Physics 5456 – Problem set 1

1. **Three particles in three single-particle states.** Consider three particles that may occupy the three single-particle states $|a\rangle$, $|b\rangle$, $|c\rangle$.

   (a) Count and list all of the possible states if the particles are distinguishable.

   (b) Count and list all of the possible states if the particles are identical bosons.

   (c) Count and list all of the possible states if the particles are identical fermions, taking into account the Pauli exclusion principle.

2. **White dwarf and neutron stars.** A white dwarf is essentially a very hot and extremely dense ionized Helium star. In other words, for $N/2$ Helium nuclei there are N electrons forming a degenerate relativistic Fermi gas.

   (a) Using the relativistic dispersion relation

$$\epsilon(\vec{p}) \;=\; \sqrt{(m_e c^2)^2 \;+\; (|\vec{p}|c)^2}$$

   compute the total ground state energy E_0 in terms of the dimensionless ratio

$$x_F \;=\; \frac{p_F}{m_e c}$$

   and the particle number N. You will need the integral

$$\int dx\, x^2(a^2+x^2)^{1/2} \;=\; \frac{x}{4}(a^2+x^2)^{3/2} \;-\; \frac{a^2}{8}\left[x(a^2+x^2)^{1/2} \;+\; a^2\ln\left(x+\sqrt{a^2+x^2}\right)\right]$$

   (b) Consider the non-relativistic case $x_F \ll 1$, and show that

$$E_0 \;\cong\; Nm_e c^2\left(1 \;+\; \frac{3}{10}x_F^2\right)$$

   Argue that the star's mass $M \cong 2Nm_p$, and if the star's radius is R, show that

$$E_0 \;-\; Nm_e c^2 \;\propto\; M^{5/3}R^{-2}$$

   The pressure P_0 is given by

$$P_0 \;=\; -\left.\frac{\partial E_0}{\partial V}\right|_N$$

   Show that $P_0 \propto M^{5/3}R^{-5}$ in this limit.

   (c) Now consider the ultrarelativistic limit $x_F \gg 1$. Show that

$$E_0 \;\cong\; \frac{3}{4}Nm_e c^2 x_F\left(1 \;+\; x_F^{-2}\right)$$

   i.e. $E_0 \propto M^{4/3}R^{-1}$. Also show $P_0 \propto M^{4/3}R^{-4}$.

(d) In a white dwarf, the gravitational force is balanced by the Fermi pressure P_0. The gravitational potential energy is

$$U_G = -\frac{\alpha G M^2}{R}$$

Using the fact that the gravitational pressure is given by

$$P_G = -\left.\frac{\partial U_G}{\partial V}\right|_N$$

derive a mass-radius relation for non-relativistic white dwarfs.

3. Schwabl I problem 13.1(a-b).