Physics 5714 – Problem set 6

1. For the matrix $A$ below, either diagonalize or put in Jordan normal form. In other words, compute a matrix $P$ and a matrix $\Lambda$ that is either diagonal or in Jordan normal form, such that $A = P \Lambda P^{-1}$.

$$A = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}$$

2. For the matrix $A$ below, either diagonalize or put in Jordan normal form. In other words, compute a matrix $P$ and a matrix $\Lambda$ that is either diagonal or in Jordan normal form, such that $A = P \Lambda P^{-1}$.

$$A = \begin{bmatrix} 1 & 1 \\ -4 & 5 \end{bmatrix}$$

3. For the matrix $A$ below, either diagonalize or put in Jordan normal form. In other words, compute a matrix $P$ and a matrix $\Lambda$ that is either diagonal or in Jordan normal form, such that $A = P \Lambda P^{-1}$.

$$A = \begin{bmatrix} 5/2 & -1/2 & 0 \\ 1/2 & 3/2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

4. Compute

$$\exp\begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}$$

5. Compute

$$\exp\begin{bmatrix} 1 & 1 \\ -4 & 5 \end{bmatrix}$$

6. (AWH 2.2.34) If $A, B$ are Hermitian matrices, show that $(AB + BA), i(AB - BA)$ are also Hermitian.

7. (AWH 2.2.37) Two matrices $A, B$ are each Hermitian. Find a necessary and sufficient condition for their product $AB$ to be Hermitian.

8. Two matrices $U, H$ are related by $U = \exp(iaH)$ with $a$ real. If $H$ is Hermitian, show that $U$ is unitary.

9. (AWH 1.1.5a-c) Test the following for convergence:
   
   (a)
   $$\sum_{n=2}^{\infty} (\log n)^{-1}$$
\[ \sum_{n=1}^{\infty} \frac{n!}{10^n} \]

\[ \sum_{n=1}^{\infty} \frac{1}{(2n)(2n+1)} \]

10. (AWH 1.1.9) (Olbers’ paradox) Assume a static universe in which the stars are uniformly distributed. Divide all space into shells of constant thickness; the stars in any one shell by themselves subtend a solid angle of \( \omega_0 \). Allowing for the blocking out of distant stars by nearer stars, show that the total net solid angle subtended by all stars, shells extending to infinity, is exactly \( 4\pi \). (Therefore the night sky should be ablaze with light.)

11. (AWH 1.1.15a) Show that
\[ \sum_{n=2}^{\infty} (\zeta(n) - 1) = 1 \]
where \( \zeta(n) \) is the Riemann zeta function.

12. (AWH 1.3.10) The displacement \( x \) of a particle of rest mass \( m_0 \), resulting from a constant force \( m_0 g \) along the \( x \) axis, is
\[ x = \frac{c^2}{g} \left\{ \left[ 1 + \left( \frac{gt}{c} \right)^2 \right]^{1/2} - 1 \right\} \]
including relativistic effects. Find the displacement \( x \) as a power series in time \( t \). Compare with the classical result, \( x = (1/2)gt^2 \).