Refer to § 9.6

8.7 A resonant cavity consists of the empty space between two perfectly conducting concentric spherical shells, the smaller of outer radius a and the larger of inner radius b. The azimuthal magnetic field has a radial dependence given by spherical Bessel functions.

a) Write down transcendental equations for the characteristic frequencies of the cavity for arbitrary l.

$$B_\phi(r,\theta) = \frac{u_\ell(r)}{r} P_\ell^1(\cos\theta) \qquad (8.102)$$

where

$$\frac{d^2 u_\ell(r)}{dr^2} + \left(\frac{\omega^2}{c^2} - \frac{\ell(\ell+1)}{r^2}\right) u_\ell(r) = 0$$

Sol'ns: $\frac{u_\ell(r)}{r} = A_\ell j_\ell(kr) + B_\ell n_\ell(kr)$

Boundary conditions:

$$\left.\frac{du_\ell(r)}{dr}\right|_{r=a} = 0 = \left.\frac{du_\ell(r)}{dr}\right|_{r=b} \qquad (8.104)$$

Here,

$$\frac{du_\ell(r)}{dr} = A_\ell\left(j_\ell(kr) + kr\, j_\ell'(kr)\right) + B_\ell\left(n_\ell(kr) + kr\, n_\ell'(kr)\right)$$

Boundary conditions are then

$$\begin{bmatrix} j_\ell(ka) + ka\, j_\ell'(ka) & n_\ell(ka) + ka\, n_\ell'(ka) \\ j_\ell(kb) + kb\, j_\ell'(kb) & n_\ell(kb) + kb\, n_\ell'(kb) \end{bmatrix} \begin{bmatrix} A_\ell \\ B_\ell \end{bmatrix} = 0$$

Sol'ns exist when the determinant vanishes,

$$\Rightarrow \boxed{\frac{j_\ell(ka) + ka\, j_\ell'(ka)}{n_\ell(ka) + ka\, n_\ell'(ka)} = \frac{j_\ell(kb) + kb\, j_\ell'(kb)}{n_\ell(kb) + kb\, n_\ell'(kb)}}$$

8.7, cont'd

b) For $l=1$ use the explicit form of the spherical Bessel functions to show that the characteristic frequencies are given by

$$\frac{\tan kh}{kh} = \frac{k^2 + \frac{1}{ab}}{k^2 + ab\left(k^2 - \frac{1}{a^2}\right)\left(k^2 - \frac{1}{b^2}\right)}$$

where $h = b - a$.

Use $j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$, $n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$ (9.87)

$$\Rightarrow \frac{j_1(x) + x\, j_1'(x)}{n_1(x) + x\, n_1'(x)} :$$

$$j_1'(x) = \frac{\cos x}{x^2} - 2\frac{\sin x}{x^3} + \frac{\sin x}{x} + \frac{\cos x}{x^2}$$

$$= 2\frac{\cos x}{x^2} - 2\frac{\sin x}{x^3} + \frac{\sin x}{x}$$

$$n_1'(x) = \frac{\sin x}{x^2} + 2\frac{\cos x}{x^3} - \frac{\cos x}{x} + \frac{\sin x}{x^2}$$

$$= 2\frac{\sin x}{x^2} + 2\frac{\cos x}{x^3} - \frac{\cos x}{x}$$

$$\frac{j_1(x) + x\, j_1'(x)}{n_1(x) + x\, n_1'(x)} = \frac{\cancel{\frac{\sin x}{x^2}} - \cancel{\frac{\cos x}{x}} + \cancel{2}\frac{\cos x}{x} - \cancel{2}\frac{\sin x}{x^2} + \sin x}{-\cancel{\frac{\cos x}{x^2}} - \cancel{\frac{\sin x}{x}} + \cancel{2}\frac{\sin x}{x} + \cancel{2}\frac{\cos x}{x^2} - \cos x}$$

$$= \frac{\frac{\cos x}{x} - \frac{\sin x}{x^2} + \sin x}{\frac{\sin x}{x} + \frac{\cos x}{x^2} - \cos x}$$

$$= \frac{x\cos x - \sin x + x^2 \sin x}{x\sin x + \cos x - x^2 \cos x}$$

$$= \frac{x\cos x - (1 - x^2)\sin x}{x\sin x + (1 - x^2)\cos x}$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

8.7 b), cont'd

The transcendental eq'n from (a) may then be written as

$$\frac{ka \cos ka - (1-(ka)^2)\sin ka}{ka \sin ka + (1-(ka)^2)\cos ka} = \frac{kb \cos kb - (1-(kb)^2)\sin kb}{kb \sin kb + (1-(kb)^2)\cos kb}$$

$$\Rightarrow (ka\,kb)\cos ka \sin kb - (1-(ka)^2)(1-(kb)^2)\sin ka \cos kb$$
$$- kb(1-(ka)^2)\sin ka \sin kb + ka(1-(kb)^2)\cos ka \cos kb$$
$$= ka\,kb \sin ka \cos kb - (1-(ka)^2)(1-(kb)^2)\cos ka \sin kb$$
$$+ kb(1-(ka)^2)\cos ka \cos kb - ka(1-(kb)^2)\sin ka \sin kb$$

$$\Rightarrow +ka\,kb \sin k(b-a) + (1-(ka)^2)(1-(kb)^2)\sin k(b-a)$$
$$- kb(1-(ka)^2)\cos k(b-a) + ka(1-(kb)^2)\cos k(b-a) = 0$$

$$\tan k(b-a) = \frac{kb(1-(ka)^2) - ka(1-(kb)^2)}{ka\,kb + (1-(ka)^2)(1-(kb)^2)}$$

$$= \frac{k(b-a) + k^3ab(b-a)}{k^2ab + (k^2a^2-1)(k^2b^2-1)}$$

$$\Rightarrow \boxed{\frac{\tan kh}{kh} = \frac{1+k^2ab}{k^2ab + (k^2a^2-1)(k^2b^2-1)} = \frac{k^2 + \frac{1}{ab}}{k^2 + ab\left(k^2-\frac{1}{a^2}\right)\left(k^2-\frac{1}{b^2}\right)}}$$

8.7, cont'd

c) For $h/a \ll 1$, verify that the result of (b) yields the frequency found in § 8.9, and find the first-order correction in h/a.

Write $b = a + h = a(1 + h/a)$
and expand:

$$\frac{\tan kh}{kh} = \frac{k^2 + \frac{1}{a^2}(1+h/a)^{-1}}{k^2 + a^2(1+h/a)(k^2 - \frac{1}{a^2})(k^2 - \frac{1}{a^2}(1+h/a)^{-2})}$$

$$\simeq \frac{k^2 + \frac{1}{a^2}(1 - h/a)}{k^2 + a^2(1+h/a)(k^2 - \frac{1}{a^2})(k^2 - \frac{1}{a^2}(1-2h/a)\ )}$$

$$\simeq \frac{k^2 + \frac{1}{a^2} - h/a^3}{k^2 + a^2(k^2 - \frac{1}{a^2})(k^2 - \frac{1}{a^2} + 2\, h/a^3)(1+h/a)}$$

$$\simeq \frac{k^2 + 1/a^2 - h/a^3}{k^2 + a^2(k^2 - 1/a^2)(k^2 - 1/a^2 + \frac{h}{a}k^2 - \cancel{h/a^3} + \cancel{2}\,h/a^3)}$$

$$\simeq \frac{k^2 + 1/a^2 - \frac{h}{a}\frac{1}{a^2}}{k^2 + a^2(k^2 - 1/a^2)^2 + a^2(k^2 - 1/a^2)\frac{h}{a}(k^2 + 1/a^2)}$$

$$\simeq \frac{k^2 + 1/a^2 - \frac{h}{a}\,1/a^2}{k^2 + a^2(k^4 - 2k^2/a^2 + 1/a^4) + a^2(k^4 - 1/a^4)\,h/a}$$

$$\simeq \frac{k^2 + 1/a^2 - \frac{h}{a}\,1/a^2}{a^2k^4 - k^2 + 1/a^2 + (k^4a^2 - 1/a^2)\,h/a}$$

$$\simeq \left(k^2 + \frac{1}{a^2} - \frac{h}{a}\frac{1}{a^2}\right)\left(a^2k^4 - k^2 + \frac{1}{a^2}\right)^{-1}\left(1 + \frac{h}{a}\,\frac{k^4a^2 - 1/a^2}{k^4a^2 - k^2 + 1/a^2}\right)^{-1}$$

$$\simeq \left(k^4a^2 - k^2 + \frac{1}{a^2}\right)^{-1}\left(k^2 + \frac{1}{a^2} - \frac{h}{a}\frac{1}{a^2}\right)\left(1 - \frac{h}{a}\,\frac{k^4a^2 - 1/a^2}{k^4a^2 - k^2 + 1/a^2}\right)$$

(cont'd)

8.7 c), cont'd

$$\frac{\tan kh}{kh} \simeq \left(k^4a^2 - k^2 + \frac{1}{a^2}\right)^{-1}\left[k^2 + \frac{1}{a^2} - \frac{h}{a}\left(\frac{1}{a^2} + \left(k^2+\frac{1}{a^2}\right)\left(\frac{k^4a^2 - 1/a^2}{k^4a^2 - k^2 + 1/a^2}\right)\right)\right]$$

$$\simeq \frac{(ka)^2+1}{(ka)^4-(ka)^2+1} - \left(\frac{h}{a}\right)\left\{\frac{a^{-2}(k^4a^2-k^2+a^{-2}) + (k^2+a^{-2})(k^4a^2-a^{-2})}{(k^4a^2-k^2+a^{-2})^2}\right\}$$

$$+ O\left(\left(\frac{h}{a}\right)^2\right)$$

$$\simeq \frac{(ka)^2+1}{(ka)^4-(ka)^2+1} - \left(\frac{h}{a}\right)\left\{\frac{(ka)^4-(ka)^2+1 + ((ka)^2+1)((ka)^4-1)}{((ka)^4-(ka)^2+1)^2}\right\}$$

$$\simeq \frac{(ka)^2+1}{(ka)^4-(ka)^2+1} - \left(\frac{h}{a}\right)\left\{\frac{(ka)^4-(ka)^2+\cancel{1} + (ka)^6 - \cancel{1} + (ka)^4 - (ka)^2}{((ka)^4-(ka)^2+1)^2}\right\}$$

$$\simeq \frac{(ka)^2+1}{(ka)^4-(ka)^2+1} - \left(\frac{h}{a}\right)\left\{\frac{(ka)^2((ka)^4+2(ka)^2-2)}{((ka)^4-(ka)^2+1)^2}\right\} + O\left(\left(\frac{h}{a}\right)^2\right)$$

By comparison,

$$\frac{\tan x}{x} \simeq 1 + \frac{x^2}{3} + O(x^4)$$

Thus, for $h/a \ll 1$, the characteristic frequency eqn becomes

$$1 = \frac{(ka)^2+1}{(ka)^4-(ka)^2+1} - \left(\frac{h}{a}\right)\left\{\frac{(ka)^2((ka)^4+2(ka)^2-2)}{((ka)^4-(ka)^2+1)^2}\right\} + O\left(\left(\frac{h}{a}\right)^2\right)$$

At zeroth order,

$$1 = \frac{(ka)^2+1}{(ka)^4-(ka)^2+1}$$

8.7 c), cont'd

Zeroth order:

$$(ka)^4 - (ka)^2 + \cancel{1} = (ka)^2 + \cancel{1}$$

$$\Rightarrow (ka)^2 = 0 \quad \text{or} \quad (ka)^2 - 1 = 1$$

$$\Rightarrow ka = 0 \text{ or } \sqrt{2}$$

Note $ka = \sqrt{2} \Rightarrow k = \boxed{\frac{\omega}{c} = \frac{\sqrt{2}}{a},}$

which matches the solution for $l=1$ in § 8.9 in (8.105)

Next, we compute the first-order in h/a correction.

Write $ka = \sqrt{2} + \delta$, so that $(ka)^2 = 2 + 2\sqrt{2}\delta + O(\delta^2)$, plug in:

$$1 = \frac{2 + 2\sqrt{2}\delta + 1}{2^2 + 4\,2^{3/2}\delta - (2 + 2\sqrt{2}\delta) + 1}$$

$$- \left(\frac{h}{a}\right)\left\{ \frac{(2+2\sqrt{2}\delta)(2^2 + 4\,2^{3/2}\delta + 2(2+2\sqrt{2}\delta) - 2}{(2^2 + 4\,2^{3/2}\delta - (2+2\sqrt{2}\delta) + 1)^2} \right\}$$

$$= \frac{3 + 2\sqrt{2}\delta}{3 + 6\sqrt{2}\delta} - \left(\frac{h}{a}\right)\left\{ \frac{(2+2\sqrt{2}\delta)(6 + 12\sqrt{2}\delta)}{(3 + 6\sqrt{2}\delta)^2} \right\}$$

$$= \frac{3 + 2\sqrt{2}\delta}{3 + 6\sqrt{2}\delta} - \left(\frac{h}{a}\right)\left\{ \frac{12 + 36\sqrt{2}\delta}{9 + 36\sqrt{2}\delta} \right\}$$

(cont'd)

8.7c), cont'd

$$1 \simeq \frac{3+2\sqrt{2}\delta}{3}\left(1-2\sqrt{2}\delta\right)$$
$$\quad -\frac{h}{a}\,\frac{12+36\sqrt{2}\delta}{9}\left(1-4\sqrt{2}\delta\right)$$

$$\simeq \frac{1}{3}\left(3-4\sqrt{2}\delta\right)-\frac{h}{a}\,\frac{1}{3}\left(4-7\sqrt{2}\delta\right)$$

$$= 1-\frac{4}{3}\sqrt{2}\delta-\left(\frac{h}{a}\right)\left(\frac{4}{3}-\frac{7}{3}\sqrt{2}\delta\right)$$

$$\Rightarrow 0 \simeq -\frac{4}{3}\sqrt{2}\delta-\left(\frac{h}{a}\right)\left(-\frac{7}{3}\sqrt{2}\delta\right)-\left(\frac{h}{a}\right)\left(\frac{4}{3}\right)$$

$$\Rightarrow \sqrt{2}\delta\left(4-7\,h/a\right) = -4\,h/a$$

$$\Rightarrow \sqrt{2}\delta \simeq -4\frac{h}{a}\left(4-7\frac{h}{a}\right)^{-1} = -\frac{h}{a}+O\left((h/a)^2\right)$$

$$\Rightarrow \delta \simeq -\frac{1}{\sqrt{2}}\frac{h}{a}+O\left(\left(\frac{h}{a}\right)^2\right)$$

Thus, the first-order correction to the frequency is

$$\boxed{\begin{aligned} \omega &= \sqrt{2}\frac{c}{a}-\frac{1}{\sqrt{2}}\frac{c}{a}\frac{h}{a}+O\left((h/a)^2\right) \\ &= \sqrt{2}\frac{c}{a}\left(1-\frac{1}{2}\frac{h}{a}\right)+O\left((h/a)^2\right) \\ &\simeq \sqrt{2}\,\frac{c}{a+h/2+\cdots} \end{aligned}}$$

(a) only

8.18 a) From Green's theorem in two dimensions show that the TM and TE modes in a waveguide defined by the boundary-value problems

$$(\nabla_t^2 + \gamma_\lambda^2)\psi_\lambda = 0 \qquad (8.34)$$

where

$$\text{TM}: \ \psi|_S = 0$$
$$\text{TE}: \ \frac{\partial \psi}{\partial n}\Big|_S = 0, \qquad (8.36)$$

are orthogonal in the sense that

$$\int_A E_{z\lambda} E_{z\mu} \, da = 0 \quad \text{for TM modes,}$$

& a corresponding relation for H_z for TE modes.

Eigenfunctions satisfy $(\nabla_t^2 + \gamma_\lambda^2)\psi_\lambda = 0$, $(\nabla_t^2 + \gamma_\mu^2)\psi_\mu = 0$

$$\Rightarrow (\gamma_\mu^2 - \gamma_\lambda^2)\psi_\mu \psi_\lambda = \psi_\mu \nabla_t^2 \psi_\lambda - \psi_\lambda \nabla_t^2 \psi_\mu$$

$$\Rightarrow (\gamma_\mu^2 - \gamma_\lambda^2)\int_A \psi_\mu \psi_\lambda \, da = \int_A \left[\psi_\mu \nabla_t^2 \psi_\lambda - \psi_\lambda \nabla_t^2 \psi_\mu\right] da$$

$$= \oint_C \left(\psi_\mu \frac{\partial \psi_\lambda}{\partial n} - \psi_\lambda \frac{\partial \psi_\mu}{\partial n}\right) da$$

from Green's theorem

Note RHS = 0 for both TE, TM modes.

$$\Rightarrow (\gamma_\mu^2 - \gamma_\lambda^2)\int_A \psi_\mu \psi_\lambda \, da$$

so for $\gamma_\mu^2 \neq \gamma_\lambda^2$,

$$\boxed{\int_A \psi_\mu \psi_\lambda \, da = 0}$$

Recall
TM: $\psi = E_z$
TE: $\psi = H_z$

In degenerate cases, can use Gram-Schmidt.