2.7 Consider a potential problem in the half-space defined by $z \geq 0$, with Dirichlet boundary conditions on the plane.

a) Write down the appropriate Green function $G(\bar{x}, \bar{x}')$.

For a charge q at $\bar{x}'$,
there is an image charge $-q$ below the plane at $\sigma(\bar{x}')$,
where
$$\sigma(x, y, z) = (x, y, -z).$$

Thus, we can read off

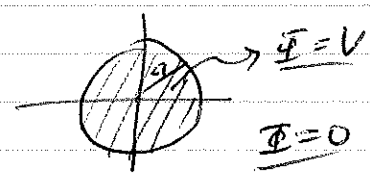
$$G(\bar{x}, \bar{x}') = \frac{1}{|\bar{x} - \bar{x}'|} - \frac{1}{|\bar{x} - \sigma(\bar{x}')|}$$

As a consistency check, note that if $z' = 0$,
then $\sigma(\bar{x}') = \bar{x}'$ and $G(\bar{x}, \bar{x}') = 0$,
consistent with Dirichlet boundary conditions.

2.7, cont'd

b) If the potential on the plane $z=0$ is specified to be $\Phi = V$ inside a circle of radius a centered at the origin and $\Phi = 0$ outside that circle, find an integral expression for the potential at the point P specified in terms of cylindrical coordinates (ρ, ϕ, z).

$z=0$:



$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\bar{x}')\, G_D(\bar{x},\bar{x}')\, d^3x' - \frac{1}{4\pi}\int \Phi(\bar{x}')\,\frac{\partial G_D}{\partial n'}\, da'$$

Here, $\rho = 0$.

Since the normal direction is $-\hat{z}$, outward from the volume of interest,

$$\left.\frac{\partial G}{\partial n'}\right|_{z'=0} = -\left.\frac{\partial G}{\partial z'}\right|_{z'=0}$$

Now,

$$G_D(\bar{x},\bar{x}') = \frac{1}{|x-x'|} - \frac{1}{|\bar{x}-\sigma(\bar{x}')|}$$

$$\frac{\partial G}{\partial z'} = -\frac{1}{2}\,\frac{1}{|\bar{x}-\bar{x}'|^3}(2)(z-z')(-) - \left(-\frac{1}{2}\right)\frac{1}{|\bar{x}-\sigma(\bar{x}')|^3}(2)(z+z')$$

$$= \frac{z-z'}{|\bar{x}-\bar{x}'|^3} + \frac{z+z'}{|x-\sigma(\bar{x}')|^3}$$

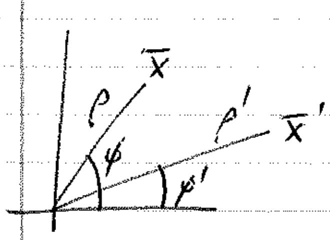
$$\left.\frac{\partial G}{\partial n'}\right|_{z'=0} = -\left.\frac{\partial G}{\partial z'}\right|_{z'=0} = -\frac{2z}{|\bar{x}-\bar{x}'|^3}$$ using the fact that along $z'=0$, $\sigma(\bar{x}') = \bar{x}'$.

27 b), cont'd

$$\Phi(\bar{x}) = -\frac{1}{4\pi}\int_0^{2\pi} d\phi' \int_0^{a} d\rho'\, \rho' V \left(-\frac{\partial G}{\partial z'}\bigg|_{z'=0}\right) da'$$

$$= +\frac{1}{4\pi}\int_0^{2\pi} d\phi' \int_0^{a} d\rho'\, \rho' V \left(\frac{2z}{|\bar{x}-\bar{x}'|^3}\bigg|_{z'=0}\right) da'$$

We need an expression for the distance between 2 points in cylindrical coordinates. Begin in polar coordinates in 2d:



$$\bar{x} = (\rho\cos\phi,\ \rho\sin\phi)$$

$$\bar{x}' = (\rho'\cos\phi',\ \rho'\sin\phi')$$

In 2d,

$$|\bar{x}-\bar{x}'|^2 = (\rho\cos\phi - \rho'\cos\phi')^2 + (\rho\sin\phi - \rho'\sin\phi')^2$$

$$= \rho^2\cos^2\phi + \rho'^2\cos^2\phi' - 2\rho\rho'\cos\phi\cos\phi'$$

$$+ \rho^2\sin^2\phi + \rho'^2\sin^2\phi' - 2\rho\rho'\sin\phi\,\sin\phi'$$

$$= \rho^2 + \rho'^2 - \rho\rho'\left(\cos(\phi-\phi') + \cancel{\cos(\phi+\phi')}\right)$$

$$- \rho\rho'\left(\cos(\phi-\phi') - \cancel{\cos(\phi+\phi')}\right)$$

$$= \rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi-\phi')$$

so in 3d,

$$|\bar{x}-\bar{x}'|^2\Big|_{z'=0} = \rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi-\phi') + (z-0)^2$$

2.7 b), cont'd

$$\boxed{\Phi(\bar{x}) = \frac{1}{4\pi}\int_0^{2\pi} d\phi' \int_0^a d\rho' \rho' (V) \frac{2z}{(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2)^{3/2}}}$$

2.7, cont'd

c) Show that, along the axis of the circle ($\rho = 0$), the potential is given by

$$\Phi = V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$

Along $\rho = 0$,

$$\Phi(\bar{x}) = \frac{1}{4\pi}\int_0^{2\pi} d\phi' \int_0^a d\rho' \rho' (V) \frac{2z}{(\rho'^2+z^2)^{3/2}}$$

$$= \frac{4\pi z V}{4\pi}\int_0^a d\rho' \rho' (\rho'^2+z^2)^{-3/2}$$

$$= Vz\left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)^{-1}(\rho'^2+z^2)^{-1/2}\Big|_0^a$$

$$= -Vz\left[(a^2+z^2)^{-1/2} - (z^2)^{-1/2}\right]$$

$$= V - \frac{Vz}{(a^2+z^2)^{1/2}}$$ using $z \geq 0$ so $(z^2)^{1/2} = z$ not $-z$

$$\boxed{= V\left(1 - \frac{z}{(a^2+z^2)^{1/2}}\right)}$$

2.7, cont'd

d) Show that at large distances ($\rho^2+z^2 \gg a^2$), the potential can be expanded in a power series in $(\rho^2+z^2)^{-1}$, and that the leading terms are

$$\Phi = \frac{Va^2}{2}\frac{z}{(\rho^2+z^2)^{3/2}}\left[1-\frac{3a^2}{4(\rho^2+z^2)}+\frac{5(3\rho^2a^2+a^4)}{8(\rho^2+z^2)^2}+\cdots\right]$$

Verify that the results of part (c), (d) are consistent with one another in their common range of validity.

From (b),

$$\Phi(\bar{x}) = \frac{1}{4\pi}\int_0^{2\pi}d\phi'\int_0^a d\rho'\rho' \frac{2Vz}{(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2)^{3/2}}$$

$$= \frac{2Vz}{4\pi}\frac{1}{(\rho^2+z^2)^{3/2}}\int_0^{2\pi}d\phi'\int_0^a d\rho'\rho'\left(1+\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2}\right)^{-3/2}$$

$$= \frac{2Vz}{4\pi}(\rho^2+z^2)^{-3/2}\int_0^{2\pi}d\phi'\int_0^a d\rho'\rho'\left[1-\frac{3}{2}\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2}\right.$$

$$\left.+\frac{1}{2!}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2}\right)^2+\cdots\right]$$

$$\int_0^{2\pi}d\phi'\int_0^a d\rho'\rho'(1) = (2\pi)\left.\frac{\rho'^2}{2}\right|_0^a = (2\pi)\frac{a^2}{2} = \pi a^2$$

(cont'd)

2.7 d), cont'd

$$\int_0^{2\pi} d\phi' \int_0^a d\rho' \rho' \left(\frac{\rho'^2 - 2\rho\rho' \cos(\phi - \phi')}{\rho^2 + z^2} \right)$$

$$= \int_0^a d\rho' \rho' \left(\frac{2\pi \rho'^2}{\rho^2 + z^2} \right) = \frac{2\pi}{\rho^2 + z^2} \left. \frac{\rho'^4}{4} \right|_0^a$$

$$= \frac{2\pi a^4}{\rho^2 + z^2} \frac{1}{4}$$

$$\int_0^{2\pi} d\phi' \int_0^a d\rho' \rho' \left(\frac{\rho'^2 - 2\rho\rho' \cos(\phi - \phi')}{\rho^2 + z^2} \right)^2$$

$$= (\rho^2 + z^2)^{-2} \int_0^{2\pi} d\phi' \int_0^a d\rho' \rho' \Big(\rho'^4 - 4\rho'^3 \rho \cos(\phi - \phi') + 4\rho^2 \rho'^2 \cos^2(\phi - \phi') \Big)$$

use $\cos^2(\phi - \phi') = \frac{1}{2} + \frac{1}{2} \cos 2(\phi - \phi')$

so $\int_0^{2\pi} d\phi' \cos^2(\phi - \phi') = \frac{1}{2}(2\pi) = \pi$

$$= (\rho^2 + z^2)^{-2} \int_0^a d\rho' \rho' \left(2\pi \rho'^4 + 4\pi \rho^2 \rho'^2 \right)$$

$$= (\rho^2 + z^2)^{-2} \left[2\pi \frac{\rho'^6}{6} + 4\pi \rho^2 \frac{\rho'^4}{4} \right]_0^a$$

$$= (\rho^2 + z^2)^{-2} \left[2\pi \frac{a^6}{6} + 4\pi \rho^2 \frac{a^4}{4} \right]$$

2.7 d), cont'd

Putting this together,

$$\boxed{\begin{aligned}\Phi(\bar{x}) &= \frac{2Vz}{4\pi}(\rho^2+z^2)^{-3/2}\Big[\pi a^2 - \frac{3}{2}\frac{2\pi a^4}{\rho^2+z^2}\frac{1}{4} \\ &\qquad + \frac{1}{2!}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(\rho^2+z^2)^{-2}(\cancel{2}\pi a^4)\left(\frac{a^2}{6}+2\frac{\rho^2}{4}\right)+\cdots\Big] \\ &= \frac{Vza^2}{2}(\rho^2+z^2)^{-3/2}\Big[1-\frac{3a^2}{4(\rho^2+z^2)} \\ &\qquad + \frac{15}{4}\frac{a^2}{(\rho^2+z^2)^2}\frac{1}{6}\frac{1}{2}(2a^2+6\rho^2)+\cdots\Big] \\ &= \frac{Va^2}{2}\frac{z}{(\rho^2+z^2)^{3/2}}\left[1-\frac{3a^2}{4(\rho^2+z^2)}+\frac{5}{8}\frac{a^2(a^2+3\rho^2)}{(\rho^2+z^2)^2}+\cdots\right]\end{aligned}}$$

Note that when $\rho = 0$, this becomes

$$\Phi(\bar{x}) = \frac{Va^2}{2}\frac{z}{z^3}\left[1-\frac{3a^2}{4z^2}+\frac{5}{8}\frac{a^4}{z^4}+\cdots\right] \quad \text{for } z \gg a$$

From (c),

$$\Phi(\bar{x}) = V\left(1-\frac{z}{\sqrt{a^2+z^2}}\right) = V\left(1-\left(1+\frac{a^2}{z^2}\right)^{-1/2}\right)$$

$$= V\left[1-\left(1-\frac{1}{2}\frac{a^2}{z^2}+\frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{a^2}{z^2}\right)^2+\frac{1}{3!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{a^2}{z^2}\right)^3+\cdots\right)\right]$$

for $z \gg a$

$$= V\left[\frac{1}{2}\frac{a^2}{z^2}-\frac{3}{8}\left(\frac{a^2}{z^2}\right)^4+\frac{5}{16}\left(\frac{a^2}{z^2}\right)^6+\cdots\right]$$

$$= V\frac{a^2}{2}\frac{1}{z^2}\left[1-\frac{3}{4}\frac{a^2}{z^2}+\frac{5}{8}\left(\frac{a^2}{z^2}\right)^2+\cdots\right]$$

matching the result in (d) for $\rho = 0$ & $z \gg a$