Physics 5714 – Problem set 8

1. Rodrigues’ formula for the Legendre polynomials $P_n(z)$ says that

$$P_n(z) = \frac{1}{2^n n!} \left( \frac{d}{dz} \right)^n (z^2 - 1)^n, \quad n = 0, 1, 2, \ldots$$

(a) Show that the Legendre polynomials can also be expressed as

$$P_n(z) = \frac{1}{2^{n+1} \pi i} \int_C \frac{(s^2 - 1)^n}{(s - z)^{n+1}} ds, \quad n = 0, 1, 2, \ldots$$

for some contour $C$ enclosing $z$. This is known as the Schlaefli integral.

(b) Show that $P_n(1) = 1$ and $P_n(-1) = (-)^n$ for all $n$, using the Schlaefli integral representation of $P_n(z)$.

2. Describe a Riemann surface for the triple-valued function

$$w = (z - 1)^{1/3}$$

and point out which third of the $w$ plane represents the image of each sheet of the surface.

3. Describe the curve, on a Riemann surface for $z^{1/2}$, whose image is the entire circle $|w| = 1$ under the transformation $w = z^{1/2}$.

4. Describe a Riemann surface for the multiple-valued function

$$f(z) = \left( \frac{z - 1}{z} \right)^{1/2}$$

5. Let $C$ describe the positively-oriented circle $|z - 2| = 1$ on the Riemann surface described for $z^{1/2}$, where the upper half of the circle lies on one sheet and the lower half on another. Note that, for each point $z$ on $C$, one can write

$$z^{1/2} = \sqrt{r} \exp \left( \pm \frac{i \theta}{2} \right), \quad \text{where} \quad 4\pi - \frac{\pi}{2} < \theta < 4\pi + \frac{\pi}{2}$$

State why it follows that

$$\int_C z^{1/2} dz = 0$$

Generalize this result to fit the case of the other simple closed curves that cross from one sheet to another without enclosing the branch point.
6. Let a function $f$ be continuous in a closed bounded region $R$, and let it be analytic and not constant in the interior of $R$. Assuming that $f(z) \neq 0$ anywhere in $R$, show that $|f(z)|$ has a minimum value in $R$ which occurs on the boundary and never in the interior. Hint: think about $g(z) = 1/f(z)$.

7. Suppose that $f(z)$ is entire and the harmonic function

$$u(x, y) = \text{Re } f(z)$$

has an upper bound, i.e., $u(x, y) \leq u_0$ for all $z = x + iy$. Show that $u(x, y)$ must be constant. Hint: apply Liouville’s theorem to $g(z) = \exp(f(z))$. 