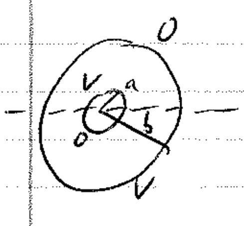
3.13 Solve for the potential between two concentric spheres of radii $a<b$, where the upper hemisphere of the inner sphere & the lower hemisphere of the outer sphere are at V, & the other two at zero, using the Green fn from the text, and verify the solution agrees with that obtained directly from the differential eqn.



The Green fn for the interior between two concentric spheres is

$$G(\bar{x},\bar{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]^{-1}\left[r_<^{l} - \frac{a^{2l+1}}{r_<^{l+1}}\right]\left[\frac{1}{r_>^{l+1}} - \frac{r_>^{l}}{b^{2l+1}}\right]$$

$$\hookrightarrow Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi) \qquad (3.125)$$

Since $\rho = 0$,

$$\Phi(\bar{x}) = -\frac{1}{4\pi}\int_{\partial V} \Phi(\bar{x}')\,\frac{\partial G}{\partial n'}\,da'$$

$$\left.\frac{\partial G}{\partial n'}\right|_{r'=b} = +\left.\frac{\partial G}{\partial r'}\right|_{r'=b} \qquad (r_> = r' = b,\ r_< = r)$$

$$= \frac{\partial}{\partial r'}\left\{\sum_{l,m}\frac{4\pi}{2l+1}\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)^{-1}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\left(\frac{1}{r'^{\,l+1}} - \frac{r'^{\,l}}{b^{2l+1}}\right) Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)\right\}\Bigg|_{r'=b}$$

$$= \sum_{l,m}\frac{4\pi}{2l+1}\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)^{-1}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\left(-\frac{l+1}{r'^{\,l+2}} - \frac{l\,r'^{\,l-1}}{b^{2l+1}}\right) Y^*_{lm}\, Y_{lm}\Bigg|_{r'=b}$$

$$= \sum_{lm}\frac{4\pi}{\cancel{2l+1}}\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)^{-1}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\left(-b^{-(l+2)}\right)\cancel{\left(l+1+l\right)} Y^*_{lm}\, Y_{lm}$$

$$= -4\pi\sum_{lm}\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)^{-1}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right) b^{-(l+2)}\, Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

3.13, cont'd

$$\left.\frac{\partial G}{\partial n'}\right|_{r'=a} = -\left.\frac{\partial G}{\partial r'}\right|_{r'=a} \qquad (r_> = r,\ r_< = r' = a)$$

$$= -\frac{\partial}{\partial r'}\left\{\sum_{l,m}\frac{4\pi}{2l+1}\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)^{-1}\left(r'^{l}-\frac{a^{2l+1}}{r'^{l+1}}\right)\left(\frac{1}{r^{l+1}}-\frac{r^{l}}{b^{2l+1}}\right)Y_{lm}^{*}Y_{lm}\right\}\Bigg|_{r'=a}$$

$$= -\sum_{l,m}\frac{4\pi}{2l+1}\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)^{-1}\left(l a^{l-1}+(l+1)\frac{a^{2l+1}}{a^{l+2}}\right)\left(\frac{1}{r^{l+1}}-\frac{r^{l}}{b^{2l+1}}\right)Y_{lm}^{*}Y_{lm}$$

$$= -a^{l-1}\sum_{l,m}4\pi\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)^{-1}\left(\frac{1}{r^{l+1}}-\frac{r^{l}}{b^{2l+1}}\right)Y_{lm}^{*}(\theta',\phi')\,Y_{lm}(\theta,\phi)$$

Putting this together,

$$\Phi(\vec{x}) = -\frac{1}{4\pi}\int_{\partial V}\Phi(x')\frac{\partial G}{\partial n'}\,da'$$

$$= -\frac{1}{4\pi}\int_{\text{upper}} da'\ V\left.\frac{\partial G}{\partial n'}\right|_{r'=a} - \frac{1}{4\pi}\int_{\text{lower}} da'\ V\left.\frac{\partial G}{\partial n'}\right|_{r'=b}$$

$$= V a^{2}\int_{0}^{1}d\cos\theta'\int_{0}^{2\pi}d\phi\sum_{l,m}a^{l-1}\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)^{-1}Y_{lm}^{*}(\theta',\phi')\ \times$$

$$\hookrightarrow\left(\frac{1}{r^{l+1}}-\frac{r^{l}}{b^{2l+1}}\right)Y_{lm}(\theta,\phi)$$

$$+Vb^{2}\int_{-1}^{0}d\cos\theta'\int_{0}^{2\pi}d\phi\sum_{l,m}b^{-(l+2)}\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)^{-1}Y_{lm}^{*}(\theta',\phi')\ \times$$

$$\hookrightarrow\left(r^{l}-\frac{a^{2l+1}}{r^{l+1}}\right)Y_{lm}(\theta,\phi)$$

The ϕ integral forces $m=0$, & we use $Y_{l0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\,P_l(\cos\theta)$

3.13, cont'd

$$\Phi(\bar{x}) = Va^2 \sum_{\ell=0}^{\infty} \int_0^1 d(\cos\theta')\, a^{\ell-1}\left(1-\left(\tfrac{a}{b}\right)^{2\ell+1}\right)^{-1} \frac{2\ell+1}{4\pi} P_\ell(\cos\theta')\, (2\pi)\, 2$$

$$\hookrightarrow \left(\frac{1}{r^{\ell+1}} - \frac{r^\ell}{b^{2\ell+1}}\right) P_\ell(\cos\theta)$$

$$+V\sum_{\ell=0}^{\infty} \int_{-1}^{0} d(\cos\theta')\, b^{-\ell}\left(1-\left(\tfrac{a}{b}\right)^{2\ell+1}\right)^{-1} \frac{2\ell+1}{4\pi} P_\ell(\cos\theta')\, (2\pi)\, 2$$

$$\hookrightarrow \left(r^\ell - \frac{a^{2\ell+1}}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

$$= \frac{V}{2}\sum_{\ell=0}^{\infty} (2\ell+1)\left(1-\left(\tfrac{a}{b}\right)^{2\ell+1}\right)^{-1} \Bigg\{$$

$$r^\ell\left(b^{-\ell}\int_{-1}^{0} d(\cos\theta')\, P_\ell(\cos\theta') - \frac{a^{\ell+1}}{b^{2\ell+1}}\int_0^1 d(\cos\theta')\, P_\ell(\cos\theta')\right)$$

$$+ r^{-(\ell+1)}\left(a^{\ell+1}\int_0^1 d(\cos\theta')\, P_\ell(\cos\theta') - \frac{a^{2\ell+1}}{b^\ell}\int_{-1}^{0} d(\cos\theta')\, P_\ell(\cos\theta')\right)\Bigg\} P_\ell(\cos\theta)$$

$$\int_{-1}^{0} d(\cos\theta')\, P_\ell(\cos\theta') = (-)^\ell \int_0^1 d(\cos\theta')\, P_\ell(\cos\theta') \quad \text{since } P_\ell(-x) = (-)^\ell P_\ell(x)$$

3.13, cont'd

Coefficient of r^{ℓ} is

$$\frac{V}{2}(2\ell+1)\left(1-\left(\frac{a}{b}\right)^{2\ell+1}\right)^{-1}\left[(-)^{\ell} b^{-\ell}-\frac{a^{\ell+1}}{b^{2\ell+1}}\right]\int_0^1 d(\cos\theta')\, P_\ell(\cos\theta')$$

$$=\frac{V}{2}(2\ell+1)\left[\frac{(-)^{\ell} b^{\ell+1}-a^{\ell+1}}{b^{2\ell+1}-a^{2\ell+1}}\right]\int_0^1 d(\cos\theta')\, P_\ell(\cos\theta')$$

= coefficient of r^{ℓ} in problem 3.1 $\equiv A_\ell$

$$=\begin{cases} V/2 & \ell=0 \\ 0 & \ell>0,\ \ell \text{ even} \\ -\frac{2\ell+1}{2} V \frac{(-b)^{\ell+1}+a^{\ell+1}}{b^{2\ell+1}-a^{2\ell+1}}\left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}} \frac{(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!} & \ell>0,\ \ell \text{ odd} \end{cases}$$

Coefficient of $r^{-(\ell+1)}$ is

$$\frac{V}{2}(2\ell+1)\left(1-\left(\frac{a}{b}\right)^{2\ell+1}\right)^{-1}\left[a^{\ell+1}-(-)^{\ell}\frac{a^{2\ell+1}}{b^{\ell}}\right]\int_0^1 d(\cos\theta')\, P_\ell(\cos\theta')$$

$$=\frac{V}{2}(2\ell+1)\ \frac{a^{-\ell}-(-b)^{-\ell}}{a^{-(2\ell+1)}-b^{-(2\ell+1)}}\int_0^1 d(\cos\theta')\, P_\ell(\cos\theta').$$

= coefficient of $r^{-(\ell+1)}$ in problem 3.1 $\equiv B_\ell$

$$=\begin{cases} 0 & \ell \text{ even} \\ \frac{2\ell+1}{2} V \frac{a^{-\ell}-(-b)^{-\ell}}{a^{-(2\ell+1)}-b^{-(2\ell+1)}}\left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}} \frac{(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!} & \ell \text{ odd} \end{cases}$$

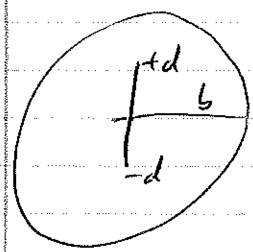
3.13, cont'd

Thus, since the coefficients match,

the potential $\Phi(\vec{x})$ obtained from the Green function matches that obtained directly from the differential eqn's in problem 3.1.

3.14 A line charge of length 2d with a total charge Q has a linear charge density varying as $d^2 - z^2$, where z is the distance from the midpoint. A grounded, conducting sphere of inner radius $b > d$ is centered at the midpoint of the line charge.

a) Find the potential everywhere inside the spherical shell as an expansion in Legendre polynomials



Find Φ inside.

From (3.125), for this case

$$G(\bar{x}, \bar{x}') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \left(\frac{r_<^l}{r_>^{l+1}} - \frac{r_<^l r_>^l}{b^{2l+1}} \right) Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$$

& since the problem is cylindrically symmetric, only the $m=0$ components will contribute,

so as $Y_{l0}(\theta, \phi) = \left(\frac{2l+1}{4\pi}\right)^{1/2} P_l(\cos\theta)$,

we might as well write

$$G(\bar{x}, \bar{x}') = \sum_{l=0}^{\infty} \left(\frac{r_<^l}{r_>^{l+1}} - \frac{r_<^l r_>^l}{b^{2l+1}} \right) P_l(\cos\theta') P_l(\cos\theta)$$

In general terms, the solution will be obtained as

$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0} \int_V \rho(\bar{x}') G(\bar{x}, \bar{x}') d^3x' - \frac{1}{4\pi} \int_{\partial V} \Phi(x') \frac{\partial G}{\partial n'} da'$$

Here, $\Phi = 0$ on ∂V, so only the first term contributes.

3.14 a), cont'd

We need an expression for ρ. We know

linear charge density $\propto d^2 - z^2$ so $\rho \propto \frac{d^2-z^2}{r^2}$
(where the r^2 corrects for the r^2 in volume measure).

Also, $\rho \propto \left(\delta(\cos\theta - 1) + \delta(\cos\theta + 1) \right) \Theta(d-r)$

where $\Theta(x) = \begin{cases} 0 & x<0 \\ 1 & x>0 \end{cases}$

Write $\rho = A \frac{d^2-z^2}{r^2} \left(\delta(\cos\theta-1) + \delta(\cos\theta+1) \right) \Theta(d-r)$

Then, $\int_V \rho(\bar{x})\, d^3\bar{x} = A \int r^2 dr\, d(\cos\theta)\, d\phi\, \frac{d^2 - r^2\cos^2\theta}{r^2}$ 2

$\hookrightarrow \left(\delta(\cos\theta-1) + \delta(\cos\theta+1) \right) \Theta(d-r)$

$$= 2\pi A (2) \int_0^d dr\, \cancel{r^2}\, \frac{d^2-r^2}{\cancel{r^2}}$$

$$= 4\pi A \left[d^2 r - \frac{r^3}{3} \right]_0^d = 4\pi A d^3 \left(\frac{2}{3}\right) = \frac{8\pi}{3} A d^3$$

demand $= Q$

$$\Rightarrow A = \frac{3}{8\pi} \frac{Q}{d^3}$$

or $\rho(\bar{x}) = \frac{3}{8\pi} \frac{Q}{d^3} \frac{d^2-z^2}{r^2} \left(\delta(\cos\theta-1) + \delta(\cos\theta+1) \right) \Theta(d-r)$

3.14 a), cont'd

$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\bar{x}')\, G(\bar{x},\bar{x}')\, d^3x'$$

$$= \frac{1}{4\pi\varepsilon_0}\int r'^2 dr'\, d(\cos\theta')\, d\phi'\ \frac{3}{8\pi}\frac{Q}{d^3}\frac{d^2-z'^2}{r'^2}\left(\delta(\cos\theta'-1)+\delta(\cos\theta'+1)\right) \theta(d-r')\sum_{l=0}^{\infty}\left(\frac{r_<^l}{r_>^{l+1}} - \frac{r_<^l r_>^l}{b^{2l+1}}\right) P_l(\cos\theta') P_l(\cos\theta)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{3}{8\pi}\frac{Q}{d^3}(2\pi)\int_0^d r'^2 dr'\ \frac{d^2-r'^2}{r'^2}\sum_{l=0}^{\infty}\left(\frac{r_<^l}{r_>^{l+1}} - \frac{r_<^l r_>^l}{b^{2l+1}}\right) P_l(\cos\theta) \left(P_l(1)+P_l(-1)\right)$$

$$P_l(1) = 1, \quad P_l(-1) = (-)^l$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{3}{8\pi}\frac{Q}{d^3}(2\pi)\sum_{l\ \text{even}}(2)\int_0^d dr'\,(d^2-r'^2)\left(\frac{r_<^l}{r_>^{l+1}} - \frac{r_<^l r_>^l}{b^{2l+1}}\right) P_l(\cos\theta)$$

3.14 a), cont'd

If $r \geq d$, then $r_< = r'$, $r_> = r$.
In this case,

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{3}{2}\frac{Q}{d^3}\sum_{\ell \text{ even}} P_\ell(\cos\theta)\underbrace{\int_0^d dr'(d^2-r'^2)\left(\frac{r'^\ell}{r^{\ell+1}} - \frac{r'^\ell r^\ell}{b^{2\ell+1}}\right)}_{*}$$

$$* = \int_0^d dr'\left[\frac{d^2 r'^\ell}{r^{\ell+1}} - \frac{r'^{\ell+2}}{r^{\ell+1}} - \frac{d^2 r'^\ell r^\ell}{b^{2\ell+1}} + \frac{r'^{\ell+2} r^\ell}{b^{2\ell+1}}\right]$$

$$= \left[\frac{d^2}{r^{\ell+1}}\frac{r'^{\ell+1}}{\ell+1} - \frac{1}{r^{\ell+1}}\frac{r'^{\ell+3}}{\ell+3} - \frac{d^2 r^\ell}{b^{2\ell+1}}\frac{r'^{\ell+1}}{\ell+1} + \frac{r^\ell}{b^{2\ell+1}}\frac{r'^{\ell+3}}{\ell+3}\right]_0^d$$

$$= \frac{d^{\ell+3}}{r^{\ell+1}}\frac{1}{\ell+1} - \frac{1}{\ell+3}\frac{d^{\ell+3}}{r^{\ell+1}} - \frac{1}{\ell+1}\frac{r^\ell}{b^{2\ell+1}}d^{\ell+3} + \frac{1}{\ell+3}\frac{r^\ell}{b^{2\ell+1}}d^{\ell+3}$$

$$= \frac{2}{(\ell+1)(\ell+3)}\left[\frac{d^{\ell+3}}{r^{\ell+1}} - \frac{r^\ell d^{\ell+3}}{b^{2\ell+1}}\right]$$

$$= \frac{2}{(\ell+1)(\ell+3)}d^{\ell+3}\left[\frac{1}{r^{\ell+1}} - \frac{r^\ell}{b^{2\ell+1}}\right]$$

so

$$\Phi(\vec{x}) = \frac{3}{8\pi\varepsilon_0}Q\sum_{\ell \text{ even}}\frac{2d^\ell}{(\ell+1)(\ell+3)}\left(\frac{1}{r^{\ell+1}} - \frac{r^\ell}{b^{2\ell+1}}\right)P_\ell(\cos\theta)$$

3.14 a), cont'd

Next, s'pose $r \le d$.

$$\Phi(\bar{x}) = \frac{3}{8\pi\epsilon_0}\frac{Q}{d^3}\sum_{\ell\,\text{even}} P_\ell(\cos\theta)\left[\int_0^r dr'\,(d^2-r'^2)\left(\frac{r'^\ell}{r^{\ell+1}} - \frac{r^\ell r'^\ell}{b^{2\ell+1}}\right) + \int_r^d dr'\,(d^2-r'^2)\left(\frac{r^\ell}{r'^{\ell+1}} - \frac{r^\ell r'^\ell}{b^{2\ell+1}}\right)\right]$$

$$\int_0^r dr'\,(d^2-r'^2)\left(\frac{r'^\ell}{r^{\ell+1}} - \frac{r^\ell r'^\ell}{b^{2\ell+1}}\right)$$

$$= \int_0^r dr'\left[\frac{d^2 r'^\ell}{r^{\ell+1}} - \frac{r'^{\ell+2}}{r^{\ell+1}} - \frac{d^2 r^\ell r'^\ell}{b^{2\ell+1}} + \frac{r^\ell r'^{\ell+2}}{b^{2\ell+1}}\right]$$

$$= \frac{1}{\ell+1}\frac{d^2 r^{\ell+1}}{r^{\ell+1}} - \frac{1}{\ell+3}\frac{r^{\ell+3}}{r^{\ell+1}} - \frac{d^2 r^\ell}{b^{2\ell+1}}\frac{1}{\ell+1}r^{\ell+1} + \frac{1}{\ell+3}\frac{r^\ell r^{\ell+3}}{b^{2\ell+1}}$$

$$= \frac{1}{\ell+1}d^2 - \frac{r^2}{\ell+3} - \frac{1}{\ell+1}\frac{d^2 r^{2\ell+1}}{b^{2\ell+1}} + \frac{1}{\ell+3}\frac{r^{2\ell+3}}{b^{2\ell+1}}$$

$$\cancel{\int_r^d dr'\,(d^2-r'^2)\left(\frac{r^\ell}{r'^{\ell+1}} - \right.}$$

3.14 a), cont'd

$$\int_r^d dr' (d^2 - r'^2)\left(\frac{r^\ell}{r'^{\ell+1}} - \frac{r^\ell r'^\ell}{b^{2\ell+1}}\right)$$

$$= \int_r^d dr' \left[\underset{A}{\frac{d^2 r^\ell}{r'^{\ell+1}}} - \underset{B}{\frac{r^\ell}{r'^{\ell-1}}} - \underset{C}{\frac{d^2 r^\ell r'^\ell}{b^{2\ell+1}}} + \underset{D}{\frac{r^\ell r'^{\ell+2}}{b^{2\ell+1}}}\right]$$

A: If $\ell = 0$,

$$= d^2 r^\ell \ln(d/r)$$

If $\ell > 0$,

$$= -\frac{d^2 r^\ell}{\ell}\left(\frac{1}{d^\ell} - \frac{1}{r^\ell}\right)$$

B: If $\ell = 0$,

$$= -\frac{r^\ell}{2}(d^2 - r^2)$$

If $\ell = 1$,

$$= -r^\ell (d - r)$$

If $\ell = 2$,

$$= -r^\ell \ln(d/r)$$

If $\ell > 2$,

$$= -\frac{1}{\ell-2} r^\ell \left(\frac{1}{d^{\ell-2}} - \frac{1}{r^{\ell-2}}\right)$$

$$C = -\frac{d^2 r^\ell}{b^{2\ell+1}} \frac{1}{\ell+1}\left(d^{\ell+1} - r^{\ell+1}\right)$$

$$D = \frac{r^\ell}{b^{2\ell+1}} \frac{1}{\ell+3}\left(d^{\ell+3} - r^{\ell+3}\right)$$

3, 14 a), cont'd

$r \le d$, cont'd

$$
\begin{aligned}
\Phi(\bar{x}) = \frac{3}{8\pi\varepsilon_0}\frac{Q}{d^3}\Bigg[& P_0(\cos\theta)\Bigg(d^2 - \frac{r^2}{3} - \cancel{\frac{d^2 r}{b}} + \cancel{\frac{1}{3}\frac{r^3}{b}} \\
& \quad + d^2 \ln(d/r) - \frac{1}{2}\left(d^2 - r^2\right) \\
& \quad - \frac{d^2}{b}\left(d - \cancel{r}\right) + \frac{1}{3}\frac{1}{b}\left(d^3 - \cancel{r^3}\right) \Bigg) \\
& + P_2(\cos\theta)\Bigg(\frac{1}{3}d^2 - \frac{1}{5}r^2 - \cancel{\frac{1}{3}\frac{d^2 r^5}{b^5}} + \cancel{\frac{1}{5}\frac{r^7}{b^5}} \\
& \quad - \frac{1}{2}d^2 r^2\left(\frac{1}{d^2} - \frac{1}{r^2}\right) - r^2 \ln(d/r) \\
& \quad - \frac{1}{3}\frac{d^2 r^2}{b^5}\left(d^3 - \cancel{r^3}\right) + \frac{1}{5}\frac{r^2}{b^5}\left(d^5 - \cancel{r^5}\right) \Bigg) \\
& + \sum_{\substack{\ell \text{ even} \\ \ell > 2}} P_\ell(\cos\theta)\Bigg(\frac{d^2}{\ell+1} - \frac{r^2}{\ell+3} - \cancel{\frac{1}{\ell+1}\frac{d^2 r^{2\ell+1}}{b^{2\ell+1}}} + \cancel{\frac{1}{\ell+3}\frac{r^{2\ell+3}}{b^{2\ell+1}}} \\
& \quad - \frac{d^2 r^\ell}{\ell}\left(\frac{1}{d^\ell} - \frac{1}{r^\ell}\right) - \frac{1}{\ell-2} r^\ell \left(\frac{1}{d^{\ell-2}} - \frac{1}{r^{\ell-2}}\right) \\
& \quad - \frac{1}{\ell+1}\frac{d^2 r^\ell}{b^{2\ell+1}}\left(d^{\ell+1} - \cancel{r^{\ell+1}}\right) + \frac{1}{\ell+3}\frac{r^\ell}{b^{2\ell+1}}\left(d^{\ell+3} - \cancel{r^{\ell+3}}\right) \Bigg) \Bigg]
\end{aligned}
$$

$r \le d$, cont'd

$$\Phi(\vec{x}) = \frac{3}{8\pi\varepsilon_0}\frac{Q}{d^3}\Big[(1)\left(\frac{1}{2}d^2 + \frac{1}{6}r^2 - \frac{2}{3}\frac{d^3}{b} + d^2 \ln(d/r)\right)$$

$$+ P_2(\cos\theta)\left(\frac{5}{6}d^2 - \frac{7}{10}r^2 - \frac{2}{15}\frac{d^5 r^2}{b^5} - r^2 \ln(d/r)\right)$$

$$+ \sum_{\substack{\ell \text{ even} \\ \ell > 2}} P_\ell(\cos\theta)\left(\frac{2\ell+1}{\ell(\ell+1)}d^2 + \frac{5}{(\ell+3)(\ell-2)}r^2 - \frac{2\ell-2}{\ell(\ell-2)}\frac{r^\ell}{d^{\ell-2}}\right.$$

$$\left. - \frac{2}{(\ell+1)(\ell+3)}\frac{r^\ell d^{\ell+3}}{b^{2\ell+1}}\right)\Big]$$

for $r \le d$

3.14, cont'd

b) Calculate the surface-charge density induced on the shell.

$$\sigma/\varepsilon_0 = \bar{E}\cdot\hat{n} = -E_r \quad (\text{sign b/c } \hat{n} \text{ points inward})$$

$$\Rightarrow \sigma = -\varepsilon_0 E_r = +\varepsilon_0 \left.\frac{\partial \Phi}{\partial r}\right|_{r=b}$$

$$= +\varepsilon_0 \frac{\partial}{\partial r}\left[\frac{3Q}{4\pi\varepsilon_0}\sum_{\ell\ \text{even}} \frac{d^\ell}{(\ell+1)(\ell+3)}\left(\frac{1}{r^{\ell+1}} - \frac{r^\ell}{b^{2\ell+1}}\right)P_\ell(\cos\theta)\right]\Bigg|_{r=b}$$

$$= \varepsilon_0 \frac{3Q}{4\pi\varepsilon_0}\sum_{\ell\ \text{even}} \frac{d^\ell}{(\ell+1)(\ell+3)}\left(-\frac{(\ell+1)}{r^{\ell+2}} - \frac{\ell r^{\ell-1}}{b^{2\ell+1}}\right)P_\ell(\cos\theta)\Bigg|_{r=b}$$

$$= -\frac{3Q}{4\pi}\sum_{\ell\ \text{even}} \frac{d^\ell}{(\ell+1)(\ell+3)}\left(\frac{\ell+1}{b^{\ell+2}} + \frac{\ell}{b^{\ell+2}}\right)P_\ell(\cos\theta)$$

$$= -\frac{3Q}{4\pi}\sum_{\ell\ \text{even}} \frac{d^\ell}{(\ell+1)(\ell+3)}\left(\frac{2\ell+1}{b^{\ell+2}}\right)P_\ell(\cos\theta)$$

3.14, cont'd

c) Discuss your answers to (a), (b) in the limit $d \ll b$.

Consider the potential for $d < r < b$, $d \ll b$.

$$\Phi(\vec{x}) \simeq \frac{3Q}{8\pi\epsilon_0}\frac{2}{(1)(3)}\left(\frac{1}{r}-\frac{1}{b}\right)P_0(\cos\theta)+\cdots$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r}-\frac{1}{b}\right) + \cdots$$

= potential due to point charge Q at origin (with constant shift), as expected if the line of charge shrinks to nearly a point.

Similarly,

$$\sigma \simeq -\frac{3Q}{4\pi}\frac{1}{(1)(3)}\left(\frac{1}{b^2}\right)P_0(\cos\theta)+\cdots$$

$$= -\frac{Q}{4\pi b^2}$$

as again would be expected for a point charge at the origin - one finds a 'mirroring' charge spread over the interior surface of the sphere.

(b, c, d) only

3.16 b) Obtain the following expression:

$$\frac{1}{|\bar{x}-\bar{x}'|} = \sum_{m=-\infty}^{\infty} \int_0^\infty dk\, e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') e^{-k(z_>-z_<)}$$

Let's solve the PDE $\nabla^2 G(\bar{x},\bar{x}') = -4\pi\delta^3(\bar{x}-\bar{x}')$.

We know components from separation of variables, so we can expand

$$G(\bar{x},\bar{x}') = \sum_{m=-\infty}^{\infty} \int_0^\infty dk\, e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho')\, g_m(k, z, z')$$

reflecting an alternative set of choices relative to the expansion derived in § 3.11, (& the fact that the expression should be finite if either $\bar{x}$ or $\bar{x}' \to 0$ separately).

Plugging in,

$$\nabla_x^2 G(\bar{x},\bar{x}') = \sum_{m=-\infty}^{\infty} \int_0^\infty dk\, e^{im(\phi-\phi')} J_m(k\rho') \Big[\left(\frac{1}{\rho} J_m'(k\rho) k + J_m''(k\rho) k^2 - \frac{m^2}{\rho^2} J_m\right) g_m + \frac{\partial^2 g_m}{\partial z^2} J_m \Big]$$

$$= \sum_{m=-\infty}^{\infty} \int_0^\infty dk\, e^{im(\phi-\phi')} J_m(k\rho') \Big[k^2 \underbrace{\left(J_m''(k\rho) + \frac{1}{k\rho} J_m'(k\rho) - \frac{m^2}{k^2\rho^2} J_m \right)}_{= -J_m(k\rho)} g_m + \frac{\partial^2 g_m}{\partial z^2} J_m \Big]$$

$$= \sum_{m=-\infty}^{\infty} \int_0^\infty dk\, e^{im(\phi-\phi')} J_m(k\rho') J_m(k\rho) \left[\frac{\partial^2 g_m}{\partial z^2} - k^2 g_m \right]$$

3.16 b), cont'd

Demand $\nabla_x^2 G(\bar{x}, \bar{x}') = -4\pi\,\delta^3(\bar{x}-\bar{x}')$:

$$\sum_{m=-\infty}^{\infty} \int_0^{\infty} dk\; e^{im(\phi-\phi')}\, J_m(k\rho')\, J_m(k\rho) \left[\frac{\partial^2 g_m}{\partial z^2} - k^2 g_m \right]$$

$$= -4\pi\, \delta(\phi-\phi')\, \frac{\delta(\rho-\rho')}{\rho}\, \delta(z-z')$$

Use $\delta(\phi-\phi') = \dfrac{1}{2\pi} \displaystyle\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}$ (3.139)

$$\frac{\delta(\rho-\rho')}{\rho} = \int_0^{\infty} dk\; k\, J_m(k\rho)\, J_m(k\rho') \qquad (3.108)$$

Hence

$$\frac{\partial^2 g_m}{\partial z^2} - k^2 g_m = -2k\,\delta(z-z')$$

For $z \neq z'$,

$$\frac{\partial^2 g_m}{\partial z^2} = k^2 g_m \Rightarrow g_m \propto e^{\pm kz}$$

We want a solution that is symmetric between z, z', and goes $\to 0$ as ~~eith~~ $|\bar{x}-\bar{x}'| \to \infty$

~~[illegible]~~

Thus, we want

$$g_m = \begin{cases} A\, e^{-k(z-z')} & z > z' \\ A\, e^{-k(z'-z)} & z' > z \end{cases}$$

$$= A\, e^{-k(z_> - z_<)}$$

3.16 b), cont'd

Next, we need to determine A.

From
$$\frac{\partial^2 g}{\partial z^2} - k^2 g = -2k\,\delta(z-z'),$$
if we integrate both side from $z = z'-\varepsilon$ to $z'+\varepsilon$ for small ε:
$$\lim_{\varepsilon\to 0}\int_{z'-\varepsilon}^{z'+\varepsilon} dz \left[\frac{\partial^2 g}{\partial z^2} - k^2 g\right] = \left.\frac{\partial g}{\partial z}\right|_{z=z'+\varepsilon} - \left.\frac{\partial g}{\partial z}\right|_{z=z'-\varepsilon}$$

$$\text{demand} = -2k$$

$$\left.\frac{\partial g}{\partial z}\right|_{z=z'+\varepsilon} = \left.\frac{\partial}{\partial z}\left(A e^{-k(z-z')}\right)\right|_{z=z'+\varepsilon} = -kA$$

$$\left.\frac{\partial g}{\partial z}\right|_{z=z'-\varepsilon} = \left.\frac{\partial}{\partial z}\left(A e^{-k(z'-z)}\right)\right|_{z=z'-\varepsilon} = +kA$$

$$\left.\frac{\partial g}{\partial z}\right|_{z'+\varepsilon} - \left.\frac{\partial g}{\partial z}\right|_{z'-\varepsilon} = -2kA$$

$$\text{demand} = -2k \Rightarrow \underline{A = 1}$$

Given that the 'boundary conditions' are at $0, \infty$,
this Green f'n should be $|\bar{x}-\bar{x}'|^{-1}$, hence

$$\boxed{\frac{1}{|\bar{x}-\bar{x}'|} = \sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\, e^{im(\varphi-\varphi')} J_m(k\rho) J_m(k\rho') e^{-k(z_>-z_<)}}$$

3.16 b), cont'd

As a consistency check, s'pose $\rho = \rho' = 0$,
so that $\bar{x}$, $\bar{x}'$ both lie along the z axis,
and

$$|\bar{x} - \bar{x}'|^{-1} = |z - z'|^{-1}.$$

From their series expansions,

$$J_0(0) = 0$$

$$J_m(0) = 0 \quad \text{for } m \neq 0$$

so in this case,

$$\sum_{m=-\infty}^{\infty} \int_0^\infty dk\, e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') e^{-k(z_> - z_<)}$$

$$= \int_0^\infty dk\, (1)(1)(1)\, e^{-k(z_> - z_<)}$$

$$= \frac{1}{z_> - z_<}$$

$$= \frac{1}{|z - z'|} = \frac{1}{|\bar{x} - \bar{x}'|} \quad \text{in this case} \checkmark$$

3.16 cont'd

c) By appropriate limiting procedure, prove the following expansions:

$$\frac{1}{\sqrt{\rho^2+z^2}} = \int_0^\infty dk\, e^{-k|z|} J_0(k\rho)$$

$$J_0\left(k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi}\right) = \sum_{m=-\infty}^{\infty} e^{im\phi} J_m(k\rho) J_m(k\rho')$$

$$e^{ik\rho\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(k\rho)$$

For the first expression, consider the special case $\bar{x}'=0$.

$$\frac{1}{|\bar{x}|} = \frac{1}{\sqrt{\rho^2+z^2}} = \sum_{m=-\infty}^{\infty} \int_0^\infty dk\, e^{im(\phi-\phi')} J_m(k\rho) J_m(0)\, e^{-k(z_>-z_<)}$$

but $J_m(0) = \delta_{m0}$, as can be seen from the series expansion,

and $z_> - z_< = |z|$

$$\boxed{\frac{1}{\sqrt{\rho^2+z^2}} = \int_0^\infty dk\, J_0(k\rho)\, e^{-k|z|}}$$

3.16 c), cont'd

In the expression just derived, replace ρ^2 by $\rho^2+\rho'^2-2\rho\rho'\cos\phi$:

$$\frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi+z^2}} = \int_0^\infty dk\, J_0\left(k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi}\right)e^{-k|z|}$$

but LHS $= \frac{1}{|\bar{x}-\bar{x}'|}$ for $\phi'=0$, $z'=0$,

hence

$$= \sum_{m=-\infty}^{\infty}\int_0^\infty dk\; e^{im\phi}\, J_m(k\rho)\, J_m(k\rho')\, e^{-k|z|}$$

Comparing the two expressions, we see

$$\boxed{J_0\left(k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi}\right) = \sum_{m=-\infty}^{\infty} e^{im\phi}\, J_m(k\rho)\, J_m(k\rho')}$$

3.16 c), cont'd

To show the last identity,
use the generating function:

$$e^{\frac{x}{2}(t - 1/t)} = \sum_{m=-\infty}^{\infty} J_m(x)\, t^m$$

Take $t = i e^{i\phi}$, $x = k\rho$

Then
$$t - t^{-1} = i e^{i\phi} - i^{-1} e^{-i\phi}$$
$$= i e^{i\phi} + i e^{-i\phi}$$
$$= 2i \cos\phi$$

$$\Rightarrow \quad e^{\frac{k\rho}{2}(2i)\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(k\rho)$$

$$\boxed{e^{ik\rho\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(k\rho)}$$

3.16, cont'd

d) From the last result obtain an integral representation of the Bessel function

$$J_m(x) = \frac{1}{2\pi i^m}\int_0^{2\pi} d\phi\, e^{ix\cos\phi - im\phi}$$

Compare the standard integral representations.

From $e^{ix\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(x)$,

we use orthogonality of the functions $e^{im\phi}$:

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{ix\cos\phi}\, e^{-im\phi} = i^m J_m(x)$$

& the result follows.

There are a number of integral representations of the Bessel f'ns. For example,

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} d\phi \cos(x\sin\phi) = \frac{1}{2\pi}\int_0^{2\pi} d\phi \cos(x\sin\phi)$$

Using the fact that $\int_0^{2\pi} \sin(x\sin\phi)\, d\phi = 0$,

we see this is equivalent to

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{ix\cos\phi} = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{ix\sin\phi}$$

The result above is similarly related to

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} d\phi \cos(x\sin\phi - n\phi)$$