Chapter 1

Quantum mechanics and path integrals

We shall begin our study of quantum field theory by learning about path integrals and path integral quantization. Path integrals (also sometimes called functional integrals) are an infinite-dimensional analogue of ordinary integrals, and so before learning about path integrals, we will first work through the corresponding derivatives, known as functional derivatives.

We will begin by studying functional derivatives and path integrals in quantum mechanics, where they were originally worked out. We will see that this will give a way of thinking about quantum mechanics from which one can derive Lagrangian classical mechanics as a limit.

1.1 Functional derivative

Let \( x(t) \) be a function of one variable \( t \). This could be the position of a point-particle along a line, as a function of time. We can define a function that depends upon \( x(t) \) – such a quantity is known as a functional of \( x(t) \). For example,

\[
S[x(t)] = \int dt \left( \frac{dx}{dt} \right)^2
\]

is a functional of \( x(t) \).

We would like to define a derivative on the space of all functions \( x(t) \). Such a derivative should vary the value of the function \( x(t) \) at a single point, and not others. Let us denote such a derivative by \( \delta/\delta x(t') \). From the property above, when \( t \neq t' \), we need

\[
\frac{\delta}{\delta x(t')} x(t) = 0
\]

However, when \( t = t' \), we need the derivative to be nonzero. In some sense, we’d want it to be
equal to 1 at that point. More precisely, we define
\[
\frac{\delta}{\delta x(t')} x(t) = \delta(t - t')
\]
where \( \delta(t - t') \) is the Dirac delta function, which recall is a function\(^1\) with the following properties:
\[
\delta(t - t') = 0 \text{ for } t \neq t'
\]
\[
\int_{-\infty}^{\infty} dt f(t') \delta(t - t') = f(t)
\]
for any ‘well-behaved’ function \( f(t) \). Furthermore, we expect this infinite-dimensional derivative or functional derivative \( \frac{\delta}{\delta x(t')} \) to have all the properties one would expect of an ordinary derivative acting on differentiable functions: there should be a product rule, a chain rule, derivatives should commute with one another, and so forth. For example:
\[
\frac{\delta}{\delta x(t')} (x(t))^2 = 2x(t) \delta(t - t')
\]
\[
\frac{\delta}{\delta x(t')} (x(t)y(t)) = \delta(t - t')y(t)
\]
A typical application of this derivative is in the ‘calculus of variations,’ where one is asked to find, for example, a function that minimizes some functional.

Here is one simple example: show that the shortest distance between two points is a straight line. For simplicity, let us assume the points lie in a fixed plane, and consider paths which can be represented as functions \( y(x) \), where \( x, y \) are the obvious coordinates on the plane. For a given path \( y(x) \) between two fixed endpoints, the length of the path is defined by
\[
s = \int \left( 1 + \left( \frac{dy}{dx} \right)^2 \right)^{1/2} dx
\]
To find the path that minimizes the arc length \( s \), let us compute \( \frac{\delta s}{\delta y(x')} \) and set it equal to zero. (Morally, this is just like first-semester calculus, where you find the extremum of a function \( f(x) \) by solving \( f'(x) = 0 \).) Let us compute:
\[
\frac{\delta}{\delta y(x')} s = \frac{\delta}{\delta y(x')} \int \left( 1 + \left( \frac{dy}{dx} \right)^2 \right)^{1/2} dx
\]
\[
= \int dx \left( 1 + \left( \frac{dy}{dx} \right)^2 \right)^{-1/2} \left( 2 \frac{dy}{dx} \right) \frac{d}{dx} \left( \frac{\delta y(x)}{\delta y(x')} \right)
\]
\[
= \int dx \left( 1 + \left( \frac{dy}{dx} \right)^2 \right)^{-1/2} \left( 2 \frac{dy}{dx} \right) \frac{d}{dx} \delta(x - x')
\]
\(\text{1More accurately, the Dirac delta ‘function’ is something called a distribution, not a function. It can be thought of as the limit of a series of functions that approximate it. For example, it turns out that}
\[
\delta(x) = \lim_{\Lambda \to \infty} \int_{-\Lambda}^{\Lambda} \frac{d\omega}{2\pi} e^{i\omega} = \lim_{\Lambda \to \infty} \frac{\sin(x\Lambda)}{\pi x}
\]
Now, the expression above contains a derivative of a Dirac delta function. To make sense of this, we integrate by parts, and use the fact that the boundary terms will vanish so long as $x'$ is not at the edges:

$$\frac{\delta}{\delta y(x')} s = - \int dx \frac{d}{dx} \left[ (1 + (y')^2)^{-1/2} y \right] \delta(x - x')$$

$$= - \frac{d}{dx} \left[ (1 + (y')^2)^{-1/2} y' \right]$$

$$= - \left( 1 + (y')^2 \right)^{-3/2} y''$$

To find the path $y(x)$ that minimizes the arc length $s$, we find the function $y(x)$ for which $\delta s/\delta y(x') = 0$. From the expression above, we see that a sufficient condition is $y'' = 0$, which implies $y = Ax + B$ for some constants $A$ and $B$. This is the equation of a straight line, so we have just verified, using functional derivatives, that the shortest distance between two points is a straight line.

**** Mention Brachistochrone, calculus of variations.

A more common physics application is a rewriting of classical mechanics due to Hamilton, and known as Hamilton’s least-action principle: the path taken by any object minimizes its action, denoted $S$, where the action is the difference of the time integrals of the kinetic and potential energies. The differential equations one obtains from expanding $\delta S[y(x)]/\delta y(x') = 0$ are known as the equations of motion.

With that in mind, let us compute the equations of motion of a (nonrelativistic) particle of mass $m$ in a potential $V(x)$. The action, the difference of the integrals of the kinetic and potential energies, is

$$S[x(t)] = \int_{-\infty}^{\infty} dt \left( \frac{m}{2} \left( \frac{dx}{dt} \right)^2 - V(x) \right)$$

Now, let us apply Hamilton’s least-action principle to compute the equations of motion. From the discussion above,

$$\frac{\delta}{\delta x(t')} S[x(t)] = \frac{\delta}{\delta x(t')} \int_{-\infty}^{\infty} dt \left( \frac{m}{2} \left( \frac{dx}{dt} \right)^2 - V(x) \right)$$

$$= \int_{-\infty}^{\infty} \left( \frac{m}{2} \frac{dx}{dt} \frac{d}{dt} \delta x(t') - V'(x) \frac{\delta x(t)}{\delta x(t')} \right)$$

$$= \int_{-\infty}^{\infty} \left( -m \frac{d^2 x}{dt^2} \delta(t - t') - V'(x) \delta(t - t') \right)$$

$$= \int_{-\infty}^{\infty} \left( -m \frac{d^2 x}{dt^2} - V'(x) \right)$$

where in the next to last step we have integrated by parts and assumed the boundary terms
vanished. In this case, the equations of motion are

\[ m \frac{d^2x}{dt^2} = -V'(x) \]

A reader who remembers a bit of classical mechanics might recall this expression. The quantity \( d^2x/dt^2 \) is the acceleration \( a \) of a particle, and the net force \( F \) on a particle in a potential \( V(x) \) is given by \( F = -V(x) \). So our equations of motion say \( F = ma \), which is one of Newton’s laws.

Let us work through a few more examples, since we will initially be using functional derivatives a great deal. Let us next consider a plane pendulum, as illustrated below.

The potential energy of the pendulum is clearly

\[ V = mgL(1 - \cos \theta) \]

The kinetic energy of the pendulum is

\[ T = \frac{1}{2}m \left( \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \right) \]

but

\[ x = L \sin \theta \]
\[ y = L - L \cos \theta \]

so \( T = (1/2)mL^2(d\theta/dt)^2 \), hence the action functional is given by

\[ S[\theta(t)] = \int dt \left[ \frac{1}{2}mL^2 \left( \frac{d\theta}{dt} \right)^2 - mgL(1 - \cos \theta) \right] \]

To minimize the action functional, we find \( \theta(t) \) such that \( \delta S/\delta \theta(t_0) = 0 \).

\[ \frac{\delta S}{\delta \theta(t_0)} = \int dt \left[ \frac{1}{2}mL^2 \left( \frac{d\theta}{dt} \right)^2 \frac{d}{dt} \delta(t - t_0) - mgL(\sin \theta)\delta(t - t_0) \right] \]

\[ = \int dt \left[ -mL^2 \frac{d^2\theta}{dt^2} - mgL \sin \theta \right] \delta(t - t_0) \]
\[ = -mL^2 \frac{d^2\theta}{dt^2} - mgL \sin \theta \]
so the equations of motion are

$$L \frac{d^2 \theta}{dt^2} = -g \sin \theta$$

Thus, any $\theta(t)$ that satisfies the differential equation above should minimize the action functional and satisfy Hamilton’s principal.

Let us work through another example. A bead of mass $m$ slides along a smooth wire bent in the shape of a pendulum $z = cr^2$. When the wire rotates at $\omega_0$, the bead revolves in a circle of radius $R$, as illustrated below. Given that information, we shall compute the value of $c$.

We shall work in cylindrical coordinates $r, \theta, z$. The kinetic energy is given by

$$T = \frac{1}{2} m \left( \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\theta}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2 \right)$$

$$= \frac{1}{2} m \left( \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\theta}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2 \right)$$

after converting to cylindrical coordinates. The potential energy $V = mgz$. Now, there is a relation between $r$, $\theta$, and $z$, imposed by the fact that the bead is constrained to lie on the wire: $z = cr^2$. Eliminating $z$, the kinetic and potential energies are given by

$$T = \frac{1}{2} m \left( \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\theta}{dt} \right)^2 + 4c^2r^2 \left( \frac{dr}{dt} \right)^2 \right)$$

$$V = mgcr^2$$

There is a second relation between the remaining coordinates $r$, $\theta$ imposed by the fact that the wire spins at rate $\omega$: $d\theta/dt = \omega$. Eliminating $\theta$ yields

$$T = \frac{1}{2} m \left( \left( \frac{dr}{dt} \right)^2 + r^2 \omega^2 + 4c^2r^2 \left( \frac{dr}{dt} \right)^2 \right)$$

$$V = mgcr^2$$

The action functional is then given by

$$S[r(t)] = \int dt \left[ \frac{1}{2} m \left( \left( \frac{dr}{dt} \right)^2 + r^2 \omega^2 + 4c^2r^2 \left( \frac{dr}{dt} \right)^2 \right) - mgcr^2 \right]$$
From $\delta S \delta r(t) = 0$ one gets the equations of motion

$$\left( \frac{d^2 r}{dt^2} \right) \left( 1 + 4c^2 r^2 \right) + 4c^2 r \left( \frac{dr}{dt} \right)^2 + r(2gc - \omega^2) = 0$$

For a solution with $r(t) = R$, a constant, this reduces to

$$R(2gc - \omega^2) = 0$$

or

$$c = \frac{\omega^2}{2g}$$

Let us work through one more example. This time, we will compute the equations of motion of a double pulley system with three masses, as illustrated below.

For mass $m_1$,

$$T = \frac{1}{2} m_1 \left( \frac{dx}{dt} \right)^2$$

$$V = -m_1 g x$$

For mass $m_2$:

$$T = \frac{1}{2} m_2 \left( \frac{d}{dt} (\ell_1 - x + y) \right)^2$$

$$V = -m_2 g (\ell_1 - x + y)$$

For mass $m_3$:

$$T = \frac{1}{2} m_3 \left( \frac{d}{dt} (\ell_1 - x + \ell_2 - y) \right)^2$$

$$V = -m_3 g (\ell_1 - x + \ell_2 - y)$$

Since there are now two independent functions, Hamilton’s principle now becomes the statement that

$$\frac{\delta S}{\delta x(t)} = \frac{\delta S}{\delta y(t')} = 0$$
– just as, in a vector calculus class, one finds extrema of a function \( f(x, y) \) by solving \( \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \). From Hamilton’s principle, we derive the equations of motion

\[
\begin{align*}
    m_1 \frac{d^2 x}{dt^2} - m_2 \left( -\frac{d^2 x}{dt^2} + \frac{d^2 y}{dt^2} \right) + m_3 \left( \frac{d^2 x}{dt^2} + \frac{d^2 y}{dt^2} \right) &= (m_1 - m_2 - m_3)g \\
    m_2 \left( -\frac{d^2 x}{dt^2} + \frac{d^2 y}{dt^2} \right) + m_3 \left( \frac{d^2 x}{dt^2} + \frac{d^2 y}{dt^2} \right) &= (m_2 - m_3)g
\end{align*}
\]

***** Insert something on Euler-Lagrange equations? Or minimize? I’m tending towards the latter – prefer students to know the general method, not memorize an equation.

### 1.2 Functional integrals

Hamilton’s least-action principle is a very elegant rephrasing of classical mechanics – no more mucking about with force diagrams, just minimize an action functional. But, you might ask, why should it describe classical mechanics?

When Richard Feynman was a graduate student, he asked himself the same question. He eventually showed, as part of his Ph.D. thesis, the Hamilton’s least-action principle expresses the leading effect in an approximation to quantum mechanics. In order to do this, he defined the integrals that go hand-in-hand with the functional derivatives introduced in the last section. Known as functional integrals or path integrals, these integrals – formally over spaces of functions – can be used to define quantum mechanics and quantum field theory.

Given a classical mechanical system described by some action functional \( S \), quantum mechanics is encoded in the functional integral denoted

\[
\int [Dx] \exp(iS/\hbar)
\]

where \( \hbar \) is Planck’s constant, which encodes the strength of quantum corrections. When \( \hbar \) is small, quantum effects are suppressed. We have not yet tried to make sense of this expression, but we can already see – at least formally – how Hamilton’s least-action principle is going to emerge.

Classical mechanics will arise from applying a version of the method of steepest descent described in section D.4. Briefly, for an integral of the form

\[
\int g(z) \exp(sf(z))dz
\]

for large \( s \) the dominant contribution to the integral is from \( z \) such that \( f'(z) = 0 \), known as the saddle points. Up to overall factors, resulting from approximating the integral by a Gaussian around each saddle point \( z_0 \), the integral is approximated by \( g(z_0) \exp(sf(z_0)) \). This gives the leading approximation to what is called an “asymptotic series expansion” in \( s \) for the value of the integral. We will return to this matter in the next subsection, where we shall discuss the method of steepest descent in greater detail.
Returning to the path integral (1.1), applying the basic idea of the method of steepest descent, one would expect that for small $\hbar$, the dominant contribution to the path integral will come from functions $x(t)$ such that $\delta S/\delta x = 0$, and other contributions should be, comparatively, exponentially suppressed. But $\delta S/\delta x = 0$ is exactly the statement of Hamilton’s least-action principle.

Experts at quantum mechanics may recognize that, in hindsight, there are a few places in quantum mechanics where closely analogous structures arise, for example:

- Eikonal approximation in scattering. When the de Broglie wavelength is much smaller than the scale over which a potential varies, and $\hbar$ is small, one can take the wavefunction $\psi$ to be approximately $\exp(iS/\hbar)$, where $S$ satisfies the Hamilton-Jacobi equation **** FILL IN ****.
- WKB approximation. This is a semiclassical approximation, valid for $\hbar$ small, in which again the wavefunction $\psi$ is given approximately by $\exp(iS/\hbar)$.

We will not study either the eikonal or WKB approximations further, but thought that it was important to at least mention these points.

*** NOTE TO SELF: ought to clarify the statements above slightly.

Now, how does one define a path integral, at least formally? Let us consider functions of a single variable $t$. Suppose the action is an integral over the interval $[a, b]$. Just as an ordinary integral can be defined as the limit of a sequence of Riemann sums, we can formally try to define the path integral as the limit of a series of approximations. Let $N$ be an integer, and split the interval $[a, b]$ into $N$ equal-size pieces. Define $t_k = a + k(b - a)/N$, and define $x_k = x(t_k)$. To try to integrate over a space of functions $x(t)$, let us integrate over the values of $x(t)$ at the $t_k$, and interpolate function values in between those points. We have tried to schematically illustrate this below:

In the picture above, we have broken up the interval $[a, b]$ into 5 regions, and we approximate a function on $[a, b]$ by its values on the boundaries between subintervals, taking the values at intermediate points to be defined by straight lines joining the values at the boundaries.

In other words, approximate $\int [Dx]$ by

$$C^N \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{N-1}$$
for some suitable normalizing constant $C$. Suppose the action functional $S$ is given by
\[ \int_a^b \frac{1}{2} \left( \frac{dx}{dt} \right)^2 dt \]
then we could approximate the functional by
\[ \sum_{j=0}^{N-1} \frac{1}{2} \left( \frac{x_{j+1} - x_j}{(b-a)/N} \right)^2 \left( \frac{b-a}{N} \right) \]
In this form, we could then try to define
\[ \int [Dx] \exp\left( \frac{iS}{\hbar} \right) = \lim_{N \to \infty} C^N \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{N-1} \exp \left[ \frac{i}{\hbar} \sum_{j=0}^{N-1} \frac{1}{2} \left( \frac{x_{j+1} - x_j}{(b-a)/N} \right)^2 \left( \frac{b-a}{N} \right) \right] \]
If we try to pursue this program, then there are some problems one will run into. For example, it is far from clear that different approximation schemes for any given action functional $S$ will result in the same path integral. In fact, there is indeed such an ambiguity in quantum mechanics – a given quantum-mechanical system is not uniquely specified by a classical mechanical system. In formal discussions, this boils down to the Stone-von Neumann theorem. This is not a limitation of path integrals, but rather reflects the reality that quantum mechanics is a more nearly correct description of the physics, so to be precise, one must specify the quantum mechanical system, for which classical mechanics is then just an approximation.

**** pp 21-28 contain a much more detailed discussion/derivation of path integrals from Heisenberg/Schrodinger picture.

1.3 Derivation of classical physics from path integrals

Part of Feynman’s insight is the way his description of quantum mechanics also includes a derivation of Lagrangian mechanics and Hamilton’s least-action principle. In fact, according to various sources, Feynman originally developed path integrals as his way of understanding Hamilton’s least-action principle: although Hamilton’s principle is very pretty, Feynman did not have a clear conceptual understanding of why it should necessarily reproduce classical mechanics, and so did not fully trust Hamilton’s principle until he found a more direct way of understanding it. (Young graduate students should note this is often an excellent approach to research.)

Feynman’s derivation of Hamilton’s least-action principle rests on the method of steepest descent, a method used to approximate contour integrals. Let us briefly outline how this works for real-valued functions along the real line. Briefly, given an integral of the form
\[ I(z) = \int dx g(x) \exp(zf(x)) \]
the method of steepest descent says that for large values of $z$, the dominant contribution to the integral $I(z)$ will come from values of $x$ such that $f'(x) = 0$. If $x_0$ is such an $x$, and $2 f''(x_0) < 0$,

\footnote{For a contour in the complex plane, if $f''(x_0)$ does not have real part of the correct sign, one can rotate the contour locally until the exponent becomes a Gaussian to leading-order. The rotation necessarily involves the contour locally moving along a steepest-descent path, thus the name of the method.}
then close to $x_0$, for large $z$, the integral can be approximated by a Gaussian, as follows. Expand both $g(x)$ and $f(x)$ in power series:

$$
g(x) = g(x_0) + (x - x_0)g'(x_0) + \frac{1}{2}(x - x_0)^2g''(x_0) + \cdots
$$

$$
f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2f''(x_0)
$$

so that we can write

$$
I(z) = \int dx \left( g(x_0) + (x - x_0)g'(x_0) + \cdots \right) \exp \left( z \left( f(x_0) - \frac{1}{2}(x - x_0)^2|f''(x_0)| + \cdots \right) \right)
$$

Now, define $y = \sqrt{z}(x - x_0)$, so that the integral above becomes

$$
I(z) = \exp(zf(x_0)) \int \frac{dy}{\sqrt{z}} \left( g(x_0) + \frac{y}{\sqrt{z}} + \mathcal{O}(z^{-1}) \right) \exp \left( -\frac{1}{2}y^2|f''(x_0)| + \mathcal{O}(z^{-1/2}) \right)
$$

$$
= \exp(zf(x_0)) \sqrt{\frac{2}{|f''(x_0)|}} g(x_0) \pi + \mathcal{O}(z^{-1})
$$

Thus, to leading order for large $z$, we see that the integral $I(z)$ is proportional to the integrand evaluated at $x_0$, for an isolated critical point $x_0$. More generally, the method of steepest descent gives the leading term in an “asymptotic series” expansion, which need not converge. Asymptotic series and the method of steepest descent will be discussed in greater detail in the next chapter and in appendix ***** FILL IN *****.

**** The formal derivation of steepest descent above is written for $x$ real (though I do mention contour integrals!); I should outline briefly (then in more detail in the appendix) how it generalizes to contour integrals.

Let us apply the method of steepest descent to a path integral. Given a path integral of the form

$$
\int [Dx] \exp(iS/\hbar)
$$

the method of steepest descent tells us that for large $\hbar^{-1}$ or small $\hbar$ (i.e. the classical limit), the dominant contribution to the path integral will come from paths $x(t)$ such that $\delta S/\delta x(t) = 0$, which is precisely Hamilton’s least-action principle. Thus, Hamilton’s least-action principle can be derived as the leading-order approximation to path integrals in the classical limit.

***** Say something about summing over all paths and stuff cancelling out, as in Feynman’s QED book?
1.4 Lagrange multipliers

The path integral description of quantum mechanics suggests a way of introducing constraints. Recall that in one dimension, the Dirac delta function is given by

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\omega x) dx$$

This suggests that in path integrals, a constraint \(\varphi[x(t)] = 0\) can be introduced by adding

$$\int [DA] \exp\left(\frac{i}{\hbar} \int dt \Lambda(t) \varphi[x(t)]\right)$$

or, equivalently, adding a field \(\Lambda\), known as a Lagrange multiplier and a term \(\Lambda(t)\varphi[x(t)]\) to the action. This intuition is correct, as we shall see next.

Let us return to the example of the plane pendulum discussed previously in section **** FILL IN ****. The kinetic and potential energies are given by

$$T = \frac{1}{2} m \left( \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \right)$$
$$V = mgy$$

However, there is really only one independent, namely \(\theta(t)\). Previously, we wrote \(x(t), y(t)\) in terms of \(\theta(t)\), and then manually rewrote \(T\) and \(V\). Instead, let us use Lagrange multipliers to impose the constraint

$$x^2 + (L - y)^2 - L^2$$

(1.2)

The Lagrangian of the theory is

$$L = \frac{1}{2} m \left( \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \right) - mgy + \Lambda \left( x^2 + (L - y)^2 - L^2 \right)$$

and the action is \(S = \int L(t) dt\), where we now have the three functions \(x(t), y(t), \Lambda(t)\), instead of the one single function \(\theta(t)\) that we had in the previous treatment of this problem. We can apply Hamilton’s principle as before, which means solving

$$\frac{\delta S}{\delta x(t)} = \frac{\delta S}{\delta y(t')} = \frac{\delta S}{\delta \Lambda(t'')} = 0$$

for \(x(t), y(t),\) and \(\Lambda(t)\). This gives the equations of motion

$$m \frac{d^2 x}{dt^2} = 2x\Lambda$$
$$m \frac{d^2 y}{dt^2} + mg = -2(L - y)\Lambda$$

which we must solve together with the constraint (1.2). We can write

$$x(t) = L \sin \theta(t)$$
$$y(t) = L - L \cos \theta(t)$$
which solves (1.2). The equations of motion become

\[
\Lambda(t) = \frac{m}{2} \left[ -\left( \frac{d\theta}{dt} \right)^2 + \cot \theta \frac{d^2 \theta}{dt^2} \right]
\]

\[
\Lambda(t) = -\frac{m}{2} \left[ \left( \frac{d\theta}{dt} \right)^2 + \tan \theta \frac{d^2 \theta}{dt^2} + \frac{g}{L \cos \theta} \right]
\]

Equating the two expressions for \(\Lambda(t)\) gives

\[
L \frac{d^2 \theta}{dt^2} = -g \sin \theta
\]

the same differential equation derived previously for the pendulum.

The notion of Lagrange multipliers also appears in studies of (ordinary, finite-dimensional) vector calculus. The notion there is really just a finite-dimensional version of the Lagrange multipliers presented here. Recall, for example, that if one wants to extremize a function \(f(x)\) subject to the constraint \(\varphi(x) = 0\), where \(x\) is a constant instead of a function, then one solves

\[
\frac{\partial f}{\partial x} + \Lambda \frac{\partial \varphi}{\partial x} = 0
\]

\[
\varphi = 0
\]

which is precisely the finite-dimensional analogue of the story presented here.

For an example of the vector calculus version, consider a rectangle with sides \(a, b\). Let us extremize the area subject to the constraint \(a + b = L\). The area \(A = ab\), so we must solve

\[
\frac{\partial A}{\partial a} + \Lambda \frac{\partial \varphi}{\partial a} = 0
\]

\[
\frac{\partial A}{\partial b} + \Lambda \frac{\partial \varphi}{\partial b} = 0
\]

\[
\varphi = a + b - L = 0
\]

where \(\varphi = a + b - L\). The solution to these equations is \(a = b = L/2\), so we see that the rectangle that extremizes its area is a square.

***** pp 18-19 of qftpart1.pdf discusses Hamiltonian dynamics; where should that go in here????

***** fill in

### 1.5 Further reading

***** read PS 9.1; Ryder 5.1-5.5; Sakurai 2.5. Also see Feynman’s original paper.

***** Math track in 7650 at Utah; Munkres ch 2-1, 2-2
1.6 Notes

1. In this problem set you’ll get your first exposure to path integrals. Although the canonical quantization you learned last semester bears a closer formal resemblance to the quantum mechanics you’re familiar with, and so is a good place to start, nevertheless for many issues in QFT path integrals give a more insightful way of thinking about matters, and so I encourage you to try thinking in terms of path integrals, you should get a much clearer picture of what you’re doing and why things work.

2. Having said that, there are 2 philosophical issues that should be raised in connection with path integrals. The first issue is that mathematicians have not yet succeeded in giving a rigorous definition of a path integral, unlike the ordinary integrals you learned about back in calculus classes. So, there’s no underlying level of mathematical rigor, and in principle in complicated situations that could come back to haunt us. (Not to worry, I’m sure that mathematicians will eventually work it out. After all, it was several hundred years after Newton invented calculus before calculus was made rigorous; it might be a few hundred years before path integrals are made rigorous. That’s no reason not to use them in the meantime.)

3. The second philosophical issue concerns the form of the path integral itself. We ought to think about classical physics as a limit of and approximation to quantum mechanics, which is more fundamental; yet, the way we’ve described it so far, the path integral describes quantum mechanics using a classical action. It sounds almost as if we’re doing things the wrong way around, getting quantum mechanics from classical physics, and that shouldn’t quite be right. Worse, the paths we sum over in a path integral are continuous but usually nowhere differentiable, yet the action is phrased in terms of derivatives – how can that be consistent?

4. The resolutions of these two philosophical conundrums are probably related. We shall see in a few weeks that the description we’ve given of path integrals in this and next week’s reading is a bit too naive, and will lead to uncontrollable divergences – the famous infinities of quantum field theory. We’ll see that the classical action must be replaced by something else, a ‘regularized’ action, for which the classical action is only an approximation, which will help make those infinities controllable. This speaks to the second philosophical issue above, in that when we do things properly, we don’t really work with a classical action, but with something a bit more complicated. Presumably, when mathematicians eventually figure out how to formulate path integrals rigorously, they’ll come to the same conclusion – that to formulate a path integral rigorously, one must work with a ‘regularized’ action, and not an ordinary classical action. Thus, this also (presumably) speaks to the first philosophical issue.

1.7 Exercises

1. Fermat’s principle of optics states that light travels in a path for which the quantity \( \int n(x, y, z) ds \)
is a minimum, where $ds$ is the infinitesimal arc length and $n$ is the “index of refraction.”

Restrict to paths in a plane for simplicity, and (in polar coordinates) suppose that $n(r, \theta) = r^k$ for some integer $k$. Show that when $k = -1$, a light ray can travel in a circle about the origin.

2. A particle moving in one dimension has the classical action

$$S = \int dt \left( \frac{1}{2} m \dot{x}^2 + \lambda x (\ddot{x})^2 \right)$$

Use functional derivatives and Hamilton’s least-action principle to derive the classical equations of motion of this particle.

3. A particle moves in one dimension with an acceleration-dependent potential, as described by the action

$$S = \int dt \left( \frac{1}{2} m \dot{x}^2 - (\ddot{x})^2 - x^3 \right)$$

Use functional derivatives and Hamilton’s least-action principle to derive the classical equations of motion of this particle.

4. A simple pendulum of length $L$ and bob mass $m$ is attached to a massless support moving horizontally with constant acceleration $a$. Find the equations of motion of the bob.

5. A bead of mass $m$ slides along a smooth wire bent in the shape of a parabola $z = cr^2$.

Assume the wire rotates about the $z$-axis at angular velocity $\omega$, and find the equations of motion of the bead. Use Lagrange multipliers to enforce the constraint that $z = cr^2$ for the bead. (In the text, we solved this problem without using Lagrange multipliers; here, you are to exercise your knowledge of Lagrange multipliers.)

6. (A mathematics problem on the method of steepest descent.) The modified Bessel function $K_0(x)$ has the integral representation

$$K_0(x) = \int_0^\infty \exp(-x \cosh t) dt$$

Use the method of steepest descent, applied to the integral above, to derive the leading term in an (“asymptotic”) expansion of $K_0(x)$, valid for large $x$.

7. In this problem I’m going to have you compute the path integral for a free particle moving in one dimension. This is described in your texts, but I want to make sure you work through the details yourself at least once. So, we want to compute

$$Z = \int \left[d\mathbf{x}(t)\right] \exp \left( \frac{i}{\hbar} \int_{t_i}^{t_f} \frac{1}{2} m \dot{x}^2 \right)$$

where the initial and final positions and times $(x_i, t_i), (x_f, t_f)$ are fixed. To do this, break up the time interval $[t_i, t_f]$ into $N$ equal-size pieces, and let $x_k = x(t_j + kT/N)$, where $T = t_f - t_i$. Then, we can write

$$Z = \lim_{N \to \infty} C_N \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{N-1} \exp \left( \frac{i}{\hbar} \frac{m}{2} \sum_{j=0}^{N-1} \left( \frac{x_{j+1} - x_j}{T/N} \right)^2 \left( \frac{T}{N} \right) \right)$$
where $C$ is chosen so as to make the result finite.

Use the integral
\[
\int_{-\infty}^{\infty} dx_1 \cdots dx_n \exp \left[ i \lambda \left( (x_1 - a)^2 + (x_2 - x_1)^2 + \cdots + (b - x_n)^2 \right) \right]
\]
\[= \left[ \frac{1}{n + 1} \left( \frac{i \pi}{\lambda} \right)^n \right] \exp \left[ i \lambda \left( \frac{n}{n + 1} (b - a)^2 \right) \right]
\]
to show that if we take
\[
C = \left( \frac{2\pi i \hbar m T}{N} \right)^{-1/2}
\]
then
\[
Z = \left( \frac{m}{2 \pi i \hbar T} \right)^{1/2} \exp \left( \frac{im}{2 \hbar T} (x_f - x_i)^2 \right)
\]

Cultural aside: Let us consider alternative ways to perform this integral. One could imagine systematically evaluating each $\int dx_i$, one by one, but you can quickly convince yourself that the resulting calculation is very messy and error-prone. A more clever alternate strategy is to write
\[
\exp \left( i \lambda \sum_{j=0}^{N-1} (x_{j+1} - x_j)^2 \right) = \exp \left( i \lambda \vec{x}^T A \vec{x} \right)
\]
where $\vec{x} = (x_0, x_1, \cdots, x_N)^T$ (where $x_0 = x_i$, $x_N = x_f$), and $A$ is a matrix of the form
\[
A = \begin{bmatrix}
1 & -1 & 0 & 0 & \cdots & 0 & 0 \\
-1 & 2 & -1 & 0 & \cdots & 0 & 0 \\
0 & -1 & 2 & -1 & \cdots & 0 & 0 \\
\vdots \\
0 & 0 & 0 & 0 & \cdots & 2 & -1 \\
0 & 0 & 0 & 0 & \cdots & -1 & 1 \\
\end{bmatrix}
\]

Then, if we were integrating over $x_0$ and $x_N$ as well as the rest, we could perform a similarity transformation to diagonalize $A$, and turn this problem into a product of decoupled Gaussians, each involving an eigenvalue of $A$. The resulting integral would be proportional to $(\det A)^{-1/2}$. Here, this approach is more tricky because the initial and final values are fixed. However, in other circumstances where the initial and final conditions are at infinity, this line of thought can be very productive, and gives us an efficient way to think about path integrals. We shall see this approach in the next problem set.

8. Quantization of the plane rotator. In this problem you will be asked to calculate
\[
\langle \theta' | \exp(-iHT/\hbar) | \theta \rangle
\]
in two different ways – by solving Schrödinger’s equation and with path integrals – and check that the result is the same in both cases.

Consider the “rigid rotator” defined by the action
\[
S = \int dt \frac{1}{2} \dot{\theta}^2
\]
where $\theta$ is defined mod $2\pi$ (you should think of $\theta$ as the angle on a circle).
(a) Solve Schrödinger’s equation and calculate
\[ \langle \theta' | \exp(-iHT/\hbar)|\theta \rangle \]
In other words, using
\[ p = -\hbar \frac{\partial}{\partial \theta}, \quad H = \frac{1}{2I} p^2 = -\frac{\hbar^2}{2I} \left( \frac{\partial}{\partial \theta} \right)^2 \]
find the (normalized, periodic) solutions of \( H\psi = E\psi \), and then identifying \( \psi_n = \langle n|\theta \rangle \) and inserting a couple of copies of \( 1 = \sum_n |n\rangle\langle n| \), calculate \( \langle \theta' | \exp(-iHT/\hbar)|\theta \rangle \).
(b) Now, use your result from the previous problem for the path integral for a free particle to compute the path integral of this plane rotator. (Hint: you can do this by inspection.) (Hint: in the case of the free particle, the path integral \( Z = \langle x_f | \exp(-iHT/\hbar)|x_i \rangle \).
(c) Show that the two results you obtained for \( \langle \theta' | \exp(-iHT/\hbar)|\theta \rangle \) are the same. You may need to use the Poisson resummation formula
\[ \frac{1}{2\pi} \sum_m \exp(im\theta) = \sum_n \delta(\theta - 2\pi n) \]
and Gaussian integration.