1. Rodrigues' formula for the Legendre polynomials $P_n(x)$ says that

$$P_n(x) = \frac{1}{2^n n!} \left( \frac{d}{dx} \right)^n (x^2 - 1)^n \quad n = 0, 1, 2, \ldots$$

a) Show that the Legendre polynomials can also be expressed as

$$P_n(x) = \frac{1}{2^{n+1} \pi i} \int_c \frac{(s^2-1)^n}{(s-x)^{n+1}} \, ds \quad n = 0, 1, 2, \ldots$$

This is known as the Glaisher integral.

First, recall $f^{(n)}(x) = \frac{n!}{2\pi i} \oint \frac{f(s)}{(s-x)^{n+1}} \, ds \quad (A.W. \, (6.47))$

$$\Rightarrow \frac{1}{2^n n!} \left( \frac{d}{dx} \right)^n (x^2 - 1)^n = \frac{1}{2^n n!} \frac{n!}{2\pi i} \oint \frac{(s^2-1)^n}{(s-x)^{n+1}} \, ds$$

$$\Rightarrow \frac{1}{2^{n+1} \pi i} \oint \frac{(s^2-1)^n}{(s-x)^{n+1}} ds$$
5) Show that $P_n(1) = 1$ and $P_n(-1) = (-1)^n$ for all $n$, using the Schlömilch integral representation of $P_n(z)$.

First, note

\[(s^2 - 1)^n = (s+1)^n (s-1)^n\]

As,

\[P_n'(1) = \frac{1}{2^n \pi i} \oint \frac{(s+1)^n (s-1)^n}{(s-1)^n (s+1)^n} ds = \frac{1}{2^n \pi i} \oint \frac{(s+1)^n}{s-1} ds\]

Recall

\[f(z) = \frac{1}{2 \pi i} \oint \frac{f(s)}{s-z} ds\]

\[\Rightarrow P_n(1) = \frac{1}{2^n} \left[ \frac{(s+1)^n}{s-1} \right]_{s=1} = \frac{1}{2^n} 2^n = 1\]

Similarly,

\[P_n'(-1) = \frac{1}{2^n \pi i} \oint \frac{(s+1)^n (s-1)^n}{(s+1)^n (s-1)^n} ds = \frac{1}{2^n \pi i} \oint \frac{(s-1)^n}{s+1} ds\]

\[= \frac{1}{2^n} \left[ \frac{(s-1)^n}{s+1} \right]_{s=-1} = \frac{1}{2^n} (-2)^n = (-1)^n\]
Describe a Riemann surface for the triple-valued function
\[ w = (z - 1)^{1/3} \]
and point out which third of the \( w \)-plane represents the image of each sheet of the surface.

Take 3 copies of the complex plane, label them A, B, C. Cut each along the real axes from \( z = 1 \) to \( \infty \).

- The A sheet: \( 0 \leq \arg z < 2\pi \)
- The B sheet: \( 2\pi \leq \arg z < 4\pi \)
- The C sheet: \( 4\pi \leq \arg z < 6\pi \)

As, glue the lower edge of the cut on the A sheet to upper edge on B sheet

On the \( w \)-plane, \( 0 \leq \arg w < \frac{2\pi}{3} \) is the image of A sheet
\( \frac{2\pi}{3} \leq \arg w < \frac{4\pi}{3} \) is the image of B sheet
\( \frac{4\pi}{3} \leq \arg w < \frac{2\pi}{3} \) is the image of C sheet
Describe the curve, on a Riemann surface for \( w^{1/2} \), whose image is the entire circle \( |w| = 1 \) under the transformation \( w = z^{1/2} \).

The curve lies in \( |z| = 1 \).
It starts on the first sheet at any \( z = 0 \), revolves \( 2\pi \) around origin, goes onto the second sheet, and revolves \( 2\pi \) around the origin before returning to its starting position.

If the curve only lied on one sheet, then its image in the \( w \) plane would only be a semicircle, not a full circle.
Describe a Riemann surface for the multiple-valued function

\[ f(z) = \left( \frac{z-1}{z} \right)^{1/2} \]

To see where the branch cut lies, consider 3 cases:
- small circle enclosing $z = 1$:
  $\theta$ is constant, but $\phi$ varies, $0 \to 2\pi$ & see a cut
- small circle enclosing $z = 0$:
  $\theta$, $\phi$ constant, but $0$ varies, $0 \to 2\pi$ & see a cut
- big circle enclosing $z = 0 \& 1$:
  $z - 1 \approx z$, so $\frac{z-1}{z} \approx 1$ & no cut seen.

Thus, construct a Riemann surface for this function as follows:
Take 2 copies of the complex plane, call them $A, B$,
and cut them along the real axis between $z = 0$ & $z = 1$.

Let $A$ have $0 \leq \theta < 2\pi$

$B$ $2\pi \leq \theta < 4\pi$

so join the lower edge of the cut on $A$ to the upper edge on $B$.
Let \( C \) describe the positively-oriented circle \( |z-2| = 1 \) on the Riemann surface defined for \( z^{1/2} \), where the upper half of the circle lies on \( \mathbb{R}_+ \) and the lower half on \( \mathbb{R}_- \).

Note that, for each point \( z \) on \( C \), one can write

\[
z^{1/2} = \sqrt{r} \exp \left( \pm \frac{i \theta}{2} \right), \quad \text{where} \quad \frac{\pi}{2} - \frac{\pi}{4} < \theta < \frac{\pi}{2} + \frac{\pi}{4}
\]

State why it follows that

\[
\int_C z^{1/2} \, dz = 0
\]

Generalize this result to fit the case of other simple closed curves that cross from one sheet to another, without enclosing the branch points.

For a path, \( \int_C z^{1/2} \, dz = \frac{2}{\sqrt{3}} \left[ z^{3/2} \text{ (end)} - z^{3/2} \text{ (start)} \right] \)

Break up \( \int_C z^{1/2} \, dz \) into 2 paths, one on each sheet,

\[
\begin{align*}
\int_{R_0} z^{1/2} \, dz &= \text{Start at } \theta = 0, \, z = 2 \\
& \quad \text{and at } \theta = \frac{2\pi}{3}, \, z = 1 \\
&= \frac{2}{\sqrt{3}} \left[ z^{3/2} \text{ (end)} - z^{3/2} \text{ (start)} \right] \\
\int_{R_1} z^{1/2} \, dz &= \text{Start at } \theta = \frac{2\pi}{3}, \, z = 1 \\
& \quad \text{and at } \theta = \frac{4\pi}{3}, \, z = 2 \\
&= \frac{2}{\sqrt{3}} \left[ z^{3/2} \text{ (end)} - z^{3/2} \text{ (start)} \right]
\end{align*}
\]

\[\text{but } e^{\frac{2\pi i}{3}} = e^{\frac{2\pi i}{3}} = 1\]

\[\Rightarrow \int_{R_0} z^{1/2} \, dz + \int_{R_1} z^{1/2} \, dz = 0\]

More generally, could put branch cut elsewhere, so that curve is on a single sheet & includes no poles, \( \Rightarrow \int_C f(z) \, dz = 0 \)
6. Let a function \( f \) be continuous in a closed bounded region \( R \), and let it be analytic and not constant in the interior of \( R \). Assuming \( f(z) \neq 0 \) anywhere in \( R \), show that \( |f(z)| \) has a minimum value in \( R \) which occurs on the boundary \( \partial R \) never in the interior. (Hint: think about \( g(z) = \frac{1}{f(z)} \).

Recall from class:

If a function \( f \) is continuous in a closed bounded region \( R \) and it is analytic and not constant in the interior of \( R \), then the maximum value of \( |f(z)| \) is reached on the boundary \( \partial R \) never in the interior.

\[ g(z) = \frac{1}{f(z)} \] satisfies all requirements.

\[ g(z) \] has maximum on boundary, not interior.

\[ f(z) \] has minimum on boundary, not interior.
7. Suppose that \( f(z) \) is entire and the harmonic function
\[ u(x, y) = \text{Re} \left[ f(z) \right] \]
has an upper bound, i.e., \( u(x, y) \leq u_0 \) \( \forall z = x + iy \).
Show that \( u(x, y) \) must be constant.

 Hint: apply Liouville’s theorem to \( g(z) = e^{f(z)} \).

\[ g(z) = e^{f(z)} \text{ is entire} \]

\[ |g(z)| = |e^u| = e^{u(x, y)} \leq e^{u_0} \Rightarrow g(z) \text{ is bounded} \]

\[ \text{Liouville} \Rightarrow g(z) \text{ is constant} \]

\[ \Rightarrow e^{f(z)} \text{ is constant} \]

\[ \Rightarrow f(z) \text{ is constant}. \]