Physics 5714 – Problem set 7

1. The functions $u(x, y)$, $v(x, y)$ are the real, imaginary parts of a holomorphic (equivalently, analytic) function $w(z)$.

   (a) Assuming the required derivatives exist, show that
   \[ \nabla^2 u = \nabla^2 v = 0 \]
   Solutions of Laplace’s equation such as $u(x, y)$, $v(x, y)$ are called **harmonic functions**.

   (b) Show that
   \[ u_x u_y + v_x v_y = 0 \]

2. (AWH 11.2.2) Having shown that the real part $u(x, y)$ and the imaginary part $v(x, y)$ of an analytic function $w(z)$ each satisfy Laplace’s equation, show that $u(x, y)$, $v(x, y)$ cannot both have either a maximum or a minimum in the interior of any region in which $w(z)$ is analytic.

3. Show that
   \[ \exp(iz) = \cos z + i \sin z \]
   for every complex number $z$.

4. For $z = x + iy$, show that
   \[ |\sin z| \geq |\sin x| \]

5. Find all roots of the equation $\cos z = 2$.

6. For a complex number $z$, define
   \[ \sinh z = \frac{1}{2} (\exp(z) - \exp(-z)), \quad \cosh z = \frac{1}{2} (\exp(z) + \exp(-z)) \]
   Show that
   \[ \sinh(2z) = 2 \sinh z \cosh z \]

7. Show that
   \[ -i \sinh(iz) = \sin z, \quad \cosh(iz) = \cos z \]

8. For complex numbers $z_1$, $z_2$, show that
   \[ \sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2 \]
   \[ \cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2 \]
9. For $z = x + iy$, show that
\[
\sinh z = \sinh x \cos y + i \cosh x \sin y \\
\cosh z = \cosh x \cos y + i \sinh x \sin y
\]

10. For $z = x + iy$, show that
\[
|\sinh z|^2 = \sinh^2 x + \sin^2 y \\
|\cosh z|^2 = \sinh^2 x + \cos^2 y
\]

11. Show that the holomorphic function
\[
f_2(z) = \frac{1}{z^2 + 1} \quad (z \neq \pm i)
\]
is the analytic continuation of the function
\[
f_1(z) = \sum_{n=0}^{\infty} (-)^n z^{2n} \quad (|z| < 1)
\]
into the domain consisting of all points in the $z$ plane except $z = \pm i$.

12. Show that the function $f_2(z) = z^{-2} \ (z \neq 0)$ is the analytic continuation of the function
\[
f_1(z) = \sum_{n=0}^{\infty} (n + 1)(z + 1)^n \quad (|z + 1| < 1)
\]
into the domain consisting of all points in the $z$ plane except $z = 0$.

13. Find the analytic continuation of the function
\[
f(z) = \int_0^{\infty} t \exp(-zt)dt \quad (\text{Re} \ z > 0)
\]
into the domain consisting of all points in the $z$ plane except the origin.

14. Show that the function $(z^2 + 1)^{-1}$ is the analytic continuation of the function
\[
f(z) = \int_0^{\infty} \exp(-zt)(\sin t) \ dt \quad (\text{Re} \ z > 0)
\]
into the domain consisting of all points in the $z$ plane except $z = \pm i$. 
