Physics 4674/5674 – Problem set 5

4674 & 5674:

1. C 2.8

2. On an n-dimensional manifold, what can you say about any $(n+1)$-form?

3. Let f, g be smooth functions on $\mathbf{R}^2$. Show that

$$df \wedge dg \;=\; \det \begin{bmatrix} \partial_1 f & \partial_2 f \\ \partial_1 g & \partial_2 g \end{bmatrix} dx^1 \wedge dx^2$$

4. Determine $\omega \wedge \cdots \wedge \omega$ (n factors), where ω is given by

$$\omega \;=\; dx^1 \wedge dx^2 \;+\; dx^3 \wedge dx^4 \;+\; \cdots \;+\; dx^{2n-1} \wedge dx^{2n}$$

5. Show that the two-form $\omega_{\mu\nu} dx^\mu \wedge dx^\nu$ is closed if and only if

$$\partial_\rho \omega_{\mu\nu} \;-\; \partial_\nu \omega_{\mu\rho} \;+\; \partial_\mu \omega_{\nu\rho} \;=\; 0$$

for all μ, ν, ρ.

6. Show that the form

$$\omega \;=\; \frac{1}{r^n} \sum_{\mu=1}^{n} (-)^{\mu-1} x^\mu dx^1 \wedge \cdots \wedge dx^{\mu-1} \wedge dx^{\mu+1} \wedge \cdots \wedge dx^n$$

is closed, where

$$r^2 \;=\; \sum_\mu (x^\mu)^2$$

7. Show that

$$\nabla_\mu V_\nu \;-\; \nabla_\nu V_\mu \;=\; \partial_\mu V_\nu \;-\; \partial_\nu V_\mu$$

for ∇ torsion-free.

8. C 3.3

9. C 3.5

5674 only: (Lie derivatives problems, see appendix B)

10. Show that, for a tensor field T and vector fields X, Y,

(a)

$$\mathcal{L}_{aX+bY}T \;=\; a\mathcal{L}_X T \;+\; b\mathcal{L}_Y T$$

(a, b constants)

(b)

$$\mathcal{L}_X\left(\mathcal{L}_Y T\right) \;-\; \mathcal{L}_Y\left(\mathcal{L}_X T\right) \;=\; \mathcal{L}_{[X,Y]}T$$

11. Show that for vector fields X, Y, Z,

$$\mathcal{L}_X\mathcal{L}_Y Z \;+\; \mathcal{L}_Y\mathcal{L}_Z X \;+\; \mathcal{L}_Z\mathcal{L}_X Y \;=\; 0$$

12. Show that for vector fields X, Y and f a function,

$$\mathcal{L}_{fX}Y \;=\; f\mathcal{L}_X Y \;-\; (Yf)X$$

13. Show that if ω is a p-form,

$$\mathcal{L}_X(d\omega) \;=\; d\left(\mathcal{L}_X\omega\right)$$