Physics 5714 – Problem set 10

1. (A WH 19.1.2) In the analysis of a complex waveform (ocean tides, earthquakes, musical tones, etc), it might be more convenient to have the Fourier series written as

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos(nx - \theta_n) \]

Show that this is equivalent to

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \]

with

\[ a_n = \alpha_n \cos \theta_n, \quad b_n = \alpha_n \sin \theta_n, \]
\[ \alpha_n^2 = a_n^2 + b_n^2, \quad \tan \theta_n = b_n/a_n \]

2. (A WH 19.1.3) A function \( f(x) \) is expanded in an exponential Fourier series

\[ f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \]

If \( f(x) \) is real, \( f(x) = f(x)^* \), what restriction is imposed on the coefficients \( c_n \)?

3. (A WH 19.1.6) Sum the trigonometric series

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n} \]

and show it equals \( x/2 \).

4. (A WH 19.2.13)

(a) Find the Fourier series representation of

\[ f(x) = \begin{cases} 0 & -\pi \leq x \leq 0 \\ x & 0 \leq x \leq \pi \end{cases} \]

(b) From the Fourier series expansion show that

\[ \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \]

5. (A WH 19.1.13a-b)
(a) Assuming that the Fourier expansion of $f(x)$ is uniformly convergent, show that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} (f(x))^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left( a_n^2 + b_n^2 \right)$$

This is Parseval’s identity.

(b) Given

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} (-)^n \frac{\cos nx}{n^2}$$

for $-\pi \leq x \leq \pi$, apply Parseval’s identity to obtain $\zeta(4)$ in closed form.

6. (AWH 19.2.17a) Show that the Dirac delta function $\delta(x - a)$, expanded in a Fourier sine series in the half-interval $(0, L)$ ($0 < a < L$) is given by

$$\delta(x - a) = \frac{2}{L} \sum_{n=1}^{\infty} \sin \left( \frac{n \pi a}{L} \right) \sin \left( \frac{n \pi x}{L} \right)$$

Note that this series actually describes $-\delta(x + a) + \delta(x - a)$ in the interval $(-L, L)$.

7. An example of convolution Consider a function

$$f(x) = \cos x + \cos 3x = \frac{1}{2} \left( e^{ix} + e^{-ix} \right) + \frac{1}{2} \left( e^{3ix} + e^{-3ix} \right)$$

Imagine passing this function through a low-pass filter that multiplies the coefficients of $1$, $e^{\pm ix}$ by 1 and all other Fourier components by zero.

If we write

$$f(x) = \sum_{n=-\infty}^{\infty} (c_f)_n e^{inx},$$

then the action of the low-pass filter can be described as

$$\sum_n (c_f)_n (c_g)_n e^{inx}$$

for

$$(c_g)_n = \begin{cases} 1 & n = 0, \pm 1, \\ 0 & n \neq 0, \pm 1 \end{cases}$$

(a) Show that

$$\sum_n (c_g)_n e^{inx} = 1 + 2 \cos x$$

(b) For $g(x) = 1 + 2 \cos x$, compute

$$(f \ast g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s)g(x-s)ds$$
(c) Show that the convolution product above matches

\[ \sum_n (c_f)_n (c_g)_n e^{inx} \]

(In other words, compute this directly in this example and compare. I’m not looking for an abstract general argument, just an explicit verification in this particular example.)