2.12 Starting with the series solution (2.71) for the two-dim'l potential problem with the potential specified on the surface of a cylinder of radius b, evaluate the coefficients formally, substitute them into the series, and sum it to obtain the potential inside the cylinder in the form of Poisson's integral

$$\Phi(\rho, \phi) = \frac{1}{2\pi}\int_0^{2\pi} d\phi' \, \Phi(b, \phi') \frac{b^2 - \rho^2}{b^2 + \rho^2 - 2b\rho\cos(\phi - \phi')}$$

What modification is necessary if the potential is desired in the region of space bounded by the cylinder and infinity?

Since we want a solution inside the cylinder,

$$\Phi(\rho, \phi) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \rho^n \cos(n\phi) + \sum_{n=1}^{\infty} b_n \rho^n \sin(n\phi)$$

The solution (2.71) also included terms proportional to ρ^{-n}, but since we want a solution that is well behaved at $\rho = 0$, the coefficients of the ρ^{-n} terms must vanish. (The opposite would be true if we wanted a solution outside the cylinder, vanishing as $\rho \to \infty$.)

Using standard methods,

$$a_n = \frac{1}{\pi b^n}\int_0^{2\pi} d\phi' \, \Phi(b, \phi') \cos n\phi'$$

$$b_n = \frac{1}{\pi b^n}\int_0^{2\pi} d\phi' \, \Phi(b, \phi') \sin n\phi'$$

(cont'd)

2.12, cont'd

Plug in:

$$\Phi(\rho,\phi) = \frac{1}{\pi}\int_0^{2\pi} d\phi'\,\Phi(b,\phi')\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^n \cos n\phi \cos n\phi' + \sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^n \sin n\phi \sin n\phi'\right]$$

$$= \frac{1}{\pi}\int_0^{2\pi} d\phi'\,\Phi(b,\phi')\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^n \cos n(\phi-\phi')\right]$$

$$= \frac{1}{\pi}\int_0^{2\pi} d\phi'\,\Phi(b,\phi')\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^n \frac{e^{in(\phi-\phi')} + e^{-in(\phi-\phi')}}{2}\right]$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi'\,\Phi(b,\phi')\left[1 + \sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^n e^{in(\phi-\phi')} + \sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^n e^{-in(\phi-\phi')}\right]$$

Recall if $A = \sum_{n=1}^{\infty} r^n$ then $rA = A - r \Rightarrow A = \frac{r}{1-r}$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi'\,\Phi(b,\phi')\left[1 + \frac{\frac{\rho}{b}e^{i(\phi-\phi')}}{1-\frac{\rho}{b}e^{i(\phi-\phi')}} + \frac{\frac{\rho}{b}e^{-i(\phi-\phi')}}{1-\frac{\rho}{b}e^{-i(\phi-\phi')}}\right]$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi'\,\Phi(b,\phi')\left[1-\frac{\rho}{b}e^{i(\phi-\phi')}\right]^{-1}\left[1-\frac{\rho}{b}e^{-i(\phi-\phi')}\right]^{-1}$$
$$\cdot\left[\left(1-\frac{\rho}{b}e^{i(\phi-\phi')}\right)\left(1-\frac{\rho}{b}e^{-i(\phi-\phi')}\right) + \frac{\rho}{b}e^{i(\phi-\phi')}\left(1-\frac{\rho}{b}e^{-i(\phi-\phi')}\right) + \frac{\rho}{b}e^{-i(\phi-\phi')}\left(1-\frac{\rho}{b}e^{i(\phi-\phi')}\right)\right]$$

(cont'd)

2.12, cont'd

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} d\phi'\, \Phi(b,\phi') \left[1 + \frac{\rho^2}{b^2} - 2\frac{\rho}{b}\cos(\phi-\phi')\right]^{-1}$$

$$\cdot \left[1 + \cancel{\frac{\rho^2}{b^2}} - \cancel{\frac{\rho^2}{b^2}} - \frac{\rho^2}{b^2}\right]$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi'\, \Phi(b,\phi') \, \frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi-\phi')}$$

If we wanted to describe the potential <u>outside</u> the cylinder, we would initially have chosen

$$\Phi(\rho,\phi) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \rho^{-n}\cos n\phi + \sum_{n=1}^{\infty} b_n \rho^{-n}\sin n\phi$$

$$\Rightarrow \quad a_n = \frac{b^n}{\pi}\int_0^{2\pi} d\phi'\, \Phi(b,\phi')\cos n\phi'$$

$$b_n = \frac{b^n}{\pi}\int_0^{2\pi} d\phi'\, \Phi(b,\phi')\sin n\phi'$$

so that the analysis on the last page would be identical except for b/ρ instead of ρ/b factors.

Final result:

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} d\phi'\, \Phi(b,\phi') \left[1 + \frac{b^2}{\rho^2} - 2\frac{b}{\rho}\cos(\phi-\phi')\right]^{-1}\left[1 - \frac{b^2}{\rho^2}\right]$$

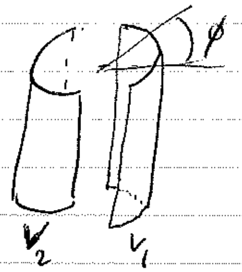
$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi'\, \Phi(b,\phi') \, \frac{\rho^2-b^2}{\rho^2+b^2-2b\rho\cos(\phi-\phi')}$$

for Φ <u>outside</u> the cylinder

2.13 a) Two halves of a long hollow conducting cylinder of inner radius b are ~~sepd~~ separated by small lengthwise gaps on each side, and are kept at different potentials V_1, V_2.
Show that the potential inside is given by

$$\Phi(\rho,\phi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi} \tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)$$

where ϕ is measured from a plane perpendicular to the plane through the gap.



We'll follow the same procedure as in exercise (2.12), since we want a solution inside the cylinder, we adopt eqn (2.71):

$$\Phi(\rho,\phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \rho^n \cos n\phi + \sum_{n=1}^{\infty} b_n \rho^n \sin n\phi$$

(Since we need the solution to be well-behaved as $\rho \to 0$, we omit terms $\propto \rho^{-n}$.)

$$a_n = \frac{1}{\pi b^n}\int_0^{2\pi} d\phi'\, \Phi(b,\phi') \cos n\phi'$$

$$= \frac{1}{\pi b^n}\left[\int_{-\pi/2}^{+\pi/2} d\phi'\, V_1 \cos n\phi' + \int_{\pi/2}^{3\pi/2} d\phi'\, V_2 \cos n\phi'\right]$$

If $n=0$, $= \frac{1}{\pi b^0}\left(V_1\pi + V_2\pi\right) = V_1 + V_2$

If $n \neq 0$:

$$= \frac{1}{\pi b^n}\left[\frac{V_1}{n}\sin n\phi'\Big|_{-\pi/2}^{\pi/2} + \frac{V_2}{n}\sin n\phi'\Big|_{\pi/2}^{3\pi/2}\right]$$

If n even, $=0$

If n odd,

$$= \frac{1}{\pi b^n}\left[\frac{V_1}{n}(2)(-)^{(n-1)/2} + \frac{V_2}{n}(2)(-)^{(n+1)/2}\right]$$

2.13 a), cont'd

a_n, cont'd

n odd:

$$a_n = -\frac{2}{n\pi b^n}(V_1 - V_2)(-)^{(n+1)/2}$$

$$b_n = \frac{1}{\pi b^n}\int_0^{2\pi} d\phi' \, \Phi(b, \phi') \sin n\phi'$$

$$= \frac{1}{\pi b^n}\left[\int_{-\pi/2}^{\pi/2} d\phi' \, V_1 \sin n\phi' + \int_{\pi/2}^{3\pi/2} d\phi' \, V_2 \sin n\phi'\right]$$

$$= \frac{1}{\pi b^n}\left[-\frac{V_1}{n}\cos n\phi'\Big|_{-\pi/2}^{\pi/2} - \frac{V_2}{n}\cos n\phi'\Big|_{\pi/2}^{3\pi/2}\right]$$

$$= 0$$

Thus,

$$\Phi(\rho, \phi) = \frac{V_1 + V_2}{2} + \sum_{\substack{n=1 \\ n\ \text{odd}}}^{\infty} \frac{1}{\pi}\left(\frac{\rho}{b}\right)^n \left(-\frac{2}{n}\right)(V_1 - V_2)(-)^{(n+1)/2} \cos n\phi$$

$$= \frac{V_1 + V_2}{2} + \left(-\frac{\cancel{2}}{\pi}\right)(V_1 - V_2)\sum_{\substack{n=1 \\ n\ \text{odd}}}^{\infty} \frac{1}{n}\left(\frac{\rho}{b}\right)^n (-)^{\frac{n+1}{2}} \frac{e^{in\phi} + e^{-in\phi}}{\cancel{2}}$$

Recall

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

hence

$$\Phi(\rho, \phi) = \frac{V_1 + V_2}{2} - i\frac{V_1 - V_2}{\pi}\left(\frac{1}{2}\ln\left[\frac{1 + i\frac{\rho}{b}e^{i\phi}}{1 - i\frac{\rho}{b}e^{i\phi}}\right] + \frac{1}{2}\ln\left[\frac{1 + i\frac{\rho}{b}e^{-i\phi}}{1 - i\frac{\rho}{b}e^{-i\phi}}\right]\right)$$

using the fact that

$$(-)^{\frac{n+1}{2}}\left(\frac{\rho}{b}\right)^n e^{in\phi} = i\left(i\frac{\rho}{b}e^{i\phi}\right)^n$$

$$\ln\left(\frac{1+i\frac{\rho}{b}e^{i\phi}}{1-i\frac{\rho}{b}e^{i\phi}}\right)+\ln\left(\frac{1+i\frac{\rho}{b}e^{-i\phi}}{1-i\frac{\rho}{b}e^{-i\phi}}\right)$$

$$=\ln\left[\frac{\left(1+i\frac{\rho}{b}e^{i\phi}\right)\left(1+i\frac{\rho}{b}e^{-i\phi}\right)}{\left(1-i\frac{\rho}{b}e^{i\phi}\right)\left(1-i\frac{\rho}{b}e^{-i\phi}\right)}\right]$$

$$=\ln\left[\frac{1-\frac{\rho^2}{b^2}+2i\frac{\rho}{b}\cos\phi}{1-\rho^2/b^2-2i\frac{\rho}{b}\cos\phi}\right]$$

$$=\ln\left[\frac{b^2-\rho^2+2ib\rho\cos\phi}{b^2-\rho^2-2ib\rho\cos\phi}\right]$$

Write $b^2-\rho^2+2ib\rho\cos\phi = re^{i\theta}$

& note $b^2-\rho^2-2ib\rho\cos\phi = re^{-i\theta}$

$$\Rightarrow \ln\left[\frac{b^2-\rho^2+2ib\rho\cos\phi}{b^2-\rho^2-2ib\rho\cos\phi}\right]=\ln\left[e^{2i\theta}\right] = 2i\theta$$

where $\theta = \tan^{-1}\left(+\frac{2b\rho\cos\phi}{b^2-\rho^2}\right)$

$$\Phi(\rho,\phi) = \frac{V_1+V_2}{2} - i\,\frac{V_1-V_2}{2\pi}\ln\left[\frac{b^2-\rho^2+2ib\rho\cos\phi}{b^2-\rho^2-2ib\rho\cos\phi}\right]$$

$$=\frac{V_1+V_2}{2} - i\,\frac{V_1-V_2}{2\pi}(2i)\tan^{-1}\left(+\frac{2b\rho\cos\phi}{b^2-\rho^2}\right)$$

$$\boxed{=\frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2b\rho\cos\phi}{b^2-\rho^2}\right)}$$

Note if $\cos\phi > 0$, then in limit $\rho\to b$, $\tan^{-1}\to +\pi/2$, & $\Phi \to V_1$

if $\cos\phi<0$, then in limit $\rho\to b$, $\tan^{-1}\to -\pi/2$, & $\Phi\to V_2$

& so this has correct boundary conditions.

2.13, cont'd

b) Calculate the surface-charge density on each half of the cylinder.

$\bar{E}\cdot\hat{n} = \sigma/\varepsilon_0$, & here $\hat{n} = -\hat{\rho}$

$$\Rightarrow \sigma = \varepsilon_0 \bar{E}\cdot\hat{n} = -\varepsilon_0 E_\rho = +\varepsilon_0 \left.\frac{\partial\Phi}{\partial\rho}\right|_{\rho=b}$$

$$\frac{\partial\Phi}{\partial\rho} = \frac{V_1-V_2}{\pi}\frac{\partial}{\partial\rho}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)$$

$$= \frac{V_1-V_2}{\pi}\left[1+\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)^2\right]^{-1}\frac{\partial}{\partial\rho}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)$$

$$= \frac{V_1-V_2}{\pi}\frac{(b^2-\rho^2)^2}{(b^2-\rho^2)^2+(2b\rho\cos\phi)^2}\left[\frac{2b}{b^2-\rho^2}\cos\phi - \frac{2b\rho}{(b^2-\rho^2)^2}(-2\rho)\cos\phi\right]$$

$$= \frac{V_1-V_2}{\pi}\left[(b^2-\rho^2)^2+(2b\rho\cos\phi)^2\right]^{-1}\left[2b(b^2-\rho^2)\cos\phi + 4b\rho^2\cos\phi\right]$$

$$= \frac{V_1-V_2}{\pi}\left[(b^2-\rho^2)^2+(2b\rho\cos\phi)^2\right]^{-1}\left[2b(b^2+\rho^2)\cos\phi\right]$$

$$= \frac{V_1-V_2}{\pi}\frac{2b(b^2+\rho^2)}{(b^2-\rho^2)^2+(2b\rho\cos\phi)^2}\cos\phi$$

$$\left.\frac{\partial\Phi}{\partial\rho}\right|_{\rho=b} = \frac{V_1-V_2}{\pi}\frac{4b^3}{4b^4\cos^2\phi}\cos\phi = \frac{V_1-V_2}{\pi}\frac{1}{b\cos\phi}$$

$$\boxed{\sigma = \varepsilon_0 \left.\frac{\partial\Phi}{\partial\rho}\right|_{\rho=b} = \varepsilon_0\frac{V_1-V_2}{\pi b}\sec\phi}$$

2.17 a) Construct the free-space Green function $G(x,y;x',y')$ for two-dimensional electrostatics by integrating $1/R$ with respect to $z'-z$ between the limits $\pm Z$, where Z is taken to be very large. Show that apart from an inessential constant, the Green function can be written alternately as

$$G(x,y;x',y') = -\ln[(x-x')^2+(y-y')^2]$$

$$= -\ln[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')]$$

As directed, we define

$$G(x,y;x',y') \approx \int_{-Z}^{Z} \frac{d\zeta}{(r^2+\zeta^2)^{1/2}}$$

for $\zeta = z'-z$, $r^2 = (x-x')^2+(y-y')^2$

Now, substitute $\zeta = r\sinh\theta$, $d\zeta = r\cosh\theta\, d\theta$

$$\int_{-Z}^{Z} \frac{d\zeta}{(r^2+\zeta^2)^{1/2}} = \int_{-Z}^{Z} \frac{r\cosh\theta\, d\theta}{(r^2+r^2\sinh^2\theta)^{1/2}} = \int_{-Z}^{Z} \frac{\cancel{r\cosh\theta}}{\cancel{r\cosh\theta}}\, d\theta$$

$$= \theta\Big|_{-Z}^{Z} = \operatorname{arcsinh}\left(\frac{Z}{r}\right) - \operatorname{arcsinh}\left(-\frac{Z}{r}\right)$$

$$= 2\operatorname{arcsinh}\left(\frac{Z}{r}\right)$$

Solve $\sinh\theta = \frac{Z}{r}$. Since $Z \gg r$ by assumption, approximate $\sinh\theta = \frac{1}{2}(e^{\theta}-e^{-\theta})$ by $\frac{1}{2}e^{\theta}$ so solve $e^{\theta} = 2Z/r$

$$\Rightarrow \theta = \operatorname{arcsinh}(Z/r) \approx \ln(2Z/r)$$

Thus, up to constants independent of the polar coord's,

$$\boxed{G(x,y;x',y') \simeq 2\ln(1/r) = -2\ln r = -\ln\left[(x-x')^2+(y-y')^2\right]}$$

Write $(x,y) = (\rho\cos\phi, \rho\sin\phi)$
$(x',y') = (\rho'\cos\phi', \rho'\sin\phi')$

$$\Rightarrow (x-x')^2+(y-y')^2 = (\rho\cos\phi-\rho'\cos\phi')^2+(\rho\sin\phi-\rho'\sin\phi')^2$$

$$= \rho^2\cos^2\phi+\rho'^2\cos^2\phi' - 2\rho\rho'\cos\phi\cos\phi'$$

$$+\rho^2\sin^2\phi+\rho'^2\sin^2\phi' - 2\rho\rho'\sin\phi\sin\phi'$$

$$= \rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')$$

Hence,

$$\boxed{G(x,y;x',y') = -\ln\left[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')\right]}$$

2.17, cont'd

b) Show explicitly by separation of variables in polar coordinates that the Green function can be expressed as a Fourier series in the azimuthal coordinate:

$$G = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} g_m(\rho, \rho'),$$

where the radial Green functions satisfy

$$\frac{1}{\rho'} \frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial g_m}{\partial \rho'}\right) - \frac{m^2}{\rho'^2} g_m = -4\pi \frac{\delta(\rho-\rho')}{\rho}$$

Note that $g_m(\rho, \rho')$ for fixed ρ is a different linear combination of the solutions of the homogeneous radial eqn's for $\rho' < \rho$ and for $\rho' > \rho$, with a slope discontinuity at $\rho' = \rho$ determined by the source delta function.

Recall in polar coordinates

$$\nabla^2 \psi = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2}$$

We require the Green function obey

$$\nabla'^2 G = -4\pi \frac{\delta(\rho-\rho')}{\rho} \delta(\phi-\phi')$$

where the $\frac{1}{\rho}$ is to ensure that $\int (\nabla'^2 G) \underset{\sim}{\rho}\, d\rho\, d\phi = -4\pi$,

and by Poisson resummation,

$$\delta(\phi-\phi') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}$$

2.17 b), cont'd

For the G given,

$$\nabla'^2 G = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\left[e^{im(\phi-\phi')}\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g_m}{\partial\rho'}\right) + \frac{1}{\rho'^2}(-m^2)\,e^{im(\phi-\phi')}g_m\right]$$

Require $= -4\pi\,\dfrac{\delta(\rho-\rho')}{\rho}\left(\dfrac{1}{2\pi}\right)\displaystyle\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}$

Clearly, we must require

$$\boxed{\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g_m}{\partial\rho'}\right) - \frac{m^2}{\rho'^2}g_m = -4\pi\,\frac{\delta(\rho-\rho')}{\rho}}$$

2.17, cont'd

c) Complete the solution and show that the free-space Green function has the expansion

$$G(\rho, \phi; \rho', \phi') = -\ln(\rho_>^2) + 2\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cos(m(\phi-\phi'))$$

where $\rho_<$ ($\rho_>$) is the smaller (larger) of ρ, ρ'.

S'pose $\rho' \neq \rho$,
so that

$$\frac{1}{\rho'}\frac{\partial}{\partial \rho'}\left(\rho'\frac{\partial g_m}{\partial \rho'}\right) = \frac{m^2}{\rho'^2} g_m$$

For the moment, assume $m \neq 0$ & WLOG $m > 0$.
It is straightforward to check that the solutions of the eq'n above are of the form

$$g_m(\rho, \rho') = A\rho'^m + B\rho'^{-m}$$

for constants A, B.

For $\rho' < \rho$, we want the solution to be finite as $\rho' \to 0$, so $B = 0$ & we take

$$g_m = A_m \rho'^m$$

For $\rho' > \rho$, we want the solution to go to zero as $\rho' \to \infty$, so $A = 0$ & we take

$$g_m = B_m \rho'^{-m}$$

Hence

$$g_m = \begin{cases} A_m \rho'^m & \rho' < \rho \\ B_m \rho'^{-m} & \rho' > \rho \end{cases}$$

2.17, cont'd

c), cont'd

Next, we need to understand the behavior across $\rho' = \rho$.

~~We can~~

1) Demand ~~[illegible]~~ continuity:

$$\lim_{\rho' \to \rho^-} g_m(\rho, \rho') = \lim_{\rho' \to \rho^+} g_m(\rho, \rho')$$

$$\Rightarrow \underline{A_m \rho^m = B_m \rho^{-m}}$$

2) Return to the differential equation

$$\frac{1}{\rho'} \frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial g_m}{\partial \rho'}\right) - \frac{m^2}{\rho'^2} g_m = -4\pi \frac{\delta(\rho - \rho')}{\rho}$$

$$\Rightarrow \frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial g_m}{\partial \rho'}\right) - \frac{m^2}{\rho'} g_m = -4\pi\, \delta(\rho - \rho')$$

$$\Rightarrow \int_{\rho-\varepsilon}^{\rho+\varepsilon} d\rho' \frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial g_m}{\partial \rho'}\right) - \int_{\rho-\varepsilon}^{\rho+\varepsilon} d\rho' \frac{m^2}{\rho'} g_m = -4\pi$$

Take a limit as $\varepsilon \to 0$.

This becomes

$$\lim_{\varepsilon \to 0} \left(\rho' \frac{\partial g_m}{\partial \rho'}\right)\Bigg|_{\rho-\varepsilon}^{\rho+\varepsilon} = -4\pi$$

or

$$\rho \frac{\partial}{\partial \rho'}\left(B_m \rho'^{-m}\right)\Big|_{\rho'=\rho} - \rho \frac{\partial}{\partial \rho'}\left(A_m \rho'^{m}\right)\Big|_{\rho'=\rho} = -4\pi$$

$$\Rightarrow \underline{\rho\left(-m B_m \rho^{-m-1}\right) - \rho\left(m A_m \rho^{m-1}\right) = -4\pi}$$

2.17 c), cont'd

Solve $\begin{cases} A_m \rho^m = B_m \rho^{-m} \\ -m B_m \rho^{-m} - m A_m \rho^m = -4\pi \end{cases}$

$\Rightarrow -m(A_m \rho^m) - m A_m \rho^m = -4\pi$

$\Rightarrow -2m A_m \rho^m = -4\pi$ or $A_m = +\frac{2\pi}{m}\frac{1}{\rho^m}$

$B_m \rho^{-m} = A_m \rho^m \Rightarrow B_m = A_m \rho^{2m}$

$= \frac{2\pi}{m}\frac{1}{\rho^m}\rho^{2m} = \frac{2\pi}{m}\rho^m$

Putting this together, we find

$$\boxed{g_m(\rho,\rho') = \begin{cases} A_m \rho'^m & \rho' < \rho \\ B_m \rho'^{-m} & \rho' > \rho \end{cases} = \begin{cases} \frac{2\pi}{m}\left(\frac{\rho'}{\rho}\right)^m & \rho' < \rho \\ \frac{2\pi}{m}\left(\frac{\rho}{\rho'}\right)^m & \rho' > \rho \end{cases} = \frac{2\pi}{m}\left(\frac{\rho_<}{\rho_>}\right)^m}$$

2.17 c), cont'd

If $m<0$:

$$g_m = \begin{cases} A_m \rho^{1-m} & \rho'<\rho \\ B_m \rho^{1+m} & \rho'>\rho \end{cases}$$

& otherwise exchange m for $-m$.

Solution:

$$g_m(\rho,\rho') = -\frac{2\pi}{m}\left(\frac{\rho_<}{\rho_>}\right)^{-m}$$

2.17 c), cont'd

It remains to consider the case $m=0$.

As before, solve for $\rho' \neq \rho$ initially:

$$\frac{1}{\rho'}\frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial g_0}{\partial \rho'}\right) = 0$$

$$\Rightarrow \rho' \frac{\partial g_0}{\partial \rho'} = \text{constant}$$

$$\Rightarrow \frac{\partial g_0}{\partial \rho'} \propto \frac{1}{\rho'} \Rightarrow g_0 \propto \ln \rho' \quad \underline{\text{or}} \quad g_0 = \text{constant}$$

We also need to impose continuity at $\rho' = \rho$, & finite as $\rho' \to \infty$.

Solution: $$g_0 = \begin{cases} C \ln \rho' & \rho' < \rho \\ C \ln \rho & \rho' > \rho \end{cases}$$

$$= C \ln \rho_>$$

Integrating across the delta function as before implies

$$\lim_{\rho' \to \rho} \underbrace{\rho' \frac{\partial}{\partial \rho'}(C \ln \rho)}_{=0} - \lim_{\rho' \to \rho} \underbrace{\rho' \frac{\partial}{\partial \rho'}(C \ln \rho')}_{=\rho \frac{C}{\rho} = C} = -4\pi$$

$$\Rightarrow C = 4\pi$$

$$\text{so } g_0 = 4\pi \ln \rho_>$$

2.17, c) cont'd

Putting all of this together,

$$G = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} g_m(\rho,\rho')$$

$$= \frac{1}{2\pi} g_0 + \frac{1}{2\pi}\sum_{m=1}^{\infty} e^{im(\phi-\phi')} g_m + \frac{1}{2\pi}\sum_{m=-\infty}^{-1} e^{im(\phi-\phi')} g_m$$

$$= \frac{1}{2\pi}\left(4\pi \ln \rho_>\right) + \frac{1}{2\pi}\sum_{m=1}^{\infty} e^{im(\phi-\phi')} \frac{2\pi}{m}\left(\frac{\rho_<}{\rho_>}\right)^m$$

$$+ \frac{1}{2\pi}\sum_{m=-\infty}^{-1} e^{im(\phi-\phi')} (-)\frac{2\pi}{m}\left(\frac{\rho_<}{\rho_>}\right)^{-m}$$

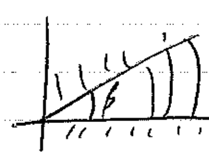
$$= 2\ln \rho_> + \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \left(e^{im(\phi-\phi')} + e^{-im(\phi-\phi')}\right)$$

$$= \ln \rho_>^2 + 2\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cos m(\phi-\phi')$$

2.24 In the two-dimensional region shown in figure 2.12, the angular functions appropriate for Dirichlet boundary conditions at $\phi=0$ and $\phi=\beta$ are $\Phi(\phi) = A_m \sin(m\pi\phi/\beta)$. Show that the completeness rel'n for these function is

$$\delta(\phi-\phi') = \frac{2}{\beta}\sum_{m=1}^{\infty} \sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right) \quad \text{for } 0<\phi,\phi'<\beta$$

Fig. 2.12:



Define $U_n = C_n \sin\left(\frac{n\pi\phi}{\beta}\right)$, for $n>0$.

First, let us determine the normalization & orthogonality relations.

S'pose $m \neq n$:

$$\int_0^\beta U_n^*(\phi)\, U_m(\phi)\, d\phi = C_n C_m \int_0^\beta \sin\left(\frac{n\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi}{\beta}\right) d\phi$$

$$= \frac{C_n^* C_m}{2}\int_0^\beta \left[\cos\left(\frac{(n-m)\pi\phi}{\beta}\right) - \cos\left(\frac{(n+m)\pi\phi}{\beta}\right)\right] d\phi$$

$$= \frac{C_n^* C_m}{2}\left[\frac{\beta}{\pi(n-m)}\sin\left(\frac{(n-m)\pi\phi}{\beta}\right) - \frac{\beta}{\pi(n+m)}\sin\left(\frac{(n+m)\pi\phi}{\beta}\right)\right]_0^\beta$$

$$= \frac{C_n^* C_m}{2}(0) = 0 \quad \text{so they are orthogonal.}$$

Normalization: s'pose $m=n$:

$$\int_0^\beta U_n^*(\phi)\, U_n(\phi)\, d\phi = |C_n|^2 \int_0^\beta \sin^2\left(\frac{n\pi\phi}{\beta}\right) d\phi$$

$$= \frac{|C_n|^2}{2}\int_0^\beta \left[1 - \cos\left(\frac{2n\pi\phi}{\beta}\right)\right] d\phi = \frac{|C_n|^2}{2}\beta$$

so we take $C_n = \sqrt{\frac{2}{\beta}}$

2.24, cont'd

Then, we can expand

$$f(\phi) = \sqrt{\frac{2}{\beta}} \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi\phi}{\beta}\right)$$

where $a_n = \sqrt{\frac{2}{\beta}} \int_0^{\beta} f(\phi) \sin\left(\frac{n\pi\phi}{\beta}\right) d\phi$

Combining:

$$f(\phi) = \frac{2}{\beta} \sum_{n=1}^{\infty} \int_0^{\beta} d\phi' \, f(\phi') \, \sin\left(\frac{n\pi\phi'}{\beta}\right) \sin\left(\frac{n\pi\phi}{\beta}\right)$$

& since this must hold for any function f,
we deduce

$$\delta(\phi-\phi') = \frac{2}{\beta} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\phi'}{\beta}\right) \sin\left(\frac{n\pi\phi}{\beta}\right)$$

2.24, cont'd

Alternatively, if we write

$$\delta(\phi-\phi') = \sqrt{\frac{2}{\beta}}\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi\phi}{\beta}\right)$$

then

$$a_n = \sqrt{\frac{2}{\beta}}\int_0^{\beta} \delta(\phi-\phi') \sin\left(\frac{n\pi\phi}{\beta}\right) d\phi$$

$$= \sqrt{\frac{2}{\beta}} \sin\left(\frac{n\pi\phi'}{\beta}\right)$$

& the result follows.