( \sigma \) on \( \partial V \), and

\( \Phi' \) is the potential due to volume charge density \( \rho' \) and surface charge density \( \sigma' \) where the boundary is a conductor, then

\[
\int_V \rho \, \Phi \, d^3x + \int_{\partial V} \sigma \, \Phi' \, da = \int_V \rho' \, \Phi \, d^3x + \int_{\partial V} \sigma' \, \Phi' \, da
\]

Apply Green's second identity since \( \rho = \Phi \), \( \Phi = \Phi' \):

\[
\int_V (\Phi' \nabla^2 \Phi - \Phi \nabla^2 \Phi') \, d^3x = \int_{\partial V} \left[ \nabla \cdot (\Phi' \nabla \Phi) - \Phi \nabla \cdot (\Phi' \nabla \Phi') \right] \, da
\]

Now, \( \nabla^2 \Phi = -\frac{1}{\epsilon_0} \),

and since the boundary is a conductor, from Gauss' law, \( E_n = -\Phi \nabla \Phi = 0 / \epsilon_0 \)

\[
\Rightarrow \int_V (\Phi' \rho - \Phi \rho') \, d^3x = \int_{\partial V} (\Phi \sigma' - \Phi' \sigma) \, da
\]

Result follows.