Physics 5714 – Problem set 10

1. Compute the following improper integrals with residues:

(a) 
\[ \int_{0}^{\infty} \frac{dx}{x^2 + 1} \]

(b) 
\[ \int_{0}^{\infty} \frac{dx}{(x^2 + 1)^2} \]

2. Compute 
\[ \int_{-\infty}^{\infty} \frac{\cos x dx}{(x^2 + a^2)(x^2 + b^2)}, \quad (a > b > 0) \]

3. Compute 
\[ \int_{-\infty}^{\infty} \frac{x \sin ax}{x^4 + 4} dx, \quad (a > 0) \]

4. Compute 
\[ \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + 1)^2} \]

5. Compute 
\[ \int_{-\infty}^{\infty} \frac{\sin x dx}{x^2 + 4x + 5} \]

6. Compute 
\[ \int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2 \theta} \]

7. Compute 
\[ \int_{0}^{\pi} \frac{d\theta}{(a + \cos \theta)^2}, \quad (a > 1) \]

8. Show that 
\[ \int_{0}^{\infty} \frac{dx}{\sqrt{x}(x^2 + 1)} = \frac{\pi}{\sqrt{2}} \]

9. (a) By considering the integral of \( \exp(iz^2) \) around the positively-oriented boundary of the sector \( 0 \leq r \leq R, \ 0 \leq \theta \leq \pi/4 \), show that
\[ \int_{0}^{R} \exp(ix^2)dx = \exp(i\pi/4) \int_{0}^{R} \exp(-r^2)dr - \int_{C_R} \exp(iz^2)dz \]
where \( C_R \) is the arc \( z = R \exp(i\theta), \ (0 \leq \theta \leq \pi/4) \).
(b) Show that the integral above along $C_R$ tends to zero as $R \to \infty$.

(c) Use the results in parts (a) and (b), together with the known integration formula

$$\int_0^\infty \exp(-x^2)dx = \frac{\sqrt{\pi}}{2}$$

to evaluate the Fresnel integrals

$$\int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

which are important in diffraction theory.