Physics 5714 – Problem set 1

(AWH = Arfken-Weber-Harris, 7th edition)

1. Verify the expansion of the triple vector product

\[ \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \]

2. (AWH 3.5.1) If \( S(x, y, z) = (x^2 + y^2 + z^2)^{-3/2} \), find

(a) \( \nabla S \) at the point (1, 2, 3)
(b) the magnitude of the gradient of \( S \), \( |\nabla S| \), at (1, 2, 3)
(c) the direction cosines of \( \nabla S \) at (1, 2, 3)

3. (AWH 3.5.2)

(a) Find a unit vector perpendicular to the surface

\[ x^2 + y^2 + z^2 = 3 \]

at the point (1, 1, 1)
(b) Derive the equation of the plane tangent to the surface at (1, 1, 1)

4. (AWH 3.5.6) For a particle moving in a circular orbit \( \vec{r} = \hat{x}r \cos(\omega t) + \hat{y}r \sin(\omega t) \), \( r \), \( \omega \) constant

(a) evaluate \( \vec{r} \times \dot{\vec{r}} \)
(b) Show that

\[ \frac{d^2}{dt^2} \vec{r} + \omega^2 \vec{r} = 0 \]

5. (AWH 3.5.9) Show

\[ \nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \]

(Hint: treat as a triple scalar product.)

6. (AWH 3.6.5) Verify the vector identity

\[ \nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B} - \vec{B}(\nabla \cdot \vec{A}) + \vec{A}(\nabla \cdot \vec{B}) \]
7. The velocity of a two-dimensional flow of liquid is given by
\[ \vec{V} = \hat{x}u(x, y) - \hat{y}v(x, y) \]
If the liquid is incompressible and the flow is irrotational, show that
\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \]

8. (AWH 3.6.10) Show that \( \nabla \times (\varphi \nabla \varphi) = 0 \).

9. (AWH 3.6.15) Show that any solution of the equation
\[ \nabla \times (\nabla \times \vec{A}) - k^2 \vec{A} = 0 \]
automatically satisfies the vector Helmholtz equation
\[ \nabla^2 \vec{A} + k^2 \vec{A} = 0 \]
and the solenoidal condition \( \nabla \cdot \vec{A} = 0 \). (Hint: let \( \nabla \cdot \) operate on the first equation.)

10. Compute the line integral
\[ \int_C (x^3 + y) \, ds \]
for \( C \) the curve described by \( x = 3t, y = t^3, \ t \in [0, 1] \).

11. Compute the line integral
\[ \int_C (\sin x + \cos y) \, ds \]
where \( C \) is the line segment from \( (0, 0) \) to \( (\pi, 2\pi) \).

12. Compute the line integral
\[ \int_C (y \, dx + x \, dy) \]
for \( C \) the curve \( y = x^2, \ x \in [0, 1] \).

13. Compute the line integral
\[ \int_C (xz \, dx + (y + z) \, dy + x \, dz) \]
for \( C \) the curve \( x = \exp t, \ y = \exp(-t), \ z = \exp(2t), \ 0 \leq t \leq 1 \).