Physics 5456 – Problem set 8

The Klein-Gordon equation coupled to an electromagnetic field is

$$\left(i\hbar\frac{\partial}{\partial t} - e\Phi\right)^2 \psi = c^2(p^2 + m^2c^2)\psi$$

where

$$\vec{p} = \frac{\hbar}{i}\nabla - \frac{e}{c}\vec{A}.$$

Show that in terms of the fields

$$\begin{aligned}
\varphi(\vec{x},t) &= \frac{1}{2}\psi(\vec{x},t) + \frac{1}{2mc^2}\left(i\hbar\frac{\partial}{\partial t} - e\Phi\right)\psi(\vec{x},t), \\
\chi(\vec{x},t) &= \frac{1}{2}\psi(\vec{x},t) - \frac{1}{2mc^2}\left(i\hbar\frac{\partial}{\partial t} - e\Phi\right)\psi(\vec{x},t),
\end{aligned}$$

it can be written in the form

$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix} \varphi(\vec{x},t) \\ \chi(\vec{x},t) \end{bmatrix} = H \begin{bmatrix} \varphi(\vec{x},t) \\ \chi(\vec{x},t) \end{bmatrix},$$

with the (non-Hermitian) Hamiltonian

$$H = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\frac{\vec{p}^2}{2m} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} mc^2 + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} e\Phi.$$