(a, b) only

6.2 The charge and current densities for a single point charge q can be written formally as

$$\rho(\bar{x}', t') = q\,\delta(\bar{x}' - \bar{r}(t')), \quad \bar{J}(\bar{x}', t') = q\,\bar{v}(t')\,\delta(\bar{x}' - \bar{r}(t'))$$

where $\bar{r}(t')$ is the charge's position at time t' and $\bar{v}(t')$ is ~~the~~ its velocity. In evaluating expressions involving the retarded time, one must put $t' = t_{ret} = t - R(t')/c$, where $\bar{R} = \bar{x} - \bar{r}(t')$.

a) Show that

$$\int d^3x'\,\delta(\bar{x}' - \bar{r}(t_{ret})) = \frac{1}{\kappa}$$

where $\kappa = 1 - \bar{v}\cdot\bar{R}/c$, evaluated at the retarded time.

Recall in one dimension

$$\delta(f(x)) = \frac{\delta(x - x_0)}{|f'(x_0)|} \quad \text{where } f(x_0) = 0, \; f'(x_0) \neq 0.$$

In higher dimensions,

$$\delta(\bar{y}(\bar{x})) = \frac{\delta(\bar{x} - \bar{x}_0)}{J} \quad \text{for } J = \det\left(\frac{\partial y_i}{\partial x_j}\right)\Bigg|_{x_0}$$

Here, $y_i = x_i' - r_i(t - c^{-1}|\bar{x} - \bar{x}'|)$

$$\frac{\partial y_i}{\partial x_j'} = \delta_{ij} - \left[\frac{dr_i}{dt}\right]_{ret} \frac{\partial}{\partial x_j'}\left(t - \frac{|\bar{x} - \bar{x}'|}{c}\right)$$

$$= \delta_{ij} - \left[\frac{dr_i}{dt}\right]_{ret}\left(\frac{x_j - x_j'}{c|\bar{x} - \bar{x}'|}\right)$$

(cont'd)

6.2 a), cont'd

$$\frac{\partial y_i}{\partial x_j'} = \delta_{ij} - [v_i]_{ret}\, \frac{1}{c} \left(\frac{x_j - r_j}{|\bar{x} - \bar{r}|} \right)_{ret}$$

$$= \delta_{ij} - [v_i]_{ret}\, \frac{1}{c} [\hat{R}_j]_{ret}$$

$$\det\left(\frac{\partial y_i}{\partial x_j'}\right) = \det \begin{pmatrix} 1 - v_1 \hat{R}_1/c & -v_2 \hat{R}_1/c & -v_3 \hat{R}_1/c \\ -v_1 \hat{R}_2/c & 1 - v_2 \hat{R}_2/c & -v_3 \hat{R}_2/c \\ -v_1 \hat{R}_3/c & -v_2 \hat{R}_3/c & 1 - v_3 \hat{R}_3/c \end{pmatrix}$$

$$= (1 - v_1 \hat{R}_1/c)\left[1 - v_2 \hat{R}_2/c - v_3 \hat{R}_3/c\right]$$

$$- \left[(1 - v_1 \hat{R}_2/c)(1 - v_3 \hat{R}_3/c) - v_1 v_3 \hat{R}_2 \hat{R}_3/c^2\right]\left[-v_2 \hat{R}_1/c\right]$$

$$+ \left[-v_3 \hat{R}_1/c\right]\left[v_1 \hat{R}_3/c\right]$$

$$= 1 - \bar{v} \cdot \hat{R}/c + \cancel{v_1 v_2 \hat{R}_1 \hat{R}_2/c^2} + \cancel{v_1 v_3 \hat{R}_1 \hat{R}_3/c^2}$$

$$- \cancel{v_1 v_2 \hat{R}_1 \hat{R}_2/c^2} - \cancel{v_1 v_3 \hat{R}_1 \hat{R}_3/c^2}$$

$$= 1 - \bar{v} \cdot \hat{R}/c$$

$$= \kappa$$

$$\Rightarrow \boxed{\int d^3x' \, \delta(\bar{x}' - \bar{r}(t_{ret})) = \frac{1}{\kappa}}$$

6.2, cont'd

b) Starting with the Jefimenko generalization of the Coulomb and Biot-Savart laws, use the expressions for the charge and current densities for a point charge and the result of part (a) to obtain the Heaviside-Feynman expressions for the electric and magnetic fields of a point charge,

$$\bar{E} = \frac{q}{4\pi\varepsilon_0}\left\{\left[\frac{\hat{R}}{\kappa R^2}\right]_{ret} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{R}}{\kappa R}\right]_{ret} - \frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{\bar{v}}{\kappa R}\right]_{ret}\right\}$$

$$\bar{B} = \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\bar{v}\times\hat{R}}{\kappa R^2}\right]_{ret} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\bar{v}\times\hat{R}}{\kappa R}\right]_{ret}\right\}$$

Use $\rho(\bar{x}',t') = q\,\delta^3(\bar{x}'-\bar{r}(t'))$, $\bar{J}(\bar{x}',t') = \rho\bar{v}(t')$, and

$$\bar{E}(\bar{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\left\{\frac{\hat{R}}{R^2}\left[\rho(\bar{x}',t')\right]_{ret} + \frac{\hat{R}}{cR}\left[\frac{\partial\rho(\bar{x}',t')}{\partial t'}\right]_{ret} - \frac{1}{c^2 R}\left[\frac{\partial\bar{J}(\bar{x}',t')}{\partial t'}\right]_{ret}\right\} \qquad (6.55)$$

Use $\left[\frac{\partial f}{\partial t'}\right]_{ret} = \frac{\partial}{\partial t}[f]_{ret}$,

$$= \frac{1}{4\pi\varepsilon_0}\int d^3x'\left\{\frac{\hat{R}}{R^2}\left[\rho(\bar{x}',t')\right]_{ret} + \frac{\hat{R}}{cR}\frac{\partial}{\partial t}\left[\rho(\bar{x}',t')\right]_{ret} - \frac{1}{c^2 R}\frac{\partial}{\partial t}\left[\bar{J}(\bar{x}',t')\right]_{ret}\right\}$$

$$= \frac{q}{4\pi\varepsilon_0}\int d^3x'\,\frac{\hat{R}}{R^2}\left[\delta(\bar{x}'-\bar{r}(t'))\right]_{ret} + \cancel{\frac{q}{4\pi\varepsilon_0}\int d^3x'}$$

$$+ \frac{q}{4\pi\varepsilon_0}\frac{\partial}{\partial t}\int d^3x'\,\frac{\hat{R}}{cR}\left[\delta(\bar{x}'-\bar{r}(t'))\right]_{ret} \cancel{-\frac{q}{4\pi}}$$

$$- \frac{q}{4\pi\varepsilon_0}\frac{\partial}{\partial t}\int d^3x'\,\frac{1}{c^2 R}\left[\delta(\bar{x}'-\bar{r}(t'))\,\bar{v}\right]_{ret}$$

6.2 b), cont'd

In the last lines of the previous page we used the fact that before integration, $\hat{R}$ & R depend on t' but not t, so the t derivative can be pulled out of the integral.

$$\Rightarrow \boxed{\bar{E}(\bar{x},t) = \frac{q}{4\pi\varepsilon_0}\left\{\left[\frac{\hat{R}}{\kappa R^2}\right]_{ret} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{R}}{\kappa R}\right]_{ret} - \frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{\bar{v}}{\kappa R}\right]_{ret}\right\}}$$

as desired, using part (a).

Next, we compute the magnetic field similarly:

$$\bar{B}(\bar{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\left\{[\bar{J}(\bar{x}',t')]_{ret} \times \frac{\hat{R}}{R^2} + \left[\frac{\partial \bar{J}(\bar{x}',t')}{\partial t'}\right]_{ret} \times \frac{\hat{R}}{cR}\right\} \qquad (6.56)$$

$$= \frac{\mu_0}{4\pi}\int d^3x'\left\{[\bar{J}(\bar{x}',t')]_{ret} \times \frac{\hat{R}}{R^2} + \frac{\partial}{\partial t}[\bar{J}(\bar{x}',t')]_{ret} \times \frac{\hat{R}}{cR}\right\}$$

$$= \frac{\mu_0}{4\pi}\int d^3x'\left[q\,\delta(\bar{x}'-\bar{r}(t'))\,\bar{v}\right]_{ret} \times \frac{\hat{R}}{R^2}$$

$$+ \frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\int d^3x'\left[q\,\delta(\bar{x}'-\bar{r}(t'))\,\bar{v}\right]_{ret} \times \frac{\hat{R}}{cR}$$

$$= \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\bar{v}\times\hat{R}}{\kappa R^2}\right]_{ret} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\bar{v}\times\hat{R}}{\kappa R}\right]_{ret}\right\}$$

using part (a)

6.4 A uniformly magnetized & conducting sphere of radius R and total magnetic moment $m = 4\pi M R^3/3$ rotates about its magnetization axis with angular speed ω. In the steady-state, no current flows in the conductor. The motion is nonrelativistic, and the sphere has no excess charge on it.

a) By considering Ohm's law in the moving conductor, show that the motion induces an electric field and a uniform volume charge density in the conductor,

$$\rho = -m\omega/\pi c^2 R^3.$$

Since the motion is nonrelativistic, we can approximate $\frac{\partial \bar{D}}{\partial t} = 0$,

& since there is no current in lab frame, $\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} = 0$.

In effect, if we are careful, we can treat this as a magnetostatics & electrostatics problem.

From §5.10, a uniformly magnetized sphere has $\bar{B} = \frac{2\mu_0}{3}\bar{M}$ (5.105) in its interior, & here,

$\bar{m} = \bar{M}\left(\frac{4\pi}{3}R^3\right)$, for $\bar{M}$ the magnetization, $\bar{M} = M\hat{k}$

Let $\bar{E}'$ denote the electric field in the rotating frame of the sphere, then,

$$\bar{E}' = \bar{E} + \bar{v}\times\bar{B} \qquad (5.142)$$

where $\bar{E}, \bar{B}$ are the fields in the lab frame, & $\bar{v} = \bar{\omega}\times\bar{r}$.

Ohm's law: $\bar{J} = \sigma \bar{E}' = \sigma(\bar{E} + \bar{v}\times\bar{B})$.

However, in the steady state, we are told no current flows

$\Rightarrow \bar{J} = 0$

$$\bar{E} = -\bar{v}\times\bar{B}$$

$$= -(\bar{\omega}\times\bar{r}) \times \frac{2\mu_0}{3}\left(\frac{4\pi}{3}R^3\right)^{-1}\bar{m}$$

6.4 d, cont'd

$$\bar{\omega} = \omega \hat{k}$$

$$(\bar{\omega} \times \bar{r}) \times \bar{m} = \omega m (\hat{k} \times \bar{r}) \times \hat{k}$$

$$= -\omega m \left((\hat{k} \cdot \bar{r}) \hat{k} - (\hat{k} \cdot \hat{k}) \bar{r} \right)$$

$$= \omega m (\bar{r} - z\hat{k})$$

$$\Rightarrow \boxed{\bar{E} = -\omega m \frac{2\mu_0}{4\pi} \frac{1}{R^3} (\bar{r} - z\hat{k}) = -\frac{\mu_0}{2\pi} \frac{\omega m}{R^3} \rho \hat{\rho} \quad \text{in cylindrical coordinates}}$$

From Gauss's law,

$$\rho = \varepsilon_0 \nabla \cdot \bar{E}$$

$$= \frac{\varepsilon_0}{\rho} \frac{\partial}{\partial \rho} (\rho E_\rho) = \frac{\varepsilon_0}{\rho} \frac{\partial}{\partial \rho} \left(-\frac{\mu_0}{2\pi} \frac{\omega m}{R^3} \rho^2 \right)$$

$$= -\frac{\varepsilon_0}{\cancel{\rho}} \frac{\mu_0}{2\pi} \frac{\omega m}{R^3} (2\cancel{\rho})$$

$$= -\frac{\varepsilon_0 \mu_0}{\pi} \frac{\omega m}{R^3} = -\frac{\omega m}{\pi c^2 R^3}$$

6.4, cont'd

b) Because the sphere is electrically neutral, there is no monopole electric field outside. Use symmetry argument to show that the lowest possible electric multipolarity is quadrupole.
Show that only a quadrupole field exists outside ~~and~~ and that the quadrupole moment tensor has nonvanishing components,
$Q_{33} = -4m\omega R^2/3c^2, \quad Q_{11} = Q_{22} = -Q_{33}/2.$

Symmetry argument:

- no electric monopole contribution
- Consider dipoles:

$$\bar{p} = \int d^3x' \; \bar{x}' \rho(\bar{x}') \qquad (4.8)$$

but ρ even, $\bar{x}$ odd under $z \mapsto -z$, hence $\bar{p}$ odd

Recall

$$E(\bar{x}) = \frac{3\hat{n}(\bar{p}\cdot\hat{n}) - \bar{p}}{4\pi\epsilon_0 |\bar{x}-\bar{x}_0|^3} \qquad (4.13)$$

$\Rightarrow$ if there is a dipole contribution, it must be odd under $z \to -z$

However, the result for $\bar{E}$ is even under $z \to -z$, hence <u>no</u> dipole contribution.

- Consider quadrupoles:

$$Q_{ij} = \int d^3x' \left(3x_i' x_j' - r'^2 \delta_{ij}\right) \rho(\bar{x}') \qquad \text{(omitting surface charge)}$$

$\rightarrow$ diagonal elements are <u>even</u> under $z \to -z$, consistent w/ symmetry properties of $\bar{E}$ in this case

$\boxed{\therefore \text{ lowest possible contribution is from } \underline{\text{quadrupole}}.}$

6.4 b), cont'd

Let's compute the diagonal component of Q_{ij}.
Now, in principle this should receive contributions from both the bulk charge density ρ as well as surface charges, but we don't know the surface charges, so we can't compute Q_{ij} directly.

Instead, we can infer it from the electric field.

We have seen that inside the sphere,

$$\bar{E} = -\frac{\mu_0}{2\pi}\frac{\omega m}{R^3}\left(\bar{r} - z\hat{k}\right)$$

$$= -\frac{\mu_0}{2\pi}\frac{\omega m}{R^3} r\left(\sin\theta\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath}\right)$$

$$= -\frac{\mu_0}{2\pi}\frac{\omega m}{R^3} r\left(\sin^2\theta\,\hat{r} + \sin\theta\cos\theta\,\hat{\theta}\right)$$

using $\hat{r} = \sin\theta\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{k}$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{\imath} + \cos\theta\sin\phi\,\hat{\jmath} - \sin\theta\,\hat{k}$$

We need to find $\bar{E}$ outside the sphere.

The general solution, valid as $r \to \infty$, (omitting the monopole term) has the form

$$\bar{E} = -\nabla\left[\sum_{\ell=1}^{\infty} B_\ell\, r^{-(\ell+1)} P_\ell(\cos\theta)\right]$$ (using cylindrical symmetry to restrict to $m=0$)

((As previously remarked, we can approximate this problem as electrostatics.)

$$= -\hat{r}\frac{\partial}{\partial r}\left[\sum_{\ell=1}^{\infty} B_\ell\, r^{-(\ell+1)} P_\ell(\cos\theta)\right] - \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}\left[\sum_{\ell=1}^{\infty} B_\ell\, r^{-(\ell+1)} P_\ell(\cos\theta)\right]$$

6.4 b), cont'd

To solve for the B_ℓ, we require that the tangential component of $\vec{E}$ be continuous across the boundary. So, match $\hat{\theta}$ component:

$$\Rightarrow -\frac{\mu_0}{2\pi}\frac{\omega m}{R^3} r\sin\theta\cos\theta\Big|_{r=R} = -\frac{1}{r}\frac{\partial}{\partial\theta}\left[\sum_{\ell=1}^{\infty} B_\ell r^{-(\ell+1)} P_\ell(\cos\theta)\right]\Big|_{r=R}$$

Use $P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2-1)$
$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$$\Rightarrow \frac{\partial}{\partial\theta} P_1(\cos\theta) = -\sin\theta$$

$$\frac{\partial}{\partial\theta} P_2(\cos\theta) = \frac{1}{2}\frac{\partial}{\partial\theta}(3\cos^2\theta - 1) = \frac{3}{2}(2)\cos\theta(-\sin\theta)$$

$$\frac{\partial}{\partial\theta} P_3(\cos\theta) = \frac{1}{2}\frac{\partial}{\partial\theta}(5\cos^3\theta - 3\cos\theta)$$

$$= \frac{1}{2}\left(5\cdot 3\cos^2\theta(-\sin\theta) + 3\sin\theta\right)$$

$$\Rightarrow -\frac{\mu_0}{2\pi}\frac{\omega m}{R^2}\sin\theta\cos\theta = -\frac{1}{R}\Big[B_1 R^{-2}(-\sin\theta)$$
$$+ B_2 R^{-3}(-3)\sin\theta\cos\theta$$
$$+ B_3 R^{-4}\left(\tfrac{1}{2}\right)\left[-15\cos^2\theta + 3\sin\theta\right] + \cdots\Big]$$

Only way to match both sides for general θ:
only B_2 is nonzero, and

$$+3B_2 R^{-4} = -\frac{\mu_0}{2\pi}\frac{\omega m}{R^2} \Rightarrow B_2 = -\frac{\mu_0}{6\pi}\omega m R^2$$

6.4 b), cont'd

Then outside the sphere,

$$E_r = -\frac{\partial}{\partial r}\left[B_2 r^{-3} P_2(\cos\theta)\right] = 3B_2 r^{-4} P_2(\cos\theta)$$

$$= -\frac{\mu_0}{2\pi}\omega m \frac{R^2}{r^4} P_2(\cos\theta)$$

$$E_\theta = \frac{B_2}{r^4}(-3)\sin\theta\cos\theta$$

$$= +\frac{\mu_0}{2\pi}\omega m \frac{R^2}{r^4}\sin\theta\cos\theta$$

Let's compare to the electric field from a multipole:

$$E_r = \frac{(l+1)}{2l+1}\frac{1}{\varepsilon_0} q_{lm} \frac{Y_{lm}(\theta,\phi)}{r^{l+2}} \qquad (4.11)$$

Hence for $l=2$, $m=0$,

$$E_r = \frac{3}{5}\frac{1}{\varepsilon_0} q_{20}\frac{Y_{20}}{r^4} = \frac{3}{5}\frac{1}{\varepsilon_0} q_{20}\frac{1}{r^4}\left(\frac{5}{4\pi}\right)^{1/2} P_2(\cos\theta)$$

which matches the field above if

$$\frac{3}{5}\frac{1}{\varepsilon_0} q_{20}\left(\frac{5}{4\pi}\right)^{1/2} = -\frac{\mu_0}{2\pi}\omega m R^2$$

$$\Rightarrow q_{20} = -\frac{5}{3}\frac{\varepsilon_0\mu_0}{2\pi}\left(\frac{4\pi}{5}\right)^{1/2}\omega m R^2 = -\frac{1}{3}\left(\frac{5}{\pi}\right)^{1/2}\frac{\omega m R^2}{c^2}$$

$$= -\frac{2}{3}\left(\frac{5}{4\pi}\right)^{1/2}\frac{\omega m R^2}{c^2}$$

6.4 b), cont'd

Let's convert q_{20} to Q_{ij}, from their def'ns:

$$Q_{ij} = \int d^3x' \rho(\bar{x}')\left(3x_i' x_j' - r'^2 \delta_{ij}\right)$$

$$q_{lm} = \int d^3x' \rho(\bar{x}') r'^l Y_{lm}^*(\theta', \phi')$$

$$\Rightarrow q_{20} = \int d^3x' \rho(\bar{x}') r'^2 Y_{20}^*(\theta', \phi')$$

$$= \int d^3x' \rho(\bar{x}') r'^2 \left(\frac{5}{4\pi}\right)^{1/2} P_2(\cos\theta)$$

$$= \left(\frac{5}{4\pi}\right)^{1/2} \int d^3x' \rho(\bar{x}') \frac{r'^2}{2} \left(3\cos^2\theta - 1\right)$$

$$= \frac{1}{2}\left(\frac{5}{4\pi}\right)^{1/2} \int d^3x' \rho(\bar{x}') \left(3z'^2 - r'^2\right)$$

$$= \frac{1}{2}\left(\frac{5}{4\pi}\right)^{1/2} Q_{33}$$

$$\Rightarrow \boxed{Q_{33} = -\frac{4}{3}\frac{\omega m R^2}{c^2}}$$

From symmetry, $Q_{11} = Q_{22}$,
and tracelessness implies

$$Q_{11} + Q_{22} + Q_{33} = 0$$

$$\Rightarrow \boxed{Q_{11} = Q_{22} = -\frac{1}{2} Q_{33}}$$

6.4, cont'd

c) By considering the radial electric fields inside & outside the sphere, show that the necessary surface-charge density $\sigma(\theta)$ is

$$\sigma(\theta) = \frac{1}{4\pi R^2}\frac{4m\omega}{3c^2}\left[1 - \frac{5}{2}P_2(\cos\theta)\right]$$

Use $\quad (\bar{E}_2 - \bar{E}_1)\cdot\hat{n} = \dfrac{\sigma}{\varepsilon_0} \qquad (1.22)$

$$\bar{E}_{out}\cdot\hat{n} = E_r\Big|_R = -\frac{\mu_0}{2\pi}\,\omega m\,\frac{1}{R^2}\,P_2(\cos\theta)$$

$$\bar{E}_{in}\cdot\hat{n} = E_r\Big|_R = -\frac{\mu_0}{2\pi}\,\frac{\omega m}{R^2}\,\sin^2\theta$$

$$\Rightarrow \boxed{\sigma =} \ \varepsilon_0\left(\bar{E}_{out}\cdot\hat{n} - \bar{E}_{in}\cdot\hat{n}\right)$$

$$= -\frac{\mu_0\varepsilon_0}{2\pi R^2}\,\omega m\left(P_2(\cos\theta) - \sin^2\theta\right)$$

$$= \frac{1}{2\pi R^2}\frac{\omega m}{c^2}\left[1 - \cos^2\theta - \frac{1}{2}(3\cos^2\theta - 1)\right]$$

$$= \frac{1}{2\pi R^2}\frac{\omega m}{c^2}\left[\frac{3}{2} - \frac{5}{2}\cos^2\theta\right]$$

$$= \frac{1}{2\pi R^2}\frac{\omega m}{c^2}\left[-\frac{5}{3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \frac{3}{2} - \frac{5}{6}\right]$$

$$= \frac{1}{2\pi R^2}\frac{\omega m}{c^2}\left[\frac{2}{3} - \frac{5}{3}P_2(\cos\theta)\right] = \frac{1}{2\pi R^2}\frac{\omega m}{c^2}\frac{2}{3}\left[1 - \frac{5}{2}P_2(\cos\theta)\right]$$

$$\boxed{= \frac{1}{4\pi R^2}\frac{4m\omega}{3c^2}\left[1 - \frac{5}{2}P_2(\cos\theta)\right]}$$

6.4, cont'd

d) The rotating sphere serves as a unipolar induction device if a stationary circuit is attached by a slip ring to the pole and a sliding contact to the equator. Show that the line integral of the electric field from the equator contact to the pole contact (by any path) is

$$\mathcal{E} = \mu_0 m \omega / 4\pi R.$$

$$\mathcal{E} = -\int \bar{E} \cdot d\bar{l}$$

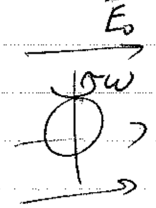
$$= -\int_0^{\pi/2} E_\theta \, (R\,d\theta)$$

$$= \frac{\mu_0}{2\pi} \frac{m\omega}{R^2} R \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta$$

$$= \frac{\mu_0}{2\pi} \frac{m\omega}{R} \left. \frac{\sin^2\theta}{2} \right|_0^{\pi/2}$$

$$= \boxed{\frac{\mu_0}{4\pi R} m\omega}$$

6.8 A dielectric sphere of dielectric constant ε and radius a is located at the origin. There is a uniform applied electric field E_0 in the x direction. The sphere rotates with an angular velocity ω about the z axis. Show that there is a magnetic field $\bar{H} = -\nabla \Phi_M$, where



$$\Phi_M = \frac{3}{5}\left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)\varepsilon_0 E_0 \omega \left(\frac{a}{r_>}\right)^5 xz$$

where $r_> = \max(r, a)$. The motion is nonrelativistic.

From §4.4, for a dielectric sphere in a constant electric field background, the polarization is

$$\bar{P} = 3\varepsilon_0 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) E_0 \hat{i} \qquad (4.57)$$

Corresponding volume charge density $\propto \nabla \cdot \bar{P} = 0$

Surface charge density $\sigma = \hat{n} \cdot \bar{P} = \hat{r} \cdot \bar{P}$, use $\hat{r} \cdot \hat{i} = \sin\theta \cos\phi$

Given these charges, we can use the rotational velocity of the sphere to derive current, then magnetic fields.

Surface ~~con~~ current density is

$$\bar{K} = \sigma \bar{v} = (\hat{r} \cdot \bar{P})(\omega \hat{k} \times \hat{r} r)$$

$$= \bar{M} \times \hat{r}$$

for $\bar{M} = \omega r (\hat{r} \cdot \bar{P}) \hat{k}$

$$= 3\varepsilon_0 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) E_0 \omega r \sin\theta \cos\phi \, \hat{k}$$

$$= 3\varepsilon_0 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) E_0 \omega x \, \hat{k}$$

6.8, cont'd

$$\Phi_M(\bar{x}) = -\frac{1}{4\pi}\int_V d^3x' \frac{\nabla'\cdot\bar{M}(\bar{x}')}{|\bar{x}-\bar{x}'|} + \frac{1}{4\pi}\int_S da' \frac{\hat{n}'\cdot\bar{M}(\bar{x}')}{|\bar{x}-\bar{x}'|} \qquad (5.100)$$

Here,

$$\nabla\cdot\bar{M} = 0$$

$$\hat{n}\cdot\bar{M} = \hat{r}\cdot\bar{M}$$

$$= 3\varepsilon_0\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)E_0\,\omega\, x\cos\theta$$

thus

$$\Phi_M = \frac{1}{4\pi}\int_S da' \frac{\alpha\, x'\cos\theta'}{|\bar{x}-\bar{x}'|} \qquad \text{for } \alpha = 3\varepsilon_0\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)E_0\,\omega$$

$$= \alpha\frac{a^2}{4\pi}\int d\Omega' \frac{x'\cos\theta'}{|\bar{x}-\bar{x}'|}$$

Use $\dfrac{1}{|\bar{x}-\bar{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\dfrac{1}{2l+1}\dfrac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$

$$x\cos\theta = -\sqrt{\frac{8\pi}{15}}\left(Y_{21} + Y_{21}^*\right)\left(\frac{1}{2}\right)$$

$$= -\sqrt{\frac{2\pi}{15}}\left(-Y^*_{2,-1} + Y^*_{21}\right)$$

hence

$$\int d\Omega' \frac{x'\cos\theta'}{|\bar{x}-\bar{x}'|} = \frac{4\pi r'}{5}\frac{r_<^2}{r_>^3}\left(-\sqrt{\frac{2\pi}{15}}\right)\Big(\underbrace{-Y_{2,-1}(\theta,\phi)}_{=+Y^*_{21}(\theta,\phi)} + Y_{2,1}(\theta,\phi)\Big)$$

$$(r' = a)$$

6.8, cont'd

$$\Phi_M = \alpha \frac{a^2}{4\pi} \int d\Omega' \frac{x' \cos\theta'}{|\vec{x} - \vec{x}'|}$$

$$= \alpha \frac{a^2}{\cancel{4\pi}} \frac{\cancel{4\pi} a}{5} \frac{r_<^2}{r_>^3} \left(-\sqrt{\frac{2\pi}{15}}\right) (2) \left(-\sqrt{\frac{15}{8\pi}}\right) \sin\theta \cos\theta \cos\phi$$

$$= 3\varepsilon_0 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) E_0 \omega \frac{a^3}{5} \frac{r_<^2}{r_>^3} \underbrace{\sin\theta \cos\theta \cos\phi}_{= \frac{xz}{r^2}}$$

Note:

$a < r \Rightarrow r_< = a, \; r_> = r$

$$\frac{a^3}{r^2} \frac{r_<^2}{r_>^3} = \frac{a^3}{r^2} \frac{a^2}{r^3} = \left(\frac{a}{r}\right)^5 = \left(\frac{a}{r_>}\right)^5$$

$a > r \Rightarrow r_< = r, \; r_> = a$

$$\frac{a^3}{r^2} \frac{r_<^2}{r_>^3} = \frac{a^3}{r^2} \frac{r^2}{a^3} = 1 = \left(\frac{a}{r_>}\right)^5$$

Thus,

$$\boxed{\Phi_M = \frac{3}{5} \varepsilon_0 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) E_0 \omega \left(\frac{a}{r_>}\right)^5 xz}$$

6.15 If a conductor or semiconductor has current flowing in it because of an applied electric field, and a transverse magnetic field is applied, there develops a component of the electric field in the direction orthogonal to both the applied electric field and the magnetic field, resulting in a voltage difference between the sides of the conductor. This is the Hall effect.

a) Use the known properties of electromagnetic fields under rotations and spatial reflections and the assumption of Taylor series expansions about zero magnetic field strength to show that for an isotropic medium, the generalization of Ohm's law, correct to second order in the magnetic field, must have the form

$$\bar{E} = \rho_0 \bar{J} + R(\bar{H}\times\bar{J}) + \beta_1 H^2 \bar{J} + \beta_2 (\bar{H}\cdot\bar{J})\bar{H}$$

All possible terms to linear order in $\bar{J}$, second order in $\bar{H}$:

$$\bar{J},\ \bar{H},\ \bar{H}\times\bar{J},\ (\bar{H}\cdot\bar{H})\bar{J},\ (\bar{H}\cdot\bar{J})\bar{H}$$

Know $\bar{E}$ is a (true) vector, as is $\bar{J}$, but $\bar{H}$ is an axial vector.

Since $\bar{H}$ is an axial vector, it can<u>not</u> contribute to $\bar{E}$.

$\bar{J}$ and $\bar{H}\times\bar{J}$ are both true vectors, so they <u>can</u> contribute to $\bar{E}$.

$\bar{H}\cdot\bar{H}$ is a true scalar, so $(\bar{H}\cdot\bar{H})\bar{J}$ <u>can</u> contribute to $\bar{E}$.

$\bar{H}\cdot\bar{J}$ is a pseudoscalar, so $(\bar{H}\cdot\bar{J})\bar{H}$ is a true vector, so it can contribute to $\bar{E}$.

Thus, $\bar{E} = \rho_0\bar{J} + R(\bar{H}\times\bar{J}) + \beta_1(\bar{H}\cdot\bar{H})\bar{J} + \beta_2(\bar{H}\cdot\bar{J})\bar{H}$

for constant coefficients $\rho_0, R, \beta_1, \beta_2$.

6.15, cont'd

b) What about the requirement of time-reversal invariance?

$\bar{E}$ is even under time reversal,
whereas both $\bar{J}$, $\bar{H}$ are odd.
However, the coefficients might also transform.

Since ρ_0 is resistivity, it should be odd under time reversal,
hence $\rho_0 \bar{J}$ is even and can contribute to $\bar{E}$.

$\bar{H}\times\bar{J}$ is even under time reversal, and can contribute to $\bar{E}$.

$(\bar{H}\cdot\bar{H})\bar{J}$, $(\bar{H}\cdot\bar{J})\bar{H}$ are both odd under time reversal,
so unless the coefficients β_1, β_2 are also odd,
those terms cannot contribute,

$$\Rightarrow \boxed{\bar{E} = \rho_0 \bar{J} + R\, \bar{H}\times\bar{J}}$$

6.18 Consider the Dirac expression

$$\bar{A}(\bar{x}) = \frac{g}{4\pi}\int_L d\bar{l}' \times \frac{(\bar{x}-\bar{x}')}{|\bar{x}-\bar{x}'|^3}$$

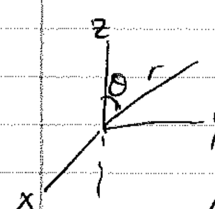
for the vector potential of a magnetic monopole and its associated string L. Suppose for definiteness that the monopole is located at the origin and the string along the negative z axis.

a) Calculate $\bar{A}$ explicitly and show that in spherical coordinates it has components

$$A_r = 0, \quad A_\theta = 0, \quad A_\phi = \frac{g(1-\cos\theta)}{4\pi r \sin\theta} = \frac{g}{4\pi r}\tan\left(\frac{\theta}{2}\right)$$

Since $\hat{r}$, $\hat{\theta}$ are in the plane spanned by $d\bar{l}'$, $(\bar{x}-\bar{x}')$, we see immediately that

$$A_r = A_\theta = 0$$



$$\bar{A} = \frac{g}{4\pi}\int_{-\infty}^{0} dz'\, \hat{k} \times \frac{(x\hat{\imath} + y\hat{\jmath} + (z-z')\hat{k})}{|x\hat{\imath}+y\hat{\jmath}+(z-z')\hat{k}|^3}$$

Use

$$\hat{k} \times (x\hat{\imath} + y\hat{\jmath} + (z-z')\hat{k}) = x\hat{\jmath} - y\hat{\imath}$$

$$= r\sin\theta\underbrace{(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})}_{=\hat{\phi}}$$

$$|x\hat{\imath} + y\hat{\jmath} + (z-z')\hat{k}|^3$$

$$= \left[x^2+y^2+(z-z')^2\right]^{3/2}$$

$$= \left[r^2 - 2zz' + z'^2\right]^{3/2}$$

$$= \left[r^2 - 2rz'\cos\theta + z'^2\right]^{3/2}$$

6.18 a), cont'd

$$\vec{A} = \frac{g\hat{\phi}}{4\pi}\int_{-\infty}^{0} dz' \frac{r\sin\theta}{(r^2 - 2rz'\cos\theta + z'^2)^{3/2}}$$

$$= \frac{g\hat{\phi}}{4\pi} r\sin\theta \int_{-\infty}^{-r\cos\theta} d\tilde{z}\left[\underbrace{(z' - r\cos\theta)}_{\equiv \tilde{z}}{}^2 + r^2 - r^2\cos^2\theta\right]^{-3/2}$$

$$= \frac{g\hat{\phi}}{4\pi} r\sin\theta \int_{-\infty}^{-r\cos\theta} d\tilde{z}\left[\tilde{z}^2 + r^2\sin^2\theta\right]^{-3/2}$$

$$= \frac{g\hat{\phi}}{4\pi} r\sin\theta \left[\frac{\tilde{z}}{r^2\sin^2\theta}\frac{1}{\sqrt{\tilde{z}^2 + r^2\sin^2\theta}}\right]_{-\infty}^{-r\cos\theta}$$

$$= \frac{g\hat{\phi}}{4\pi}\frac{r\sin\theta}{r^2\sin^2\theta}\left[\frac{-r\cos\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}} + 1\right]$$

$$= \frac{g\hat{\phi}}{4\pi}\frac{1}{r\sin\theta}\left[-\frac{r\cos\theta}{r} + 1\right]$$

$$= \frac{g\hat{\phi}}{4\pi r}\frac{1-\cos\theta}{\sin\theta}$$

$$= \boxed{\frac{g\hat{\phi}}{4\pi r}\tan(\theta/2)}$$

6.18, cont'd

b) Verify that $\bar{B} = \nabla \times \bar{A}$ is the Coulomb-like field of a point charge, except perhaps at $\theta = \pi$.

$$\bar{B} = \nabla \times \bar{A}$$

$$= \hat{r}\,\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) - \frac{\hat{\theta}}{r}\underbrace{\frac{\partial}{\partial r}\left(rA_\phi\right)}_{=0 \text{ since } rA_\phi \text{ is independent of } r}$$

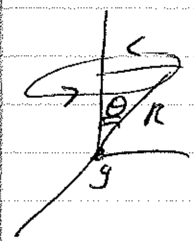
$$= \frac{\hat{r}}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{g}{4\pi r}\left(1-\cos\theta\right)\right)$$

$$= \frac{\hat{r}\,g}{4\pi r^2 \cancel{\sin\theta}}\left(+\cancel{\sin\theta}\right) = \boxed{\frac{g}{4\pi r^2}\hat{r}}$$

exactly what one would expect of a point charge.

6.18 cont'd

c) With the $\bar{B}$ determined in part (b), evaluate the total magnetic flux passing through the circular loop of radius $R\sin\theta$ shown.



Consider $\theta < \pi/2$ and $\theta > \pi/2$ separately, but always calculate the upward flux.

$\theta < \pi/2$:

$$\text{Flux} = \int \bar{B}\cdot\hat{n}\,da = \int\left(\frac{g}{4\pi r^2}\cos\theta'\right) r^2\,d\theta'\,d\phi \qquad \text{using } \hat{r}\cdot\hat{z} = \cos\theta$$

$$= \frac{g}{4\pi}\int d\Omega = \frac{g}{4\pi}\int_0^{2\pi} d\phi \int_\theta^0 d(\cos\theta')$$

$$= 2\pi\,\frac{g}{4\pi}\left(1-\cos\theta\right)$$

$$= \boxed{\frac{g}{2}\left(1-\cos\theta\right)}$$

$\theta > \pi/2$:

$$\text{Flux} = \int\left(\frac{g}{4\pi r^2}(-)\cos\theta'\right) r^2\,d\theta'\,d\phi$$

$$= -\frac{g}{4\pi}(2\pi)\int_{-1}^{\cos\theta} d(\cos\theta')$$

$$= \boxed{-\frac{g}{2}\left(1+\cos\theta\right)}$$

6.18, cont'd

d) From $\oint \vec{A} \cdot d\vec{l}$ around the loop, determine the total magnetic flux through the loop. Compare the result with that of (c). Show that they are equal for $0 < \theta < \pi/2$, but have a constant difference for $\pi/2 < \theta < \pi$. Interpret the difference.

$$\boxed{\oint \vec{A} \cdot d\vec{l} = \int_0^{2\pi} d\phi \, A_\phi \, r \sin\theta = \int_0^{2\pi} d\phi \, \frac{g}{4\pi \cancel{r}} \frac{1-\cos\theta}{\cancel{\sin\theta}} \cancel{r \sin\theta} = \frac{g}{2}(1-\cos\theta)}$$

This matches the result of (c) for $\theta < \pi/2$.
The difference between this result and that for $\theta > \pi/2$ is

$$\frac{g}{2}(1-\cancel{\cos\theta}) - \left[-\frac{g}{2}(1+\cancel{\cos\theta})\right] = \boxed{g}$$

The difference is the contribution from the string.

6.20 An example of the preservation of causality and finite speed of propagation in spite of the use of Coulomb gauge is afforded by a dipole source that is flashed on and off at $t=0$. The effective charge and current densities are

$$\rho(\bar{x},t) = \delta(x)\,\delta(y)\,\delta'(z)\,\delta(t)$$
$$J_z(\bar{x},t) = -\,\delta(x)\,\delta(y)\,\delta(z)\,\delta'(t)$$

where a prime means differentiation with respect to the argument. This dipole is of unit strength and it points in the negative z direction.

a) Show that the instantaneous Coulomb potential (6.23)

$$\Phi(\bar{x},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\bar{x}',t)}{|\bar{x}-\bar{x}'|}\,d^3x' \qquad (6.23)$$

is given in this case by

$$\Phi(\bar{x},t) = -\frac{1}{4\pi\varepsilon_0}\,\delta(t)\,\frac{z}{r^3}$$

$$\Phi(\bar{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\frac{1}{|\bar{x}-\bar{x}'|}\,\delta(x')\,\delta(y')\,\delta'(z')\,\delta(t)$$

$$= \frac{\delta(t)}{4\pi\varepsilon_0}\int dz'\,\left[x^2+y^2+(z-z')^2\right]^{-1/2}\frac{d\delta(z')}{dz'}$$

$$= -\frac{\delta(t)}{4\pi\varepsilon_0}\int dz'\,\delta(z')\left(-\tfrac{1}{2}\right)\left[x^2+y^2+(z-z')^2\right]^{-3/2}(2)(z-z')(-)$$

$$= -\frac{\delta(t)}{4\pi\varepsilon_0}\left[x^2+y^2+z^2\right]^{-3/2} z$$

$$= -\frac{1}{4\pi\varepsilon_0}\,\delta(t)\,\frac{z}{r^3}$$

6.20, cont'd

b) Show that the transverse current $\bar{J}_t$ is

$$\bar{J}_t(\bar{x},t) = -\delta'(t)\left[\frac{2}{3}\hat{k}\,\delta^3(\bar{x}) - \frac{\hat{k}}{4\pi r^3} + \frac{3}{4\pi r^3}\,\hat{n}\,(\hat{k}\cdot\hat{n})\right].$$

where the factor of 2/3 multiplying the delta function comes from treating the gradient of z/r^3 as in (4.20).

$$\bar{J}_t = \bar{J} - \bar{J}_l \qquad (6.25)$$

$$= \bar{J} - \varepsilon_0 \nabla \frac{\partial \Phi}{\partial t} \qquad (6.29)$$

$$= -\hat{k}\,\delta(x)\,\delta(y)\,\delta(z)\,\delta'(t) + \frac{1}{4\pi}\delta'(t)\,\nabla\left(\frac{z}{r^3}\right)$$

$$\nabla\left(\frac{z}{r^3}\right) = -\frac{3(\hat{k}\cdot\hat{r})\hat{r} - \hat{k}}{r^3} + \frac{4\pi}{3}\delta^3(\bar{x})\,\hat{k}$$

$$\Rightarrow \boxed{\bar{J}_t = -\hat{k}\,\delta^3(\bar{x})\,\delta'(t) + \frac{1}{3}\delta'(t)\,\delta^3(\bar{x})\,\hat{k} - \frac{1}{4\pi}\delta'(t)\,\frac{3(\hat{k}\cdot\hat{r})\hat{r} - \hat{k}}{r^3} = -\delta'(t)\left[\frac{2}{3}\hat{k}\,\delta^3(\bar{x}) + \frac{1}{4\pi r^3}\left(3(\hat{k}\cdot\hat{r})\hat{r} - \hat{k}\right)\right]}$$

6.20, cont'd

c) Show that the electric and magnetic fields are causal and that the electric field components are

$$E_x(\bar{x},t) = \frac{1}{4\pi\varepsilon_0}\frac{c}{r}\left[-\delta''(r-ct) + \frac{3}{r}\delta'(r-ct) - \frac{3}{r^2}\delta(r-ct)\right]\sin\theta\cos\theta\cos\phi$$

E_y the same as E_x but with $\cos\phi$ replaced by $\sin\phi$

$$E_z(\bar{x},t) = \frac{1}{4\pi\varepsilon_0}\frac{c}{r}\left[\sin^2\theta\,\delta''(r-ct) + (3\cos^2\theta-1)\left(\frac{\delta'(r-ct)}{r} - \frac{\delta(r-ct)}{r^2}\right)\right]$$

$$\bar{A}(\bar{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{1}{R}\left[\bar{J}(\bar{x}',t')\right]_{ret} \qquad (6.48)$$

for $\bar{R} = \bar{x} - \bar{x}'$

Use

$$\bar{J}_t = -\delta'(t)\left[\hat{z}\,\delta^3(\bar{x}) + \frac{1}{4\pi}\nabla\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right]$$

& the fact that the source for the wave eqn for $\bar{A}$ can be taken to be the transverse part of the current.

$$\bar{A}(\bar{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{1}{R}\left[\bar{J}_t(\bar{x}',t')\right]_{ret}$$

$$= \frac{-\mu_0}{4\pi}\int d^3x'\,\frac{1}{R}\left[\delta'\left(t-\frac{R}{c}\right)\left(\hat{z}\,\delta^3(\bar{x}') + \frac{1}{4\pi}\nabla'\frac{\partial}{\partial z'}\frac{1}{r'}\right)\right]$$

$$\frac{\partial\bar{A}}{\partial t} = -\frac{\mu_0}{4\pi}\int d^3x'\,\frac{1}{R}\,\frac{\partial}{\partial t}\left[\delta'\left(t-\frac{R}{c}\right)\left(\hat{z}\,\delta^3(\bar{x}') + \frac{1}{4\pi}\nabla'\frac{\partial}{\partial z'}\frac{1}{r'}\right)\right]$$

6.20 d), cont'd

$$\frac{\partial \bar{A}}{\partial t} = -\frac{\mu_0}{4\pi}\frac{1}{r}\,\delta''\left(t-\frac{r}{c}\right)\hat{n}$$
$$-\frac{\mu_0}{4\pi}\int d^3x'\,\frac{1}{R}\,\delta''\left(t-\frac{R}{c}\right)\frac{1}{4\pi}\nabla'\frac{\partial}{\partial z'}\frac{1}{r'}$$

Write $\delta\left(t-\frac{R}{c}\right) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\,e^{ik(t-R/c)}$

$$\frac{\partial \bar{A}}{\partial t} = -\frac{\mu_0}{4\pi r}\,\delta''\left(t-\frac{r}{c}\right)\hat{n}$$
$$-\frac{\mu_0}{4\pi}\int d^3x'\int\frac{dk}{2\pi}\,\frac{1}{R}\,\frac{\partial^2}{\partial t^2}e^{ik(t-R/c)}\,\frac{1}{4\pi}\nabla'\frac{\partial}{\partial z'}\frac{1}{r'}$$

Note

$$\nabla'^2\,\frac{e^{ik(t-R/c)}}{R} = \left(-\frac{ik}{c}\right)^2\frac{e^{ik(t-R/c)}}{R} - e^{ik(t-R/c)}(4\pi)\,\delta^3(\bar{x}-\bar{x}')$$
$$= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{e^{ik(t-R/c)}}{R} - e^{ikt}(4\pi)\,\delta^3(\bar{x}-\bar{x}')$$

thus

$$\frac{\partial \bar{A}}{\partial t} = -\frac{\mu_0}{4\pi r}\,\delta''\left(t-\frac{r}{c}\right)\hat{n} - \frac{\mu_0 c^2}{4\pi}\int d^3x'\int\frac{dk}{2\pi}\left(\nabla'^2\,\frac{e^{ik(t-R/c)}}{R} + e^{ikt}\,\delta^3(x-x')(4\pi)\right)_2$$
$$\hookrightarrow \frac{1}{4\pi}\nabla'\frac{\partial}{\partial z'}\frac{1}{r'}$$

$$= -\frac{\mu_0}{4\pi r}\,\delta''\left(t-\frac{r}{c}\right)\hat{n} - \frac{1}{4\pi\varepsilon_0}\left[\int d^3x'\int\frac{dk}{2\pi}\,\frac{e^{ik(t-R/c)}}{4\pi R}\,\nabla'\frac{\partial}{\partial z'}\nabla'^2\frac{1}{r'}\right.$$
$$\left. + \delta(t)\,\nabla\frac{\partial}{\partial z}\frac{1}{r}\right]$$

6.20 c), cont'd

$$\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0}{4\pi r}\delta''\left(t-\frac{\hat{R}}{c}\right)\hat{k} + \frac{1}{4\pi\varepsilon_0}\delta(t)\,\nabla\left(\frac{z}{r^3}\right) + \frac{1}{4\pi\varepsilon_0}\int d^3x'\int\frac{dk}{2\pi}\,\frac{e^{ik(t-R/c)}}{R}\nabla'\frac{\partial}{\partial z'}\delta^3(\vec{x}')$$

$$= -\frac{\mu_0}{4\pi r}\delta''\left(t-\frac{\hat{R}}{c}\right)\hat{k} + \frac{1}{4\pi\varepsilon_0}\delta(t)\,\nabla\left(\frac{z}{r^3}\right) + \frac{1}{4\pi\varepsilon_0}\int\frac{dk}{2\pi}\,\nabla\frac{\partial}{\partial z}\,\frac{e^{ik(t-r/c)}}{r}$$

Now,

$$\nabla\frac{\partial}{\partial z}\frac{e^{-ikr/c}}{r} = \nabla\left(\left(-\frac{ik}{c}\frac{e^{-ikr/c}}{r} - \frac{e^{-ikr/c}}{r^2}\right)\frac{\partial r}{\partial z}\right)$$

$$= -\nabla\left[\left(\frac{ik}{cr}+\frac{1}{r^2}\right)e^{-ikr/c}\cos\theta\right]$$

Use the fact that $\nabla r = \hat{r}$

$$= +\left[\frac{2}{r^3}+\frac{2ik}{cr^2}-\frac{k^2}{c^2 r}\right](\cos\theta)\,e^{-ikr/c}\,\hat{r}$$
$$+\left(\frac{ik}{cr}+\frac{1}{r^2}\right)e^{-ikr/c}\,\frac{\sin\theta}{r}\,\hat{\theta}$$

$$\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0}{4\pi r}\delta''\left(t-\frac{\hat{R}}{c}\right)\hat{k} + \frac{1}{4\pi\varepsilon_0}\delta(t)\,\nabla\left(\frac{z}{r^3}\right)$$
$$+\frac{1}{4\pi\varepsilon_0}\int\frac{dk}{2\pi}\,e^{ik(t-r/c)}\left[\left(\frac{2}{r^3}+\frac{2ik}{cr^2}-\frac{k^2}{c^2r}\right)\cos\theta\,\hat{r} + \left(\frac{ik}{cr}+\frac{1}{r^2}\right)\frac{\sin\theta}{r}\hat{\theta}\right]$$

$$= -\frac{\mu_0}{4\pi r}\delta''\left(t-\frac{\hat{R}}{c}\right)\hat{k} + \frac{1}{4\pi\varepsilon_0}\delta(t)\,\nabla\left(\frac{z}{r^3}\right)$$
$$+\frac{1}{4\pi\varepsilon_0}\left[\left(\frac{1}{r^3}\delta(t-\hat{r}/c) - \frac{1}{cr^2}\delta'(t-r/c)\right)(2\cos\theta\,\hat{r}+\sin\theta\,\hat{\theta})\right.$$
$$\left.+\frac{1}{c^2r}\delta''(t-r/c)\cos\theta\,\hat{r}\right]$$

6.20 c), cont'd

$$\bar{E} = -\nabla\Phi - \frac{\partial \bar{A}}{\partial t}$$

$$= \cancel{+\frac{1}{4\pi\varepsilon_0}\delta(t)\nabla\left(\frac{z}{r^3}\right)} + \frac{\mu_0}{4\pi r}\delta''\left(t-\frac{r}{c}\right)\hat{k} \cancel{- \frac{1}{4\pi\varepsilon_0}\delta(t)\nabla\left(\frac{z}{r^3}\right)}$$

$$-\frac{1}{4\pi\varepsilon_0}\left[\left(\frac{1}{r^3}\delta(t-r/c) - \frac{1}{cr^2}\delta'\left(t-\frac{r}{c}\right)\right)(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) + \frac{1}{c^2 r}\delta''(t-r/c)\cos\theta\,\hat{r}\right]$$

Use $\hat{r} = \sin\theta\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{k}$

$\hat{\theta} = \cos\theta\cos\phi\,\hat{\imath} + \cos\theta\sin\phi\,\hat{\jmath} - \sin\theta\,\hat{k}$

$\Rightarrow 2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}$

$$= 2\cos\theta\sin\theta\cos\phi\,\hat{\imath} + 2\cos\theta\sin\theta\sin\phi\,\hat{\jmath} + 2\cos^2\theta\,\hat{k}$$

$$+ \sin\theta\cos\theta\cos\phi\,\hat{\imath} + \sin\theta\cos\theta\sin\phi\,\hat{\jmath} - \sin^2\theta\,\hat{k}$$

$$= 3\cos\theta\sin\theta\cos\phi\,\hat{\imath} + 3\sin\theta\cos\theta\sin\phi\,\hat{\jmath} + (2\cos^2\theta - \sin^2\theta)\hat{k}$$

$$\cos\theta\,\hat{r} = \sin\theta\cos\theta\cos\phi\,\hat{\imath} + \sin\theta\cos\theta\sin\phi\,\hat{\jmath} + \cos^2\theta\,\hat{k}$$

6.20 c), cont'd

$$\overline{E} = \frac{\mu_0}{4\pi r}\,\delta''(t - r/c)\,\hat{z}$$
$$-\frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r^3}\,\delta(t-r/c) - \frac{1}{cr^2}\,\delta'(t-r/c)\right)\Big(3\cos\theta\sin\theta\cos\phi\,\hat{x}$$
$$+\,3\cos\theta\sin\theta\sin\phi\,\hat{y}$$
$$+\,(2\cos^2\theta - \sin^2\theta)\,\hat{z}\Big)$$
$$-\frac{1}{4\pi\varepsilon_0}\,\frac{1}{c^2 r}\,\delta''(t-r/c)\left(\sin\theta\cos\theta\cos\phi\,\hat{x} + \sin\theta\cos\theta\cos\phi\,\hat{y} + \cos^2\theta\,\hat{z}\right)$$

$$E_x = \frac{1}{4\pi\varepsilon_0}\,\frac{1}{c^2 r}\left[-\delta''(t-r/c) + \frac{3c}{r}\,\delta'(t-r/c) - \frac{3c^2}{r^2}\,\delta(t-r/c)\right]_e$$

$\hookrightarrow \sin\theta\cos\theta\cos\phi$

$$E_y = \frac{1}{4\pi\varepsilon_0}\,\frac{1}{c^2 r}\left[-\delta''(t-r/c) + 3\,\frac{c}{r}\,\delta'(t-r/c) - 3\,\frac{c^2}{r^2}\,\delta(t-r/c)\right]_e$$

$\hookrightarrow \sin\theta\cos\theta\cos\phi$

$$E_z = \frac{1}{4\pi\varepsilon_0}\,\frac{1}{c^2 r}\left[\sin^2\theta\,\delta''(t-r/c) + (3\cos^2\theta - 1)\left(\frac{c}{r}\,\delta'(t-r/c) - \frac{c^2}{r^2}\,\delta(t-r/c)\right)\right]$$

These match the textbook after using $\delta(t-r/c) = c\,\delta(r-ct)$

Since the only time dependence is in the δ functions, $\overline{E}$ is causal.

As discussed in § 6.3, $\overline{A}$ is causal, hence $\overline{B}$ is causal.