Physics 5714 – Problem set 4

1. Let $V$ be the vector space of all $2 \times 2$ matrices over a field $F$. Show that $V$ has dimension 4 by exhibiting a basis with 4 elements.

2. Let $V$ be a vector space over a subfield $F$ of the complex numbers. Suppose $\alpha, \beta, \gamma$ are linearly independent vectors in $V$. Show that $(\alpha + \beta), (\beta + \gamma), (\gamma + \alpha)$ are linearly independent.

3. Show that the vectors
   
   \begin{align*}
   \alpha_1 &= (1, 1, 0, 0) \\
   \alpha_2 &= (0, 0, 1, 1) \\
   \alpha_3 &= (1, 0, 0, 4) \\
   \alpha_4 &= (0, 0, 0, 2)
   \end{align*}

   form a basis for $\mathbb{R}^4$. Find the coordinates of each of the standard basis vectors in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

4. Let $B = \{\alpha_1, \alpha_2, \alpha_3\}$ be the ordered basis for $\mathbb{R}^3$ consisting of
   
   \begin{align*}
   \alpha_1 &= (1, 0, -1) \\
   \alpha_2 &= (1, 1, 1) \\
   \alpha_3 &= (1, 0, 0)
   \end{align*}

   What are the coordinates of the vector $(a, b, c)$ in the ordered basis $B$?

5. Let $V$ be the vector space over the complex numbers of all functions $\mathbb{R} \rightarrow \mathbb{C}$, i.e. the space of all complex-valued functions on the real line. Let $f_1(x) = 1$, $f_2(x) = \exp(ix)$, $f_3(x) = \exp(-ix)$.
   
   (a) Show that $f_1$, $f_2$, $f_3$ are linearly independent.
   
   (b) Let $g_1(x) = 1$, $g_2(x) = \cos x$, $g_3(x) = \sin x$. Find an invertible $3 \times 3$ matrix $P$ such that

   \[ g_j(x) = \sum P_{ij} f_i(x) \]

6. Let $V$ be the real vector space of all polynomial functions $\mathbb{R} \rightarrow \mathbb{R}$ of degree 2 or less, i.e. the space of all functions $f$ of the form

   \[ f(x) = c_0 + c_1 x + c_2 x^2 \]

   Let $t$ be a fixed real number, and define

   \begin{align*}
   g_1(x) &= 1, \\
   g_2(x) &= x + t, \\
   g_3(x) &= (x + t)^2
   \end{align*}

   Show that $B = \{g_1, g_2, g_3\}$ is a basis for $V$. If $f(x) = c_0 + c_1 x + c_2 x^2$, what are the coordinates of $f$ in this ordered basis $B$?
7. Let \( \alpha_1 = (1,1,-2,1), \alpha_2 = (3,0,4,-1), \alpha_3 = (-1,2,5,2) \). Let \( \alpha = (4,-5,9,-7), \beta = (3,1,-4,4), \gamma = (-1,1,0,1) \). Which of the vectors \( \alpha, \beta, \gamma \) are in the subspace of \( \mathbb{R}^4 \) spanned by the \( \alpha_i \)?

8. Find the range, rank, null space, and nullity for the zero transformation and the identity transformation on a finite-dimensional vector space \( V \).

9. Describe the range and null space of the differentiation transformation on the vector space of polynomials.

10. Is there a linear transformation \( T : \mathbb{R}^3 \to \mathbb{R}^2 \) such that \( T(1,-1,1) = (1,0), T(1,1,1) = (0,1) \)?

11. Let \( V \) be the vector space of all \( n \times n \) matrices over a field \( F \), and let \( B \) be a fixed \( n \times n \) matrix. If
\[
T(A) = AB - BA
\]
verify that \( T \) is a linear transformation \( V \to V \).

12. Let \( V \) be the set of all complex numbers regarded as a vector space over the field of real numbers. Find a function \( V \to V \) which is a linear transformation on the above vector space, but which is not a linear transformation on \( \mathbb{C} \), i.e. which is not complex linear.

13. Let \( V \) be an \( n \)-dimensional vector space over the field \( F \), \( T \) a linear transformation \( V \to V \) such that range \( T = \) null space \( T \). Show that \( n \) is even.

14. Let \( V \) be a vector space, \( T : V \to V \) a linear transformation. Show that the following statements are equivalent:

(a) the intersection of the range of \( T \) and the null space of \( T \) is the zero subspace of \( V \)

(b) if \( T(T\alpha) = 0 \) then \( T\alpha = 0 \)

15. Find two linear operators \( T, U \) on \( \mathbb{R}^2 \) such that \( TU = 0, UT \neq 0 \).

16. Let \( T \) be the (unique) linear operator on \( \mathbb{C}^3 \) for which
\[
T\epsilon_1 = (1,0,i), \ T\epsilon_2 = (0,1,1), \ T\epsilon_3 = (i,1,0)
\]
where the \( \epsilon_i \) are the standard basis vectors. Is \( T \) invertible?