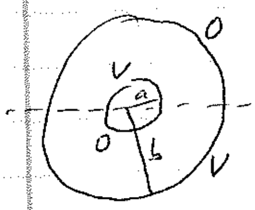
3.1 Two concentric spheres have radii a, b $(b > a)$ and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential V. The other hemispheres are at zero potential.

Determine the potential in the region $a \le r \le b$ as a series in Legendre polynomials. Include terms at least up to $l = 4$. Check your solution against known results in the limiting cases $b \to \infty$, and $a \to 0$.



Since this is cylindrically symmetric, from (3.33)

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)$$

Using the orthonormality of the Legendre polynomials, for any fixed radius ρ, if $\Phi(\rho, \theta) = V(\theta)$, then

$$\int_{-1}^{1} V(x = \cos\theta) P_l(x)\, dx = \left(\frac{2l+1}{2}\right)^{-1} \left(A_l \rho^l + B_l \rho^{-(l+1)}\right)$$

Here,

$$V \int_0^1 P_l(\cos\theta)\, d\cos\theta = \left(\frac{2l+1}{2}\right)^{-1} \left(A_l a^l + B_l a^{-(l+1)}\right)$$

$$V \int_{-1}^{0} P_l(\cos\theta)\, d\cos\theta = \left(\frac{2l+1}{2}\right)^{-1} \left(A_l b^l + B_l b^{-(l+1)}\right)$$

3.1, cont'd

Solve for A_ℓ, B_ℓ:

$$\begin{cases} A_\ell a^\ell + B_\ell a^{-\ell-1} = \frac{2\ell+1}{2} V \int_0^1 P_\ell(x)\,dx \\ A_\ell b^\ell + B_\ell b^{-\ell-1} = \frac{2\ell+1}{2} V \int_{-1}^0 P_\ell(x)\,dx = \frac{2\ell+1}{2} V (-)^\ell \int_0^1 P_\ell(x)\,dx \end{cases}$$

using the symmetry properties of $P_\ell(x)$.

Then,

$$A_\ell a^{2\ell+1} + B_\ell = \frac{2\ell+1}{2} V a^{\ell+1} \int_0^1 P_\ell(x)\,dx$$

$$A_\ell b^{2\ell+1} + B_\ell = \frac{2\ell+1}{2} V (-)^\ell b^{\ell+1} \int_0^1 P_\ell(x)\,dx$$

$$A_\ell \left(b^{2\ell+1} - a^{2\ell+1}\right) = \frac{2\ell+1}{2} V \left((-)^\ell b^{\ell+1} - a^{\ell+1}\right) \int_0^1 P_\ell(x)\,dx$$

$$A_\ell = -\frac{2\ell+1}{2} V \frac{(-b)^{\ell+1} + a^{\ell+1}}{b^{2\ell+1} - a^{2\ell+1}} \int_0^1 P_\ell(x)\,dx$$

$$= \cancel{-\frac{V}{2} \frac{(-b)^{\ell+1} + a^{\ell+1}}{b^{2\ell+1} - a^{2\ell+1}}}$$

For ℓ even, $\int_0^1 P_\ell(x)\,dx = \frac{1}{2}\int_{-1}^1 P_\ell(x)\,dx = \frac{1}{2}\int_{-1}^1 P_0(x) P_\ell(x)\,dx$

$$= \frac{1}{2} \frac{2}{2\ell+1} \delta_{\ell 0} = \delta_{\ell 0}$$

For ℓ odd, from (3.26),

$$\int_0^1 P_\ell(x)\,dx = \left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}} \frac{(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!}$$

3.1, cont'd

Thus,

$$A_\ell = \begin{cases} \frac{V}{2} & \ell = 0 \\ 0 & \ell > 0,\ \ell \text{ even} \\ -\frac{2\ell+1}{2} V \frac{(-b)^{\ell+1} + a^{\ell+1}}{b^{2\ell+1} - a^{2\ell+1}} \left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}} \frac{(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!} & \ell > 0,\ \ell \text{ odd} \end{cases}$$

We can similarly derive an expression for B_ℓ:

$$A_\ell + B_\ell a^{-2\ell-1} = \frac{2\ell+1}{2} V a^{-\ell} \int_0^1 P_\ell(x)\,dx$$

$$A_\ell + B_\ell b^{-2\ell-1} = \frac{2\ell+1}{2} V b^{-\ell} (-)^\ell \int_0^1 P_\ell(x)\,dx$$

$$B_\ell \left(a^{-2\ell-1} - b^{-2\ell-1}\right) = \frac{2\ell+1}{2} V \left(a^{-\ell} - (-b)^{-\ell}\right) \int_0^1 P_\ell(x)\,dx$$

hence

$$B_\ell = \frac{2\ell+1}{2} V \frac{a^{-\ell} - (-b)^{-\ell}}{a^{-2\ell-1} - b^{-2\ell-1}} \int_0^1 P_\ell(x)\,dx$$

3.1, cont'd

Proceeding as before,

$$B_\ell = \begin{cases} 0 & \ell \text{ even} \\ \frac{2\ell+1}{2} V \frac{a^{-\ell} - (-b)^{-\ell}}{a^{-2\ell-1} - b^{-2\ell-1}} \left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}} \frac{(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!} & \ell \text{ odd} \end{cases}$$

3.1, cont'd

Consider the limit $a \to 0$.
Then, we are describing the potential inside a sphere of radius b, of potential 0 on the upper hemisphere & V on the lower hemisphere.

In this limit,

$$A_\ell = \begin{cases} V/2 & \ell = 0 \\ 0 & \ell > 0,\ \ell \text{ even} \\ -\frac{2\ell+1}{2} V (-)^{\ell+1} b^{-\ell} \left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}} \frac{(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!} & \ell \text{ odd} \end{cases}$$

$B_\ell = 0$ for ℓ even,

& for ℓ odd,

$$B_\ell = \lim_{a \to 0} \frac{2\ell+1}{2} V \frac{a^{\ell+1} - a^{2\ell+1}(-b)^{-\ell}}{1 - a^{2\ell+1} b^{-2\ell-1}} \left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}} \frac{(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!}$$

$$= 0$$

hence $B_\ell = 0 \ \forall \ell$

so

$$\Phi(r,\theta) = \frac{V}{2} + \sum_{\ell \text{ odd}} (-)^\ell \frac{2\ell+1}{2} V \left(\frac{r}{b}\right)^\ell \left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}} \frac{(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!} P_\ell(\cos\theta)$$

$$= \frac{V}{2} - \frac{3}{4} V \left(\frac{r}{b}\right) P_1(\cos\theta) - \frac{7}{2} V \left(\frac{r}{b}\right)^3 \left(-\frac{1}{2}\right) \frac{1}{2(2!)} P_3(\cos\theta)$$

$$- \frac{11}{2} V \left(\frac{r}{b}\right)^5 \frac{1}{4} \frac{3 \cdot 1}{2(3!)} P_5(\cos\theta) + \cdots$$

$$= \frac{V}{2} - \frac{V}{2}\left[\frac{3}{2}\frac{r}{b} P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{b}\right)^3 P_3(\cos\theta) + \frac{11}{16}\left(\frac{r}{b}\right)^5 P_5(\cos\theta) + \cdots\right]$$

matching (3.36) after adjusting & rescaling potential, to describe a $\frac{V}{2}$ shift of $-\frac{V}{2}$ on top & $+\frac{V}{2}$ on bottom.

3.1, cont'd

Consider the limit $b \to \infty$.
Then, we are describing the potential outside a sphere of radius a, of potential V on the top and 0 on the bottom.

In this limit,

$$A_\ell = \begin{cases} V/2 & \text{for } \ell = 0 \\ 0 & \text{for } \ell > 0,\ \ell \text{ even} \end{cases}$$

& for ℓ odd,

$$\lim_{b\to\infty} A_\ell = \lim_{b\to\infty} -\frac{2\ell+1}{2} V \frac{(-)^{\ell+1} + (a/b)^{\ell+1}}{b^\ell - a^{2\ell+1} b^{-\ell-1}} \left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}} \frac{(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!}$$

$$= 0$$

$B_\ell = 0$ for ℓ even,
& for ℓ odd,

$$B_\ell = \lim_{b\to\infty} \frac{2\ell+1}{2} V \frac{a^{-\ell} - (-b)^{-\ell}}{a^{-2\ell-1} - b^{-2\ell-1}} \left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}} \frac{(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!}$$

$$= \frac{2\ell+1}{2} V a^{\ell+1} \left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}} \frac{(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!}$$

Thus,

$$\Phi(r,\theta) = \frac{V}{2} + \sum_{\ell \text{ odd}} \frac{2\ell+1}{2} V \left(\frac{a}{r}\right)^{\ell+1} \left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}} \frac{(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!} P_\ell(\cos\theta)$$