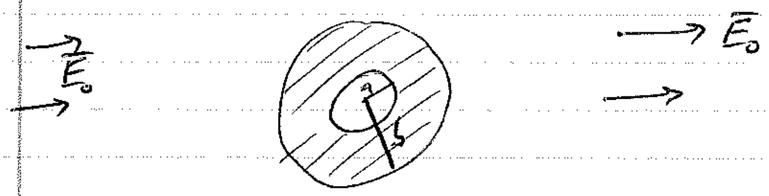
(a) only

4.8 A very long, right circular cylindrical shell of dielectric constant ϵ/ϵ_0 and inner and outer radii a and b, respectively, is placed in a previously uniform electric field E_0 with its axis perpendicular to the field. The medium inside and outside the cylinder has a dielectric constant of unity.



a) Determine the potential Φ and electric field in the three regions, neglecting end effects.

The problem is symmetric along the z axis, hence the potential and electric field should be independent of z, so the problem is effectively two-dimensional.

Solve $\nabla^2 \Phi = 0$

$$\Rightarrow \frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

Separate variables: write $\Phi = R(\rho)\, Q(\phi)$

$$\Rightarrow \frac{R''}{R} + \frac{1}{\rho}\frac{R'}{R} + \frac{1}{\rho^2}\frac{Q''}{Q} = 0$$

$$\text{or} \quad \rho^2\frac{R''}{R} + \rho\frac{R'}{R} = -\frac{Q''}{Q}$$

Sol'ns: $Q(\phi) \propto e^{im\phi}$, & $Q(\phi+2\pi) = Q(\phi) \Rightarrow m$ an integer

$$\Rightarrow \rho^2\frac{R''}{R} + \rho\frac{R'}{R} = +m^2$$

If $m \neq 0$, $R(\rho) \propto \rho^{\pm m}$
If $m = 0$, $R(\rho) \propto 1, \ln(\rho)$

4.8 a), cont'd

Label the three regions as follows:

$$A = \{\rho > b\}, \quad B = \{a < \rho < b\}, \quad C = \{\rho < a\}$$

Requiring that Φ be well-behaved as $\rho \to 0$ and as $\rho \to \infty$, we find solutions of the form

$$\rho < a: \ \Phi_C = \sum_{m=1}^{\infty} D_m \rho^m \cos(m\phi) + D_0$$

$$a < \rho < b: \ \Phi_B = B_0 + C_0 \ln\rho + \sum_{m=1}^{\infty} \left(B_m \rho^m + C_m \rho^{-m}\right) \cos(m\phi)$$

$$\rho > b: \ \Phi_A = A_0 + \sum_{m=1}^{\infty} A_m \rho^{-m} \cos(m\phi) - \underbrace{E_0 \rho \cos\phi}_{\text{from external field}}$$

In writing the above, we have used the $\phi \to -\phi$ symmetry to eliminate some phase choices, & also used the fact that phases should align across boundaries, to make this discussion more brief.

Boundary conditions:

Continuity of $E_{\parallel}$:

① $$-\frac{1}{a} \frac{\partial \Phi_C}{\partial \phi}\bigg|_{\rho=a} = -\frac{1}{a} \frac{\partial \Phi_B}{\partial \phi}\bigg|_{\rho=a}$$

② $$-\frac{1}{b} \frac{\partial \Phi_B}{\partial \phi}\bigg|_{\rho=b} = -\frac{1}{b} \frac{\partial \Phi_A}{\partial \phi}\bigg|_{\rho=b}$$

Continuity of $D_{\perp}$:

③ $$-\varepsilon_0 \frac{\partial \Phi_C}{\partial \rho}\bigg|_{\rho=a} = -\varepsilon \frac{\partial \Phi_B}{\partial \rho}\bigg|_{\rho=a}$$

④ $$-\varepsilon \frac{\partial \Phi_B}{\partial \rho}\bigg|_{\rho=b} = -\varepsilon_0 \frac{\partial \Phi_A}{\partial \rho}\bigg|_{\rho=b}$$

4.8 a), cont'd

① $\Rightarrow \displaystyle\sum_{m=1}^{\infty} D_m a^m (-m) \sin m\phi = \sum_{m=1}^{\infty} (B_m a^m + C_m a^{-m})(-m) \sin m\phi$

② $\Rightarrow \displaystyle\sum_{m=1}^{\infty} (B_m b^m + C_m b^{-m})(-m) \sin m\phi$

$$= \sum_{m=1}^{\infty} A_m b^{-m} (-m) \sin m\phi + E_0 b \sin\phi$$

③ $\varepsilon_0 \displaystyle\sum_{m=1}^{\infty} D_m a^{m-1} (m) \cos m\phi$

$$= \varepsilon \left[\frac{C_0}{a} + \sum_{m=1}^{\infty} (m B_m a^{m-1} - m C_m a^{-m-1}) \cos m\phi \right]$$

④ $\varepsilon \left[\dfrac{C_0}{b} + \displaystyle\sum_{m=1}^{\infty} (m B_m b^{m-1} - m C_m b^{-m-1}) \cos m\phi \right]$

$$= \varepsilon_0 \left[\sum_{m=1}^{\infty} A_m b^{-m-1} (-m) \cos m\phi - E_0 \cos\phi \right]$$

For $m \geq 1$, these imply the relations

① $D_m a^m = B_m a^m + C_m a^{-m}$

② $B_m b^m + C_m b^{-m} = A_m b^{-m}$

③ $\varepsilon_0 D_m a^{m-1} = \varepsilon (B_m a^{m-1} - C_m a^{-m-1})$

④ $\varepsilon (B_m b^{m-1} - C_m b^{-m-1}) = - \varepsilon_0 A_m b^{-m-1}$

4.8 a), cont'd

$m>1$:

$$①,③ \Rightarrow D_m a^m = B_m a^m + C_m a^{-m}$$
$$\text{also} = \frac{\varepsilon}{\varepsilon_0}\left(B_m a^m - C_m a^{-m}\right)$$

$$②,④ \Rightarrow A_m b^{-m} = B_m b^m + C_m b^{-m}$$
$$\text{also} = -\varepsilon/\varepsilon_0 \left(B_m b^m - C_m b^{-m}\right)$$

Thus,

$$\begin{cases} B_m\left(1-\varepsilon/\varepsilon_0\right) = -\frac{C_m}{a^{2m}}\left(1+\varepsilon/\varepsilon_0\right) \\ B_m\left(1+\varepsilon/\varepsilon_0\right) = -\frac{C_m}{b^{2m}}\left(1-\varepsilon/\varepsilon_0\right) \end{cases}$$

or

$$B_m = -\frac{C_m}{a^{2m}}\,\frac{1+\varepsilon/\varepsilon_0}{1-\varepsilon/\varepsilon_0} = -\frac{C_m}{b^{2m}}\,\frac{1-\varepsilon/\varepsilon_0}{1+\varepsilon/\varepsilon_0}$$

which require either $C_m = 0$ or $\left(\frac{a}{b}\right)^{2m}\left(\frac{1-\varepsilon/\varepsilon_0}{1+\varepsilon/\varepsilon_0}\right)^2 = 1$

For generic a, b, the only solution for $m>1$ is

$$C_m = 0 = B_m = A_m = D_m$$

Similarly, $C_0 = 0$.

4.8 d), cont'd

Now, consider the special case $m=1$.
Here, the boundary conditions reduce to the following:

① : $D_1 a = B_1 a + C_1 a^{-1}$

② : $-(B_1 b + C_1 b^{-1}) = -(A_1 b^{-1}) + E_0 b$

③ : $\varepsilon_0 D_1 = \varepsilon(B_1 - C_1 a^{-2})$

④ : $\varepsilon (B_1 - C_1 b^{-2}) = \varepsilon_0 (A_1 b^{-2}(-1))$

Solve these equations:

①, ③ $\Rightarrow$ $D_1 = B_1 + C_1 a^{-2}$

also $= (\varepsilon/\varepsilon_0)(B_1 - C_1 a^{-2})$

hence $B_1(\varepsilon/\varepsilon_0 - 1) = C_1 a^{-2}(1 + \varepsilon/\varepsilon_0)$

②, ④ $\Rightarrow$ $A_1 = -(\varepsilon/\varepsilon_0)(B_1 b^2 - C_1)$

also $= B_1 b^2 + C_1 + E_0 b^2$

hence $B_1 b^2 (1 + \varepsilon/\varepsilon_0) = C_1(\varepsilon/\varepsilon_0 - 1) - E_0 b^2$

Solving, we see

$$B_1 = \frac{C_1}{a^2} \frac{1 + \varepsilon/\varepsilon_0}{\varepsilon/\varepsilon_0 - 1} = \frac{C_1}{b^2} \frac{\varepsilon/\varepsilon_0 - 1}{1 + \varepsilon/\varepsilon_0} - \frac{E_0}{1 + \varepsilon/\varepsilon_0}$$

4.8 a), cont'd

Then,

$$C_1\left[\frac{1}{b^2}\,\frac{\varepsilon/\varepsilon_0-1}{1+\varepsilon/\varepsilon_0}-\frac{1}{a^2}\,\frac{1+\varepsilon/\varepsilon_0}{\varepsilon/\varepsilon_0-1}\right]=\frac{E_0}{1+\varepsilon/\varepsilon_0}$$

$$\|$$

$$C_1\left[\frac{a^2(\varepsilon/\varepsilon_0-1)^2-b^2(1+\varepsilon/\varepsilon_0)^2}{a^2b^2(\varepsilon/\varepsilon_0+1)(\varepsilon/\varepsilon_0-1)}\right]$$

or

$$C_1=E_0\left[\frac{a^2(\varepsilon/\varepsilon_0-1)^2-b^2(1+\varepsilon/\varepsilon_0)^2}{a^2b^2(\varepsilon/\varepsilon_0-1)}\right]^{-1}$$

$$B_1=\frac{C_1}{a^2}\,\frac{1+\varepsilon/\varepsilon_0}{\varepsilon/\varepsilon_0-1}=E_0\left[\frac{a^2(\varepsilon/\varepsilon_0-1)^2-b^2(1+\varepsilon/\varepsilon_0)^2}{b^2(1+\varepsilon/\varepsilon_0)}\right]^{-1}$$

$$D_1=B_1+C_1a^{-2}$$

$$=E_0b^2\left[a^2(\varepsilon/\varepsilon_0-1)^2-b^2(1+\varepsilon/\varepsilon_0)^2\right]^{-1}\left(\cancel{1}+\varepsilon/\varepsilon_0+\varepsilon/\varepsilon_0\cancel{-1}\right)$$

$$=2E_0b^2\left(\frac{\varepsilon}{\varepsilon_0}\right)\left[a^2(\varepsilon/\varepsilon_0-1)^2-b^2(1+\varepsilon/\varepsilon_0)^2\right]^{-1}$$

$$A_1=-(\varepsilon/\varepsilon_0)(B_1b^2-C_1)$$

$$=-E_0\left(\frac{\varepsilon}{\varepsilon_0}\right)b^2\left[a^2(\varepsilon/\varepsilon_0-1)^2-b^2(1+\varepsilon/\varepsilon_0)^2\right]^{-1}\left[b^2(1+\varepsilon/\varepsilon_0)-a^2(\varepsilon/\varepsilon_0-1)\right]$$

4.8 a), cont'd

so we find

$$\boxed{\begin{aligned} \rho<a:&\quad \Phi_C = D_1\rho\cos\phi + D_0 \\ a<\rho<b:&\quad \Phi_B = B_0 + (B_1\rho + C_1\rho^{-1})\cos\phi \\ \rho>b:&\quad \Phi_A = A_0 + A_1\rho^{-1}\cos\phi - E_0\rho\cos\phi \end{aligned}}$$

for the constants A_1, B_1, C_1, D_1 on preceding page.

(A_0, B_0, D_0 are constants that can be chosen, up to an overall shift, so as to make Φ continuous.)

4.8 a), cont'd

Electric field:

$\rho < a$

$$E_\rho = -\frac{\partial \Phi_C}{\partial \rho} = -D_1 \cos\phi$$

$$E_\phi = -\frac{1}{\rho}\frac{\partial \Phi_C}{\partial \phi} = +D_1 \sin\phi$$

$a < \rho < b$

$$E_\rho = -\frac{\partial \Phi_B}{\partial \rho} = -(B_1 - C_1\rho^{-2})\cos\phi$$

$$E_\phi = -\frac{1}{\rho}\frac{\partial \Phi_B}{\partial \phi} = +(B_1 + C_1\rho^{-2})\sin\phi$$

$\rho > b$

$$E_\rho = -\frac{\partial \Phi_A}{\partial \rho} = +A_1\rho^{-2}\cos\phi + E_0\cos\phi$$

$$E_\phi = -\frac{1}{\rho}\frac{\partial \Phi_A}{\partial \phi} = +A_1\rho^{-2}\sin\phi - E_0\sin\phi$$

5.1 Starting with the differential expression

$$d\bar{B} = \frac{\mu_0 I}{4\pi} d\bar{l}' \times \frac{\bar{x}-\bar{x}'}{|\bar{x}-\bar{x}'|^3}$$

for $\bar{B}$ at $\bar{x}$ produced by $I d\bar{l}'$ at $\bar{x}'$,
show that for a closed loop carrying a current I,

$$\bar{B}(\bar{x}) = \frac{\mu_0 I}{4\pi} \nabla \Omega$$

where Ω is the solid angle subtended by the loop at $\bar{x}$.
(Text then elaborates on the sign convention.)

First, recall $\dfrac{\bar{x}-\bar{x}'}{|\bar{x}-\bar{x}'|^3} = \nabla'\left(\dfrac{1}{|\bar{x}-\bar{x}'|}\right)$

Then, look at component:

$$\hat{x}_i \cdot \int_{\partial S} d\bar{l}' \times \nabla'\left(\frac{1}{|\bar{x}-\bar{x}'|}\right) = \int_{\partial S} d\bar{l}' \cdot \left(\nabla'\left(\frac{1}{|\bar{x}-\bar{x}'|}\right) \times \hat{x}_i\right)$$

$$= \int_S da'\, \hat{n}' \cdot \nabla' \times \left(\nabla'\left(\frac{1}{|\bar{x}-\bar{x}'|}\right) \times \hat{x}_i\right)$$

$$= \int_S da'\, \hat{n}' \cdot \left[\nabla'\left(\frac{1}{|\bar{x}-\bar{x}'|}\right) \underbrace{(\nabla' \cdot \hat{x}_i)}_{=0} - \hat{x}_i \underbrace{\nabla' \cdot \nabla'\left(\frac{1}{|\bar{x}-\bar{x}'|}\right)}_{= \nabla'^2\left(\frac{1}{|\bar{x}-\bar{x}'|}\right)} \right.$$
$$\left. + (\hat{x}_i \cdot \nabla')\, \nabla'\left(\frac{1}{|\bar{x}-\bar{x}'|}\right) - \underbrace{\left(\nabla'\left(\frac{1}{|\bar{x}-\bar{x}'|}\right) \cdot \nabla'\right) \hat{x}_i}_{=0}\right]$$

$$= \int_S da'\, \hat{n}' \cdot \left[-\hat{x}_i (-4\pi)\delta^3(\bar{x}-\bar{x}') + \frac{\partial}{\partial x_i'} \nabla'\left(\frac{1}{|\bar{x}-\bar{x}'|}\right)\right]$$

We'll assume $\bar{x}$ is not on the surface S, so that $\bar{x}'$ never $= \bar{x}$,
& so $\delta^3(\bar{x}-\bar{x}') = 0$.

So far:

$$\hat{x}_i \cdot \int_{\partial S} d\bar{l}' \times \nabla'\left(\frac{1}{|\bar{x}-\bar{x}'|}\right) = \int_S da'\, \hat{n}' \cdot \frac{\partial}{\partial x_i'} \nabla'\left(\frac{1}{|\bar{x}-\bar{x}'|}\right)$$

$$= -\frac{\partial}{\partial x_i} \int da'\, \hat{n}' \cdot \nabla'\left(\frac{1}{|\bar{x}-\bar{x}'|}\right)$$

From §1.6, below (1.25),

$$\hat{n} \cdot \nabla'\left(\frac{1}{|\bar{x}-\bar{x}'|}\right) da' = -d\Omega$$

$$\Rightarrow \hat{x}_i \cdot \int_{\partial S} d\bar{l}' \times \nabla'\left(\frac{1}{|\bar{x}-\bar{x}'|}\right) = +\frac{\partial}{\partial x_i} \int_S d\Omega = \hat{x}_i \cdot \nabla\Omega$$

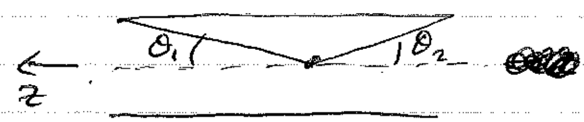
Putting all of this together,

$$\boxed{\bar{B}(\bar{x}) = \frac{\mu_0 I}{4\pi} \int_{\partial S} d\bar{l}' \times \nabla'\left(\frac{1}{|\bar{x}-\bar{x}'|}\right) = \frac{\mu_0 I}{4\pi} \nabla\Omega}$$

5.3 A right-circular solenoid of finite length L & radius a has N turns per unit length and carries a current I.
Show that the magnetic field on the cylinder axis in the limit $NL \to \infty$ is

$$B_z = \frac{\mu_0 N I}{2} (\cos\theta_1 + \cos\theta_2)$$

where θ_1, θ_2 are shown below:



Let's use the result of problem 5.1:

$$\bar{B} = \frac{\mu_0 I}{4\pi} \nabla \Omega \quad \text{for a single loop.}$$

Here, we have N loops/unit length & $\bar{B} \parallel \hat{z}$, hence

$$B_z(z) = \int_0^L dz \, \frac{\mu_0 N I}{4\pi} \frac{\partial \Omega}{\partial z} \qquad \& \ \bar{B} = B_z \hat{k}.$$

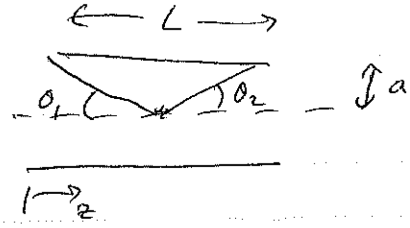
$$= \frac{\mu_0 N I}{4\pi} \left(\Omega(L) - \Omega(0) \right)$$

Now, because of the cylindrical symmetry,

$$\Omega(z) = \int_0^{2\pi} d\phi \int_{\cos\theta}^{1} d(\cos\theta) = 2\pi \left(1 - \cos\theta\right)$$

$$\Rightarrow \boxed{B_z(z) = \frac{\mu_0 N I}{4\pi} (2\pi) \left[\Big(1 - \underbrace{\cos(\pi - \theta_2)}_{= -\cos\theta_2}\Big) - \left(1 - \cos\theta_1\right) \right] = \frac{\mu_0 N I}{2} \left[\cos\theta_1 + \cos\theta_2 \right]}$$

5.3, cont'd



Alternate solution:

Recall from (5.40) that on-axis ($\theta = 0$), for a single loop,

$$dB_z = \frac{\mu_0 I a^2}{2\left(a^2 + (z'-z)^2\right)^{3/2}}\, dz'$$

Integrating over the length of the loop,
& including density of N loops / length,

$$B_z(z) = \int_0^L dz' N \, \frac{\mu_0 I a^2}{2\left(a^2 + (z'-z)^2\right)^{3/2}}$$

$$= \frac{\mu_0 N I}{2} \int_{-z}^{L-z} \frac{a^2}{(a^2 + x^2)^{3/2}}\, dx \qquad x = z' - z$$

Write $x = a \cot\theta \quad \Rightarrow dx = -a\csc^2\theta \, d\theta$

$$= \frac{\mu_0 N I}{2} (a^2) \int_{\pi-\theta_1}^{\theta_2} (-a) \cancel{\csc^2}\theta \, d\theta \, \frac{1}{a^3 \csc^3\theta} \qquad \text{using } 1 + \cot^2\theta = \csc^2\theta$$

$$= \frac{\mu_0 N I}{2} \cos\theta \Big|_{\pi-\theta_1}^{\theta_2}$$

$$= \frac{\mu_0 N I}{2} \left(\cos\theta_2 + \cos\theta_1\right)$$

5.4 A magnetic field $\bar{B}$ in a current-free region in a uniform medium is cylindrically symmetric with components $B_z(\rho,z)$, $B_\rho(\rho,z)$ and with a known $B_z(0,z)$ on the axis of symmetry. The magnitude of the axial field varies slowly in z.

a) Show that near the axis, the axial and radial components of the magnetic field are approximately

$$B_z(\rho,z) \simeq B_z(0,z) - \left(\frac{\rho^2}{4}\right)\left[\frac{\partial^2 B_z(0,z)}{\partial z^2}\right] + \cdots$$

$$B_\rho(\rho,z) \simeq -\left(\frac{\rho}{2}\right)\left[\frac{\partial B_z(0,z)}{\partial z}\right] + \left(\frac{\rho^3}{16}\right)\left[\frac{\partial^3 B_z(0,z)}{\partial z^3}\right] + \cdots$$

Expand in series:

$$B_z(\rho,z) = b_0(z) + \rho b_1(z) + \tfrac{1}{2}\rho^2 b_2(z) + \cdots = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} b_n(z)$$

$$B_\rho(\rho,z) = c_0(z) + \rho c_1(z) + \tfrac{1}{2}\rho^2 c_2(z) + \cdots = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} c_n(z)$$

Now, the field satisfies $\nabla\cdot\bar{B}=0$ and $\nabla\times\bar{B}=0$

Apply $\nabla\cdot\bar{B}=0$:

$$\nabla\cdot\bar{B} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho) + \underbrace{\frac{1}{\rho}\frac{\partial}{\partial\phi}B_\phi}_{=0} + \frac{\partial B_z}{\partial z}$$

$$= \sum_{n=0}^{\infty} \frac{n+1}{n!}\rho^{n-1} c_n(z) + \sum_{n=0}^{\infty}\frac{\rho^n}{n!} b_n'(z) \qquad = 0$$

Requiring this to hold for all ρ implies

$$c_0 = 0, \qquad \frac{n+2}{(n+1)!} c_{n+1} = \frac{-1}{n!} b_n'$$

5.4 a), cont'd

So far: $\boxed{\begin{aligned} c_0 &= 0 \\ c_{n+1} &= -\frac{n+1}{n+2} b_n' \end{aligned}}$

Next: apply $\nabla \times \bar{B} = 0$.

The only nonzero component is $\hat{\phi}$, and

$$(\nabla \times \bar{B})_\phi = \frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho}$$

$$= \sum_{n=0}^{\infty} \frac{\rho^n}{n!} c_n'(z) - \sum_{n=1}^{\infty} \frac{\rho^{n-1}}{(n-1)!} b_n(z)$$

$$= 0 \Rightarrow \boxed{b_{n+1}(z) = c_n'(z)}$$

Let's compute b_n's:

1^{st}, note $b_1 = c_0' = 0$ since $c_0 = 0$

However, b_0 is undetermined by these conditions (and is given to be $B_z(0,z)$ in the problem statement).

$$b_{n+1}(z) = c_n'(z) = -\frac{n}{n+1} b_{n-1}''(z)$$

so the higher b's can be computed from lower ones, & since $b_1 = 0$, only b_n for even n are nonzero.

5.4 a), cont'd

Thus, we have

$b_0 = B_z(0,z)$

$b_1 = 0$

$b_2 = -\frac{1}{2}\frac{\partial^2}{\partial z^2} B_z(0,z)$

$b_3 = 0$

& so on.

Similarly:

$c_0 = 0$

$c_1 = -\frac{1}{2}\frac{\partial}{\partial z} B_z(0,z)$

$c_2 = 0$

$c_3 = -\frac{3}{4}\frac{\partial}{\partial z} b_2' = +\frac{3}{8}\frac{\partial^3}{\partial z^3} B_z(0,z)$

$c_4 = 0$

& so on.

5.4 a), cont'd

Putting this together, we see

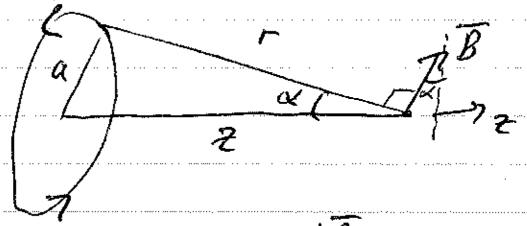
$$\boxed{\begin{aligned} B_z(\rho,z) &\simeq B_z(0,z) - \frac{1}{2}\frac{\rho^2}{2}\frac{\partial^2}{\partial z^2}B_z(0,z) + \mathcal{O}(\rho^4) \\ B_\rho(\rho,z) &= -\frac{1}{2}\rho\frac{\partial}{\partial z}B_z(0,z) + \frac{\rho^3}{3!}\,\frac{3}{8}\,\frac{\partial^3}{\partial z^3}B_z(0,z) + \mathcal{O}(\rho^5) \end{aligned}}$$

as expected.

(a, b, c, d) only

5.7 A compact circular coil of radius a, carrying a current I, lies in the xy plane with its center at the origin.

a) Using the Biot - Savart law, find the magnetic field at any point on the z axis.



$$d\bar{B} = \frac{\mu_0}{4\pi} I \, d\bar{l} \times \frac{(\bar{x}-\bar{x}_0)}{|\bar{x}-\bar{x}_0|^3}$$

By symmetry, only the z component of $\bar{B}$ will survive after integrating around the circle.

$$B_z = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} a\, d\phi \, \frac{\sin\alpha}{r^2}$$

$$= \frac{\mu_0 I a}{4\pi} (2\pi) \frac{\sin\alpha}{r^2} \qquad r^2 = a^2 + z^2$$

$$\sin\alpha = \frac{a}{r}$$

$$= \frac{\mu_0 I a}{2} \frac{a}{r^3}$$

$$\boxed{= \frac{\mu_0 I a^2}{2} \frac{1}{(a^2+z^2)^{3/2}}}$$

5.7, cont'd

b) An identical coil with the same magnitude and sense of the current is located on the same axis, parallel to, and a distance b above, the first coil. With the coordinate origin relocated to the point midway between the centers of the two coils, determine the magnetic field on the axis near the origin as an expansion in powers of z, up to z^4 inclusive:

$$B_z = \frac{\mu_0 I a^2}{d^3}\left[1 + \frac{3(b^2-a^2)z^2}{2d^4} + \frac{15(b^4 - 6b^2a^2 + 2a^4)z^4}{16d^8} + \cdots\right]$$

where $d^2 = a^2 + b^2/4$.

$$B_z = \frac{\mu_0 I a^2}{2}\left[\left(a^2 + (z+b/2)^2\right)^{-3/2} + \left(a^2 + (z-b/2)^2\right)^{-3/2}\right]$$ from superposition

$$= \frac{\mu_0 I a^2}{2}\left[\left(d^2 + z^2 + bz\right)^{-3/2} + \left(d^2 + z^2 - bz\right)^{-3/2}\right]$$

$$= \frac{\mu_0 I a^2}{2d^3}\left[\left(1 + \frac{z^2+bz}{d^2}\right)^{-3/2} + \left(1 + \frac{z^2-bz}{d^2}\right)^{-3/2}\right]$$

$$= \frac{\mu_0 I a^2}{2d^3}\left[1 - \frac{3}{2}\frac{z^2+bz}{d^2} + \frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{z^2+bz}{d^2}\right)^2\right.$$

$$+ \frac{1}{3!}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\left(\frac{z^2+bz}{d^2}\right)^3 + \frac{1}{4!}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\left(-\frac{9}{2}\right)\left(\frac{z^2+bz}{d^2}\right)^4 + \cdots$$

$$+ 1 - \frac{3}{2}\frac{z^2-bz}{d^2} + \frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{z^2-bz}{d^2}\right)^2$$

$$\left. + \frac{1}{3!}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\left(\frac{z^2-bz}{d^2}\right)^3 + \frac{1}{4!}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\left(-\frac{9}{2}\right)\left(\frac{z^2+bz}{d^2}\right)^4 + \cdots\right]$$

(cont'd)

5.7 b), cont'd

$$B_z = \frac{\mu_0 I a^2}{\cancel{2} d^3}\left[\cancel{2} - 3\frac{z^2}{d^2} + \frac{15}{8}(\cancel{2})\left(\frac{z^4 + b^2 z^2}{d^4}\right)\right.$$

$$\left. - \frac{(5)(7)}{16}(\cancel{2})\left(\frac{z^6 + 3z^2(bz)^2}{d^6}\right) + \frac{(5)(7)(9)}{(16)(8)}(\cancel{2})\frac{(bz)^4}{d^8} + \cdots\right]$$

$$= \frac{\mu_0 I a^2}{d^3}\left[1 + z^2\left(-\frac{3}{2d^2} + \frac{15}{8}\frac{b^2}{d^4}\right)\right.$$

$$\left. + z^4\left(\frac{15}{8}\frac{1}{d^4} - \frac{(5)(7)(3)}{16}\frac{b^2}{d^6} + \frac{(5)(7)(9)}{(16)(8)}\frac{b^4}{d^8}\right) + \cdots\right]$$

$$= \frac{\mu_0 I a^2}{d^3}\left[1 + \frac{3z^2}{d^4}\left(-\frac{d^2}{2} + \frac{5}{8}b^2\right)\right.$$

$$\left. + \frac{15 z^4}{16 d^8}\left(2d^4 - 7b^2 d^2 + \frac{(3)(7)}{8}b^4\right) + \cdots\right]$$

$$= \frac{\mu_0 I a^2}{d^3}\left[1 + \frac{3z^2}{d^4}\left(-\frac{a^2}{2} - \frac{b^2}{8} + \frac{5}{8}b^2\right)\right.$$

$$\left. + \frac{15}{16}\frac{z^4}{d^8}\left(2\left(a^4 + \frac{b^4}{16} + \frac{a^2b^2}{2}\right) - 7\left(a^2b^2 + \frac{b^4}{4}\right) + \frac{(3)(7)}{8}b^4\right) + \cdots\right]$$

$$\boxed{= \frac{\mu_0 I a^2}{d^3}\left[1 + \frac{3}{2}\frac{z^2}{d^4}\left(-a^2 + b^2\right) + \frac{15}{16}\frac{z^4}{d^8}\left(2a^4 - 6a^2b^2 + b^4\right) + O(z^6)\right]}$$

5.7, cont'd

c) Show that, off-axis near the origin, the axial and radial components, correct to second order in the coordinates, take the form

$$B_z = \sigma_0 + \sigma_2\left(z^2 - \frac{\rho^2}{2}\right), \quad B_\rho = -\sigma_2 z \rho$$

A convenient sol'n is to use the results of problem 5.4. There, it was argued that in circumstances such as this,

$$B_z(\rho, z) = B_z(0,z) - \frac{\rho^2}{4}\frac{\partial^2}{\partial z^2} B_z(0,z) + \dots$$

$$B_\rho(\rho, z) = -\frac{\rho}{2}\frac{\partial}{\partial z} B_z(0,z) + \dots$$

Here,

$$B_z(0,z) = \frac{\mu_0 I a^2}{d^3}\left[1 + \frac{3(b^2 - a^2)z^2}{2d^4} + \dots\right] = \sigma_0 + \sigma_2 z^2$$

hence $\frac{\partial^2}{\partial z^2} B_z(0,z) = 2\sigma_2 + \dots$, $\frac{\partial}{\partial z} B_z(0,z) = 2\sigma_2 z + \dots$

& so for $\sigma_0 = \frac{\mu_0 I a^2}{d^3}$, $\sigma_2 = \frac{3}{2}\frac{b^2 - a^2}{d^4}\sigma_0$

$$B_z(\rho, z) = \sigma_0 + \sigma_2 z^2 - \frac{\rho^2}{4}(2\sigma_2) + \dots$$

$$= \sigma_0 + \sigma_2\left(z^2 - \frac{\rho^2}{2}\right) + \dots$$

$$B_\rho(\rho, z) = -\frac{\rho}{2}(2\sigma_2 z) + \dots = -\sigma_2 z \rho + \dots$$

5.7, cont'd

d) For the two coils in part (b) show that the magnetic field on the z axis for large $|z|$ is given by the expansion in inverse odd powers of $|z|$ obtained from the small z expansion of part (b) by the formal substitution $d \to |z|$.

$$B_z = \frac{\mu_0 I a^2}{2}\left[\left(a^2 + (z + b/2)^2\right)^{-3/2} + \left(a^2 + (z - b/2)^2\right)^{-3/2}\right]$$

Expand for large $|z|$:

$$B_z = \frac{\mu_0 I a^2}{2}\frac{1}{|z|^3}\left[\left(1 + bz^{-1} + \frac{d^2}{z^2}\right)^{-3/2} + \left(1 - bz^{-1} + \frac{d^2}{z^2}\right)^{-3/2}\right]$$

$$= \frac{\mu_0 I a^2}{2|z|^3}\left[\left(1 + \frac{d^2 + bz}{z^2}\right)^{-3/2} + \left(1 + \frac{d^2 - bz}{z^2}\right)^{-3/2}\right]$$

Compare part (b):

$$B_z = \frac{\mu_0 I a^2}{2 d^3}\left[\left(1 + \frac{z^2 + bz}{d^2}\right)^{-3/2} + \left(1 + \frac{z^2 - bz}{d^2}\right)^{-3/2}\right]$$

The result follows.