Physics 5714 – Problem set 5

1. Find two linear operators $T, U$ on $\mathbb{R}^2$ such that $TU = 0, UT \neq 0$.

2. Let $T$ be the (unique) linear operator on $\mathbb{C}^3$ for which

$T\epsilon_1 = (1, 0, i), \ T\epsilon_2 = (0, 1, 1), \ T\epsilon_3 = (i, 1, 0)$

where the $\epsilon_i$ are the standard basis vectors. Is $T$ invertible?

3. Let $T$ be a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$, and let $U$ be a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$. Show that the linear transformation $UT$ is not invertible.

4. Let $V, W$ be vector spaces over a field $F$, and let $U$ be an isomorphism of $V$ onto $W$. Show that $T \mapsto UTU^{-1}$ is an isomorphism of $L(V, V)$ onto $L(W, W)$.

5. Let $\theta$ be a real number. Show that the following two matrices are similar over the field of complex numbers:

$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \ \begin{bmatrix} \exp(i\theta) & 0 \\ 0 & \exp(-i\theta) \end{bmatrix}$

6. (AWH 2.2.7) For square matrices $A, B, C$, verify the Jacobi identity


where $[A, B] = AB - BA$.

7. (AWH 2.2.11) The three Pauli spin matrices are

$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Show that

(a) $(\sigma_i)^2 = 1$

(b) $\sigma_j\sigma_k = i\sigma_\ell, \ (j, k, \ell) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$

(c) $\sigma_j\sigma_i + \sigma_i\sigma_j = 2\delta_{ij}1$

8. Using the Pauli $\sigma_i$ of the last exercise, show that

$(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = (\vec{a} \cdot \vec{b}) \ 1 + i\vec{\sigma} \cdot (\vec{a} \times \vec{b})$

where $\vec{\sigma} = \sigma_1\hat{x} + \sigma_2\hat{y} + \sigma_3\hat{z}$.  