1. Let $V$ be the vector space of all $2 \times 2$ matrices over a field $F$. Show that $V$ has dimension 4 by exhibiting a basis with 4 elements.

2. Let $V$ be a vector space over a subfield $F$ of the complex numbers. Suppose $\alpha$, $\beta$, $\gamma$ are linearly independent vectors in $V$. Show that $(\alpha + \beta)$, $(\beta + \gamma)$, $(\gamma + \alpha)$ are linearly independent.

3. Show that the vectors
\[
\alpha_1 = (1, 1, 0, 0) \\
\alpha_2 = (0, 0, 1, 1) \\
\alpha_3 = (1, 0, 0, 4) \\
\alpha_4 = (0, 0, 0, 2)
\]
form a basis for $\mathbb{R}^4$. Find the coordinates of each of the standard basis vectors in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

4. Let $B = \{\alpha_1, \alpha_2, \alpha_3\}$ be the ordered basis for $\mathbb{R}^3$ consisting of
\[
\alpha_1 = (1, 0, -1) \\
\alpha_2 = (1, 1, 1) \\
\alpha_3 = (1, 0, 0)
\]
What are the coordinates of the vector $(a, b, c)$ in the ordered basis $B$?

5. Let $V$ be the vector space over the complex numbers of all functions $\mathbb{R} \to \mathbb{C}$, i.e. the space of all complex-valued functions on the real line. Let $f_1(x) = 1$, $f_2(x) = \exp(ix)$, $f_3(x) = \exp(-ix)$.

(a) Show that $f_1$, $f_2$, $f_3$ are linearly independent.
(b) Let $g_1(x) = 1$, $g_2(x) = \cos x$, $g_3(x) = \sin x$. Find an invertible $3 \times 3$ matrix $P$ such that
\[
g_j(x) = \sum_i P_{ij} f_i(x)
\]

6. Let $V$ be the real vector space of all polynomial functions $\mathbb{R} \to \mathbb{R}$ of degree 2 or less, i.e. the space of all functions $f$ of the form
\[
f(x) = c_0 + c_1 x + c_2 x^2
\]
Let $t$ be a fixed real number, and define
\[
g_1(x) = 1, \quad g_2(x) = x + t, \quad g_3(x) = (x + t)^2
\]
Show that $B = \{g_1, g_2, g_3\}$ is a basis for $V$. If $f(x) = c_0 + c_1 x + c_2 x^2$, what are the coordinates of $f$ in this ordered basis $B$?
7. Let $\alpha_1 = (1, 1, -2, 1)$, $\alpha_2 = (3, 0, 4, -1)$, $\alpha_3 = (-1, 2, 5, 2)$. Let $\alpha = (4, -5, 9, -7)$, $\beta = (3, 1, -4, 4)$, $\gamma = (-1, 1, 0, 1)$. Which of the vectors $\alpha, \beta, \gamma$ are in the subspace of $\mathbb{R}^4$ spanned by the $\alpha_i$?

8. Find the range, rank, null space, and nullity for the zero transformation and the identity transformation on a finite-dimensional vector space $V$.

9. Describe the range and null space of the differentiation transformation on the vector space of polynomials.

10. Is there a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1, -1, 1) = (1, 0)$, $T(1, 1, 1) = (0, 1)$?

11. Let $V$ be the vector space of all $n \times n$ matrices over a field $F$, and let $B$ be a fixed $n \times n$ matrix. If $T(A) = AB - BA$ verify that $T$ is a linear transformation $V \to V$.

12. Let $V$ be the set of all complex numbers regarded as a vector space over the field of real numbers. Find a function $V \to V$ which is a linear transformation on the above vector space, but which is not a linear transformation on $\mathbb{C}$, i.e. which is not complex linear.

13. Let $V$ be an $n$-dimensional vector space over the field $F$, $T$ a linear transformation $V \to V$ such that $\text{range } T = \text{null space } T$. Show that $n$ is even.

14. Let $V$ be a vector space, $T : V \to V$ a linear transformation. Show that the following statements are equivalent:

   (a) the intersection of the range of $T$ and the null space of $T$ is the zero subspace of $V$

   (b) if $T(T\alpha) = 0$ then $T\alpha = 0$

15. Find two linear operators $T, U$ on $\mathbb{R}^2$ such that $TU = 0$, $UT \neq 0$. 