Physics 5455 – Problem set 7

1. Calculate the matrix representation of the angular momentum operators L_x, L_y, L_z, $\vec{L}^2$ on states of $\ell = 3/2$. (You should get a set of 4×4 matrices.) Verify explicitly that the matrices satisfy the commutation relations

$$[L_x, L_y] \;=\; i\hbar L_z, \quad [\vec{L}^2, L_z] \;=\; 0$$

The next several problems concern the Legendre and associated Legendre polynomials. You will need to make use of the generating function

$$\left(1 \;-\; 2yt \;+\; t^2\right)^{-1/2} \;=\; \sum_{\ell=0}^{\infty} P_\ell(y) t^\ell$$

orthogonality relation

$$\int_{-1}^{1} P_m(y) P_n(y) dy \;=\; \frac{2\delta_{m,n}}{2n+1}$$

Rogrigues' formula

$$P_n(y) \;=\; \frac{1}{2^n n!} \left(\frac{d}{dy}\right)^n (y^2 - 1)^n$$

and particular values

$$P_{2n}(0) \;=\; \frac{(-)^n (2n)!}{2^{2n}(n!)^2}, \quad P_{2n+1}(0) \;=\; 0$$

for the ordinary Legendre polynomials.

2. Show that

$$P_n'(1) \;=\; \frac{1}{2} n(n+1)$$

3. A function $f(x)$ is expanded in a Legendre series

$$f(x) \;=\; \sum_{n=0}^{\infty} a_n P_n(x)$$

for constants a_n. Show that

$$\int_{-1}^{1} (f(x))^2 dx \;=\; \sum_{n=0}^{\infty} \frac{2a_n^2}{2n+1}$$

4. Expand the Dirac delta function in a series of Legendre polynomials along the interval $[-1, 1]$. (In other words, compute the a_n's of the previous problem for $f(x) = \delta(x)$.)

5. Show that

$$\int_{-1}^{1} x^m P_n(x) dx \;=\; 0$$

for $m < n$.

6. Show that

$$P_n^{-m}(x) \;=\; (-)^m \frac{(n-m)!}{(n+m)!} P_n^m(x)$$

where the associated Legendre polynomial $P_n^m(x)$ is defined by

$$P_n^m(x) \;=\; \frac{1}{2^n n!}(1-x^2)^{m/2} \left(\frac{d}{dx}\right)^{n+m} (x^2-1)^n$$

7. The associated Legendre polynomial $P_n^m(x)$ satisfies the self-adjoint ordinary differential equation

$$(1-x^2)\frac{d^2}{dx^2}P_n^m(x) \;-\; 2x\frac{d}{dx}P_n^m(x) \;+\; \left(n(n+1) \;-\; \frac{m^2}{1-x^2}\right) P_n^m(x) \;=\; 0$$

From the differential equation show that

$$\int_{-1}^{1} P_n^m(x) P_n^k(x) \frac{dx}{1-x^2} \;=\; 0$$

for $k \neq m$.