Physics 5456 – Problem set 10

Klein-Gordon Foldy-Wouthuysen

In a previous homework assignment, you showed that the Klein-Gordon equation in an electromagnetic field could be written in the form

$$i\hbar \frac{\partial \Phi}{\partial t} \;=\; H\Phi$$

where

$$\begin{aligned}
\Phi &= \begin{bmatrix} \theta \\ \chi \end{bmatrix}, \\
\theta &= \frac{1}{2}\varphi + \frac{1}{2mc^2}\left(i\hbar\frac{\partial}{\partial t} - eV \right)\varphi, \\
\chi &= \frac{1}{2}\varphi - \frac{1}{2mc^2}\left(i\hbar\frac{\partial}{\partial t} - eV \right)\varphi,
\end{aligned}$$

φ the Klein-Gordon field, with

$$\begin{aligned}
H &= \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \frac{\vec{\pi}^2}{2m} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} mc^2 + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} eV, \\
&= \sigma^3 mc^2 + \left(eV + \sigma^3 \frac{\vec{\pi}^2}{2m} \right) + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\vec{\pi}^2}{2m}
\end{aligned}$$

and $\vec{\pi} = \vec{p} - (e/c)\vec{A}$.

In this problem you will apply that result.

Apply a Foldy-Wouthuysen transformation of the form

$$\Phi \;=\; e^{-iS}\Phi'$$

to derive

$$i\hbar \frac{\partial \Phi'}{\partial t} \;=\; H'\Phi'$$

and find an S (motivated by our work on the Dirac equation) such that, for static external fields, and for a single transformation,

$$\begin{aligned}
H' &= \sigma^3 \left(mc^2 + \frac{\vec{\pi}^2}{2m} - \frac{\vec{\pi}^4}{8m^3c^2} \right) + eV + \frac{1}{32m^4c^4}[\vec{\pi}^2, [\vec{\pi}^2, eV]] \\
&\quad + \frac{1}{mc^2}\sigma^3 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \left(\frac{1}{2}\left[\frac{\vec{\pi}^2}{2m}, eV \right] - \left(\frac{\vec{\pi}^2}{2m} \right)^2 \right) + \cdots
\end{aligned}$$

where we have omitted higher-order terms in $1/m$.