Quantum Sheaf Cohomology I

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In type II string compactifications, worldsheet corrections have a long history (Gromov-Witten, . . .), and lead to quantum cohomology.

Briefly, in heterotic strings, worldsheet instanton effects are modified by the target space gauge instantons, leading to *quantum sheaf cohomology*.

Today, I'll give an introduction to such effects in heterotic strings.

Will be continued later today by Josh Guffin, Jock McOrist.

Classes of superpotential terms

To be specific, imagine that we have compactified a heterotic string on a CY 3-fold with a rank 3 vector bundle, breaking an E_8 to E_6, and so the low-energy theory contains $\mathbf{27}$'s and $\overline{\mathbf{27}}$'s in addition to singlets.

- *Charged matter couplings e.g.* $\overline{\mathbf{27}}^3$*.* When the gauge bundle = tangent bundle, these are computed by the "A model topological field theory," and correspond to Gromov-Witten invariants, essentially. For more general gauge bundles, need (0,2) A model (also called A/2).

(cont'd ....)

Classes of superpotential terms

- *Charged matter couplings e.g.* $\mathbf{27}^3$. When the gauge bundle = tangent bundle, these are computed by the "B model topological field theory," and are purely classical – no nonperturbative (in α') corrections. For more general gauge bundles, need (0,2) B model (also called B/2).
- *Gauge singlet matter couplings.* When the gauge bundle = tangent bundle, the previous two classes could be computed by well-known math tricks, but no such tricks exist for gauge singlets. Turns out that in "many" cases, individual worldsheet instanton contributions are nonzero but cancel out when you add them all up, resulting in no net nonperturbative correction. (Dine-Seiberg-Wen-Witten, Silverstein-Witten, Candelas *et al*, Beasley-Witten)

What I'm going to talk about today are corrections to *e.g.* $\overline{\mathbf{27}}^3$ and, later, $\mathbf{27}^3$ couplings.

(These corrections will all be nonperturbative; perturbative corrections in α' forbidden by Kähler axion.)

There's another, more formal, motivation for what I'll describe today, namely: (0,2) mirror symmetry.

Ordinary mirror symmetry: $X_1 \leftrightarrow X_2$, X_1, X_2 CY's

(0,2) mirror symmetry: $(X_1, \mathcal{E}_1) \leftrightarrow (X_2, \mathcal{E}_2)$ where $\mathcal{E}_1$, $\mathcal{E}_2$ are bundles on X_1, X_2

(0,2) mirror symmetry is poorly understood at present, though progress is being made.

Progress towards (0,2) mirrors:

Example:
Adams-Basu-Sethi (2003) applied work of Morrison-Plesser / Hori-Vafa on ordinary (2,2) mirrors to (0,2) GLSM's, to make some predictions for (0,2) mirrors in some relatively simple cases.

They also made some predictions for analogues of $\overline{\mathbf{27}}^3$ superpotential terms, or equivalently product structures in heterotic chiral rings, "quantum sheaf cohomology."

Outline

- review A model TFT, half-twisted (0,2) TFT
- review correlation f'n computations in A model, describe analogue for (0,2) models
  - formal structure similar; (0,2) generalizes A model
  - compactification issues; not only $\mathcal{M}$, but bundles on $\mathcal{M}$
- apply GLSM's; not only naturally compactify $\mathcal{M}$, but also naturally extend the bundles
- Adams-Basu-Sethi prediction
- Analogue for B model
- Consistency conditions in closed string B model

As outlined before, when the gauge bundle = tangent bundle, the $\overline{\mathbf{27}}^3$ and analogous couplings are computed by a 2d TFT called the “A model.”

What I’ll be describing amounts to a (0,2) analogue or generalization of the ordinary A model.

First: review the A model....

The 2D TFT's are obtained by changing the worldsheet fermions: worldsheet spinors $\mapsto$ worldsheet scalars & vectors.

Concretely, that means if we start with the nonlinear sigma model

$$g_{i\bar{\jmath}}\overline{\partial}\phi^i\partial\phi^{\bar{\jmath}} \; + \; i g_{i\bar{\jmath}}\psi_-^{\bar{\jmath}} D_z \psi_-^i \; + \; i g_{i\bar{\jmath}}\psi_+^{\bar{\jmath}} D_{\overline{z}} \psi_+^i \; + \; R_{i\bar{\jmath}k\bar{l}}\psi_+^i \psi_+^{\bar{\jmath}} \psi_-^k \psi_-^{\bar{l}}$$

then we deform the $D\psi$'s by changing the spin connection term. Since $J \sim \overline{\psi}\psi$, this amounts to making the modification

$$L \; \mapsto \; L \; \pm \; \frac{1}{2}\omega J \quad \Longleftrightarrow \quad T \; \mapsto \; T \; \pm \; \frac{1}{2}\partial J$$

More formally (useful for computation), A model:

$$g_{i\bar{\jmath}}\overline{\partial}\phi^i\partial\phi^{\bar{\jmath}} \;+\; i g_{i\bar{\jmath}}\psi_-^{\bar{\jmath}} D_z \psi_-^i \;+\; i g_{i\bar{\jmath}}\psi_+^{\bar{\jmath}} D_{\overline{z}}\psi_+^i \;+\; R_{i\bar{\jmath}k\bar{l}}\psi_+^i\psi_+^{\bar{\jmath}}\psi_-^k\psi_-^{\bar{l}}$$

$$\psi_-^i (\equiv \chi^i) \;\in\; \Gamma((\phi^* T^{0,1}X)^\vee) \qquad \psi_+^i(\equiv \psi_z^i) \;\in\; \Gamma(K\otimes \phi^* T^{1,0}X)$$
$$\psi_-^{\bar{\imath}}(\equiv \psi_{\overline{z}}^{\bar{\imath}}) \;\in\; \Gamma(K\otimes\phi^* T^{0,1}X) \qquad \psi_+^{\bar{\imath}}(\equiv \chi^{\bar{\imath}}) \;\in\; \Gamma((\phi^* T^{1,0}X)^\vee)$$

(Action has same form, but worldsheet spinors now scalars, vectors.)

Massless states

Since we no longer have worldsheet *spinors*, we no longer sum over spin sectors. In effect, the only surviving sector in this field theory is the RR sector of the original theory.

So, part of the 2D TFT story is that we're only considering RR sectors (consistent b/c no worldsheet spinors).

So, massless spectrum computations are done in RR sector only. Otherwise, proceed much as usual – states are built as Q-invariant objects, where Q is a subset of susy, corresponding to the scalar supercharges.

Massless states

Under the scalar supercharge,

$$
\begin{aligned}
\delta \phi^i \ \propto \ \chi^i, &\quad \delta \phi^{\bar{\imath}} \ \propto \ \chi^{\bar{\imath}} \\
\delta \chi^i \ = \ 0, &\quad \delta \chi^{\bar{\imath}} \ = \ 0 \\
\delta \psi_z^i \ \neq \ 0, &\quad \delta \psi_{\bar{z}}^{\bar{\imath}} \ \neq \ 0
\end{aligned}
$$

States (Q-cohomology):

$$
\begin{aligned}
\mathcal{O} \ \sim \ b_{i_1 \cdots i_p \bar{\imath}_1 \cdots \bar{\imath}_q} \chi^{\bar{\imath}_1} \cdots \chi^{\bar{\imath}_q} \chi^{i_1} \cdots \chi^{i_p} \quad &\leftrightarrow \quad H^{p,q}(X) \\
Q \quad &\leftrightarrow \quad d
\end{aligned}
$$

The A model TFT is, first and foremost, still a QFT.

But, if you only consider correlation functions between Q-invariant massless states, then the correlation functions reduce to purely zero-mode computations – (usually) no meaningful contribution from Feynman propagators or loops, and the correlators are independent of insertion positions.

Analagous phenomena elsewhere: eg in 4d $\mathcal{N} = 1$ susy models, correlation functions involving products of chiral operators are independent of insertion position (Cachazo-Douglas-Seiberg-Witten). (Basic pt: spacetime deriv $\propto \overline{Q}^{\dot{\alpha}}$ commutators, which vanish; same idea in 2d.)

More generally, TFT’s are special kinds of QFT’s which contain a “topological subsector” of correlators whose correlation functions reduce to purely zero mode calculations. Since they reduce to zero mode calculations, we can get the *exact* answer (instead of merely some asymptotic expansion) for the correlation function merely by doing a bit of math.

Put more simply still, TFT’s allow us to reduce *a priori* computationally difficult physics problems to easy math problems.

(0,2) A model (equiv'ly, A/2 model):

$$g_{i\bar{\jmath}}\overline{\partial}\phi^i\partial\phi^{\bar{\jmath}} \;+\; i h_{a\bar{b}}\lambda_-^{\bar{b}} D_z \lambda_-^a \;+\; i g_{i\bar{\jmath}}\psi_+^{\bar{\jmath}} D_{\overline{z}}\psi_+^i \;+\; F_{i\bar{\jmath}a\bar{b}}\psi_+^i\psi_+^{\bar{\jmath}}\lambda_-^a\lambda_-^{\bar{b}}$$

$$\begin{array}{rclrcl} \lambda_-^a & \in & \Gamma(\phi^*\overline{\mathcal{E}}) & \psi_+^i & \in & \Gamma(K \otimes \phi^* T^{1,0}X) \\ \lambda_-^{\bar{b}} & \in & \Gamma(K \otimes \phi^*\overline{\mathcal{E}}) & \psi_+^{\bar{\imath}} & \in & \Gamma((\phi^* T^{1,0}X)^\vee) \end{array}$$

RR states (Q cohomology):

$$\mathcal{O} \;\sim\; b_{\bar{\imath}_1\cdots\bar{\imath}_n a_1\cdots a_p}\psi_+^{\bar{\imath}_1}\cdots\psi_+^{\bar{\imath}_n}\lambda_-^{a_1}\cdots\lambda_-^{a_p} \;\leftrightarrow\; H^n(X, \Lambda^p\mathcal{E}^\vee)$$

When $\mathcal{E} = TX$, reduces to the A model above, since $H^{p,q}(X) = H^q(X, \Lambda^q (TX)^\vee)$.

Symmetry properties of states

A model:

$$H^{p,q}(X) \;\cong\; H^{n-p,n-q}(X)^* \text{ for compact } n\text{-dim'l } X$$

(0,2) A model:

$$H^q(X, \Lambda^p \mathcal{E}^\vee) \;\cong\; H^{n-q}(X, (\Lambda^{r-p}\mathcal{E}^\vee) \otimes (\Lambda^{top}\mathcal{E} \otimes K_X))^*$$

for compact n-dim'l X, rank r $\mathcal{E}$

We'll assume $\Lambda^{\rm top}\mathcal{E}^\vee \cong K_X$, in add'n to anomaly cancellation $\text{ch}_2(\mathcal{E}) = \text{ch}_2(TX)$

- **makes path integral measure well-defined**
- recovers symmetry property $H^q(X, \Lambda^p \mathcal{E}^\vee) \cong H^{n-q}(X, \Lambda^{r-p}\mathcal{E}^\vee)$
- essential for correlation functions
- in CY compactification, guarantees a left-moving $U(1)$ that is essential for spacetime gauge symmetry

Anomaly cancellation

We just outlined why we'll assume $\Lambda^{\mathrm{top}} \mathcal{E}^\vee \cong K_X$.

We'll also assume $\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX)$.

This is the "anomaly cancellation" condition arising from the Green-Schwarz mechanism

$$dH \;=\; \mathrm{tr}\, F \wedge F \;-\; \mathrm{tr}\, R \wedge R$$

This condition also manifests itself in the worldsheet theory, and can be derived (as we'll see later) for massive 2D QFT's w/ non-CY targets.

Classical correlation functions

A model:
For X compact, n-dim'l, have n χ^i zero modes and n $\chi^{\overline{\imath}}$ zero modes, plus bosonic zero modes $\sim X$, so

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \; = \; \int_X H^{p_1,q_1}(X) \wedge \cdots \wedge H^{p_m,q_m}(X)$$

Selection rule from left-, right-moving $U(1)$'s:
$\sum_i p_i = \sum_i q_i = n$. Thus

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \; \sim \; \int_X (\text{top-form})$$

Classical correlation functions

(0,2) A model:
Here we have n $\psi_+^{\bar{\imath}}$ zero modes and r λ^a zero modes, so

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \; = \; \int_X H^{q_1}(X, \Lambda^{p_1} \mathcal{E}^\vee) \wedge \cdots \wedge H^{q_m}(X, \Lambda^{p_m} \mathcal{E}^\vee)$$

Selection rule from left-, right-moving $U(1)$'s: $\sum_i q_i = n$, $\sum_i p_i = r$. Thus

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \; \sim \; \int_X H^{\mathrm{top}}(X, \Lambda^{\mathrm{top}} \mathcal{E}^\vee)$$

When $\Lambda^{\mathrm{top}} \mathcal{E}^\vee \cong K_X$, then the integrand is a top-form.

Next: worldsheet instantons

Worldsheet instantons

A model:

Here, moduli space of bosonic zero modes = moduli space of worldsheet instantons, $\mathcal{M}$.

We'll assume $\mathcal{M}$ is smooth, and review its compactification later.

Here again, correlation f'ns

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_{\mathcal{M}} (\text{top form})$$

Worldsheet instantons

(0,2) A model:

For the moment, for simplicity, we will take $\mathcal{M} = \mathrm{Maps}$, so that there is a universal instanton $\alpha : \Sigma \times \mathcal{M} \rightarrow X$, and will describe compactification later.

In addition to $\mathcal{M}$, the bundle $\mathcal{E}$ on X induces a bundle (of λ zero modes) $\mathcal{F}$ on $\mathcal{M}$:

$$\mathcal{F} \equiv R^0 \pi_* \alpha^* \mathcal{E}$$

where $\alpha : \Sigma \times \mathcal{M} \rightarrow X$, and $\pi : \Sigma \times \mathcal{M} \rightarrow \mathcal{M}$.
On the (2,2) locus, where $\mathcal{E} = TX$, have $\mathcal{F} = T\mathcal{M}$ (fixed cpx structure on worldsheet)

Worldsheet instantons

(0,2) A model, cont'd

When no excess zero modes ($R^1 \pi_* \alpha^* \mathcal{E} = 0 = R^1 \pi_* \alpha^* TX$),

$$\left. \begin{array}{c} \Lambda^{\mathrm{top}} \mathcal{E}^\vee \cong K_X \\ \mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX) \end{array} \right\} \stackrel{GRR}{\Longrightarrow} \Lambda^{\mathrm{top}} \mathcal{F}^\vee \cong K_{\mathcal{M}}$$

Correlation functions:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_{\mathcal{M}} H^{top}(\mathcal{M}, \Lambda^{top} \mathcal{F}^\vee)$$

Classically, the integrand was a top-form b/c $\Lambda^{\mathrm{top}} \mathcal{E}^\vee \cong K_X$.
Here, the integrand is a top form b/c (GRR) $\Lambda^{\mathrm{top}} \mathcal{F}^\vee \cong K_{\mathcal{M}}$.

Cohomology on $X \mapsto$ cohomology on $\mathcal{M}$

A model:

Each element of $H^{p,q}(X)$ plus a point p on the worldsheet Σ define an element of $H^{p,q}(\mathcal{M})$,

by,

pullback along $\alpha|_{p \times \mathcal{M}}$, where $\alpha : \Sigma \times \mathcal{M} \rightarrow X$.

Cohomology on $X \mapsto$ cohomology on $\mathcal{M}$

(0,2) A model:
Each element of $H^q(X, \Lambda^p \mathcal{E}^\vee)$ plus point p on worldsheet Σ define an element of $H^q(\mathcal{M}, \Lambda^p \mathcal{F}^\vee)$:

1. first pullback along $\alpha|_{p \times \mathcal{M}}$ to get an element of $H^q(\mathcal{M}, \Lambda^p (\alpha^* \mathcal{E})^\vee |_{p \times \mathcal{M}})$
2. next use map

$$\mathcal{F} \; (\equiv \pi_* \alpha^* \mathcal{E}) \; \longrightarrow \; \alpha^* \mathcal{E}|_{p \times \mathcal{M}}$$

to define map

$$\Lambda^p \left(\alpha^* \mathcal{E} \right)^\vee |_{p \times \mathcal{M}} \; \longrightarrow \; \Lambda^p \mathcal{F}^\vee$$

When $\mathcal{E} = TX$, this reduces to the A model map.

Excess zero modes

A model:
Use 4-fermi term $\int_\Sigma R_{i\bar{\jmath}k\bar{l}} \chi^i \chi^{\bar{\jmath}} \psi^k \psi^{\bar{l}}$.
For each cpx pair of ψ zero modes, bring down one copy of 4-fermi term above.
Result:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_{\mathcal{M}} H^{\sum p_i, \sum q_i}(\mathcal{M}) \wedge c_{top}(\text{Obs})$$

where

$$\begin{aligned} \text{Obs} &= \text{bundle over } \mathcal{M} \text{ defined by } \psi \text{ zero modes} \\ &= R^1 \pi_* \alpha^* TX \\ &= \text{“obstruction bundle”} \end{aligned}$$

Excess zero modes

A model, cont'd:

Selection rules: $\sum p_i = \sum q_i = \#\chi - \#\psi$ zero modes.

$$\begin{aligned} \#\psi \text{ zero modes} &= \text{rank Obs} \\ \#\chi \text{ zero modes} &= \dim \mathcal{M} \end{aligned}$$

$$\sum p_i \; + \; (\text{rank Obs}) \; = \; \sum q_i \; + \; (\text{rank Obs}) \; = \; \dim \mathcal{M}$$

$\Longrightarrow$ integrand is a top form

Excess zero modes

(0,2) A model:

Assume rk $R^1\pi_*\alpha^*\mathcal{E} =$ rk $R^1\pi_*\alpha^*TX = n$.

Use 4-fermi term $\int_\Sigma F_{i\bar{\jmath}a\bar{b}}\psi_+^i\psi_+^{\bar{\jmath}}\lambda_-^a\lambda_-^{\bar{b}}$.

$$\psi_+^{\bar{\jmath}} \sim T\mathcal{M} = R^0\pi_*\alpha^*TX \quad \lambda_-^a \sim \mathcal{F} = R^0\pi_*\alpha^*\mathcal{E}$$
$$\psi_+^i \sim \text{Obs} = R^1\pi_*\alpha^*TX \quad \lambda_-^{\bar{b}} \sim \mathcal{F}_1 \equiv R^1\pi_*\alpha^*\mathcal{E}$$

Each 4-fermi $\sim H^1(\mathcal{M}, \mathcal{F}^\vee \otimes \mathcal{F}_1 \otimes (\text{Obs})^\vee)$.

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_{\mathcal{M}} H^{\sum q_i}\left(\mathcal{M}, \Lambda^{\sum p_i}\mathcal{F}^\vee\right) \wedge$$
$$H^n\left(\mathcal{M}, \Lambda^n\mathcal{F}^\vee \otimes \Lambda^n\mathcal{F}_1 \otimes \Lambda^n(\text{Obs})^\vee\right)$$

Excess zero modes

(0,2) A model, cont'd:
Selection rules:

$$\begin{aligned} \sum q_i + n &= \text{dim } \mathcal{M} \\ \sum p_i + n &= \text{rank } \mathcal{F} \end{aligned}$$

and by assumption, rk $\mathcal{F}_1 =$ rk Obs $= n$.

$$\left. \begin{array}{c} \Lambda^{\text{top}} \mathcal{E}^\vee \cong K_X \\ \text{ch}_2(\mathcal{E}) = \text{ch}_2(TX) \end{array} \right\}$$

$$\stackrel{GRR}{\Longrightarrow} \Lambda^{\text{top}} \mathcal{F}^\vee \otimes \Lambda^{\text{top}} \mathcal{F}_1 \otimes \Lambda^{\text{top}} (\text{Obs})^\vee \cong K_{\mathcal{M}}$$

Once again, integrand is a top-form.

We just presented an ansatz for interpreting 4-fermi terms in (0,2) models, and observed that GRR $\Rightarrow$ integrand a top-form, as needed.

But why does it reduce to (2,2) case when $\mathcal{E} = TX$?

Answer: Atiyah classes

Atiyah classes

Consider the curvature of a connection on a hol' bundle $\mathcal{E}$ on X:

$$F_{i\overline{\jmath}a\overline{b}}$$

Bianchi: $\overline{\partial} F = 0$, so $[F] \in H^1\left(X, \Omega^1_X \otimes \mathcal{E}^\vee \otimes \mathcal{E}\right)$.

Since ch$_r(\mathcal{E}) \propto$ tr $F \wedge \cdots \wedge F$ (r times), the Chern classes of $\mathcal{E}$ are encoded in

$$\begin{aligned} H^1\left(X, \Omega^1_X \otimes \mathcal{E}^\vee \otimes \mathcal{E}\right) \wedge \cdots \wedge H^1\left(X, \Omega^1_X \otimes \mathcal{E}^\vee \otimes \mathcal{E}\right) \\ = H^r\left(X, \Omega^r_X \otimes \mathcal{E}^\vee \otimes \mathcal{E}\right) \end{aligned}$$

Let's specialize for a moment to $\mathcal{E} = TX$, so $\mathcal{F} = T\mathcal{M}$.
Each (0,2) 4-fermi term generates a factor of

$$H^1\left(\mathcal{M}, \mathcal{F}^\vee \otimes \mathcal{F}_1 \otimes (\text{Obs})^\vee\right) \stackrel{\mathcal{E}=TX}{=} H^1\left(\mathcal{M}, \Omega^1_{\mathcal{M}} \otimes (\text{Obs})^\vee \otimes \text{Obs}\right)$$

$\rightarrow$ same gp that contains the Atiyah class of Obs bundle

Bringing down ($n =$ rk Obs) factors generates

$$H^n\left(\mathcal{M}, \Omega^n_{\mathcal{M}} \otimes \Lambda^{\text{top}}(\text{Obs})^\vee \otimes \Lambda^{\text{top}}\text{Obs}\right)$$

which contains $c_{\text{top}}(\text{Obs})$.

Thus, our (0,2) ansatz generalizes (2,2) obstruction bdles

Compactifications of moduli spaces

In order to make sense of expressions such as

$$\int_{\mathcal{M}} (\text{top form})$$

we need $\mathcal{M}$ to be compact.

Problem: spaces of honest holomorphic maps *not* compact

Ex: Degree 1 maps $\mathbf{P}^1 \rightarrow \mathbf{P}^1$ = group manifold of $SL(2, \mathbf{C})$

How to solve? Regularize the 2d QFT: compactify $\mathcal{M}$.

Compactifications of moduli spaces

We just argued that to make sense of formal calculations, must compactify $\mathcal{M}$, *i.e.* add some measure-zero pieces that make $\mathcal{M}$ compact.

Furthermore, in the (0,2) case, need to extend $\mathcal{F}$, $\mathcal{F}_1$ over the compactification, in a way consistent with symmetries.

How to compactify? One way (Morrison-Plesser; Givental) uses gauged linear sigma models. We'll follow their lead.

Gauged linear sigma models

(2,2) case:
A chiral superfield in 2d contains

ϕ	cpx boson
ψ_+, ψ_-	cpx fermions
F	auxiliary field

Ex: A GLSM describes $\mathbf{P}^{N-1}$ as, N chiral superfields each of charge 1 w.r.t. gauged $U(1)$.

D-terms: $\sum |\phi_i|^2 = r \Longrightarrow \phi$'s span S^{2N-1}
Gauge-invariants: $S^{2N-1}/U(1) = \mathbf{P}^{N-1}$

Gauged linear sigma models

Can use GLSM's to describe more general toric varieties; look like, some chiral superfields + gauged $U(1)$'s
Can describe CY's by adding superpotential; zero locus of bosonic potential = CY

1. massive 2D QFT's, not CFT's
2. linear kinetic terms make analysis of some aspects of QFT easier than in a NLσM

Today I'll only consider (mostly massive) theories w/ toric targets.

(0,2) GLSM's

(0,2) chiral superfield Φ	(0,2) fermi superfield Λ
ϕ (cpx boson)	ψ_- (cpx fermion)
ψ_+ (cpx fermion)	F (aux field)

Together, form (2,2) chiral multiplet.

The fermi superfields have an important quirk: Although $\overline{D}_+ \Phi = 0$ for Φ chiral, can permit $\overline{D}_+ \Lambda = E$ for nonzero E obeying $\overline{D}_+ E = 0$. This constrains the superpotential; details soon....

Can describe a toric variety target as a collection of (0,2) chiral superfields with some gauged $U(1)$'s.
The (left-moving) fermi multiplets define bundles.

Bundles on toric varieties

Ex: Reducible case, $\mathcal{E} = \oplus_a \mathcal{O}(\vec{n}_a)$.

In GLSM have fermi superfields Λ^a w/ charges $\vec{n}_a$ under some $U(1)$'s

Ex: Kernel,

$$0 \longrightarrow \mathcal{E} \longrightarrow \oplus_a \mathcal{O}(\vec{n}_a) \xrightarrow{F_a^i} \oplus_i \mathcal{O}(\vec{m}_i) \longrightarrow 0$$

Have fermi superfields Λ^a as above, plus chiral superfields p_i of charges $\vec{m}_i$, plus superpotential term $p_i F_a^i(\phi)$.

Resulting Yukawa couplings $\psi_+ i F_a^i \lambda^a$ give mass to any λ not in ker F, hence, $\mathcal{E} =$ ker F.

Bundles on toric varieties

Ex: Cokernel,

$$0 \longrightarrow \mathcal{O}^{\oplus k} \stackrel{E^i_a}{\longrightarrow} \oplus_a \mathcal{O}(\vec{n}_a) \longrightarrow \mathcal{E} \longrightarrow 0$$

Have fermi superfields Λ^a w/ charges $\vec{n}_a$ as above, plus k neutral chiral superfields Σ_i, where $\overline{D}_+ \Lambda^a = \Sigma_i E^i_a$.

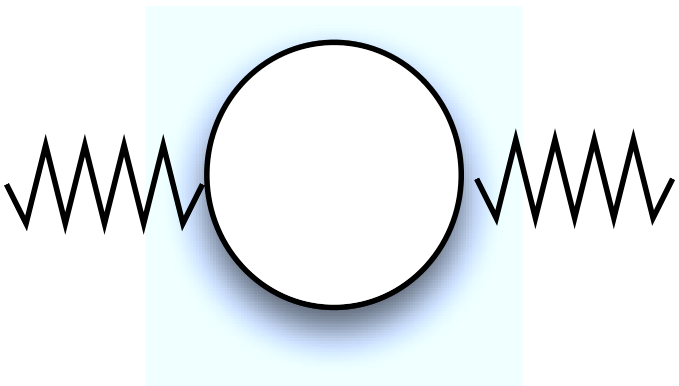
Ex: Monad,

$$0 \longrightarrow \mathcal{O}^{\oplus k} \stackrel{E^{i'}_a}{\longrightarrow} \oplus_a \mathcal{O}(\vec{n}_a) \stackrel{F^i_a}{\longrightarrow} \oplus_i \mathcal{O}(\vec{m}_i) \longrightarrow 0$$

Have $\Sigma_{i'}$, Λ^a, p_i as above, w/ superpotential and susy transformation.

Anomaly cancellation

Let $\vec{n}_a$ denote charges of left-moving fermions, $\vec{q}_i$ denote charges of right-moving fermions.



Anom' cancellation implies

$$\sum_a n_a^t n_a^s = \sum_i q_i^t q_i^s$$

for each s, t

This implies, but is slightly stronger than, $\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX)$.

We'll also assume $\sum_a n_a^t = \sum_i q_i^t$ for each t, which implies, but is slightly stronger than, $\Lambda^{\mathrm{top}} \mathcal{E}^\vee \cong K_X$.

Linear sigma model moduli spaces

For target space $V//G$, V a vector space, $G = GL(k)$, the LSM moduli space $\mathcal{M}$ is a space of pairs

$$(\text{p-pal } G-\text{bundle } E \text{ on } \mathbf{P}^1, G-\text{(sheaf) map } E \rightarrow \mathcal{O} \otimes_{\mathbf{C}} V)$$

minus maps that send all of E to the exceptional subset of V defined by the original GIT quotient $V//G$.

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Resulting $\mathcal{M}$ is a fine moduli space of sheaf quotients (though not a fine moduli space of curves).
Get Maps if restrict to bundle maps.

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Get Maps if restrict to bundle maps.

Example: Grassmannian of k-planes in $\mathbf{C}^n$,
$G(k,n) = \mathbf{C}^{kn}//GL(k)$,
$\mathcal{M}$ is a Quot scheme of rank-k bundles on $\mathbf{P}^1$
with a sheaf embedding into $\mathcal{O}^n$.

Linear sigma model moduli spaces

For target toric varieties:

1. expand fields in a basis of zero modes; if x_i has charges $\vec{q}_i$, then zero modes are $x_i \in \Gamma(\mathbf{P}^1, \mathcal{O}(\vec{q}_i \cdot \vec{d}))$
2. coefficients are homogeneous coordinates on $\mathcal{M}$
3. build $\mathcal{M}$ like a Higgs moduli space (symplectic quotient)
   (a) exclude those zero modes that force the x_i to lie in excluded set for all points on worldsheet
   (b) the zero modes of x_i have same $U(1)$ charges as the original x_i

Linear sigma model moduli spaces

Ex: $\mathbf{P}^{N-1}$

Has N chiral superfields $x_1, \cdots, x_N$, one gauged $U(1)$, each x_i has charge 1.

The *gauge* instantons of the GLSM become the *worldsheet* instantons of the NLσM.

Moduli space of degree d maps here:

$$\begin{aligned} x_i &\in \Gamma(\mathcal{O}(1 \cdot d)) \\ &= x_{i0} u^d + x_{i1} u^{d-1} v + \cdots + x_{id} v^d \end{aligned}$$

where u, v are homogeneous coordinates on worldsheet ($\mathbf{P}^1$).

Linear sigma model moduli spaces

Ex, cont'd

The (x_{ij}) are homogeneous coord's on $\mathcal{M}$. Omit point where all $x_i \equiv 0$. The (x_{ij}) have same $U(1)$ charges as x_i for each x_i, thus

$$\mathcal{M} \; = \; \mathbf{P}^{N(d+1)-1}$$

Similarly for other toric varieties.

Induced bundles

The same ideas allow us to induce bundles on LSM moduli spaces.

Just as worldsheet fields define line bundles on target, expand in zero modes, and coefficients define line bundles on $\mathcal{M}$.

Next: examples....

Induced bundles

Ex: completely reducible bundles, $\mathcal{E} = \oplus_a \mathcal{O}(\vec{n}_a)$

The left-moving fermions are completely free (mod action of the gauge group).

Expand each fermion in zero modes, take coeff's to define line bundles on $\mathcal{M}$.
Here, λ_-^a has charges $\vec{n}_a$. Expand

$$\lambda_-^a \; = \; \lambda_-^{a0} u^{\vec{n}_a \cdot \vec{d} + 1} \; + \; \lambda_-^{a1} u^{\vec{n}_a \cdot \vec{d}} v \; + \; \cdots$$

Each $\lambda_-^{ai} \sim \mathcal{O}(\vec{n}_a)$ on $\mathcal{M}$. Thus,

$$\mathcal{F} \; = \; \oplus_a H^0 \left(\mathbf{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d}) \right) \otimes_{\mathbf{C}} \mathcal{O}(\vec{n}_a)$$

Induced bundles

Ex (completely reducible bundles), cont'd

Similarly,

$$\mathcal{F}_1 \;=\; \oplus_a H^1\left(\mathbf{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d})\right) \otimes_{\mathbf{C}} \mathcal{O}(\vec{n}_a)$$

Induced bundles

Ex: Cokernel

$$0 \longrightarrow \mathcal{O}^{\oplus m} \longrightarrow \oplus_a \mathcal{O}(\vec{n}_a) \longrightarrow \mathcal{E} \longrightarrow 0$$

In add'n to fermi superfields Λ^a for the $\mathcal{O}(\vec{n}_a)$, recall have chiral superfields Σ_j for the $\mathcal{O}$'s. As before, expand fields in basis of zero modes and interpret coefficients as line bundles on $\mathcal{M}$.

$$\begin{array}{ccl} 0 & \rightarrow & \oplus_1^m H^0\left(\mathcal{O}(0 \cdot \vec{d})\right) \otimes \mathcal{O} \rightarrow \oplus_a H^0\left(\mathcal{O}(\vec{n}_a \cdot \vec{d})\right) \otimes \mathcal{O}(\vec{n}_a) \\ & & \rightarrow \mathcal{F} \\ & \rightarrow & \oplus_1^m H^1\left(\mathcal{O}(0 \cdot \vec{d})\right) \otimes \mathcal{O} \rightarrow \oplus_a H^1\left(\mathcal{O}(\vec{n}_a \cdot \vec{d})\right) \otimes \mathcal{O}(\vec{n}_a) \\ & & \rightarrow \mathcal{F}_1 \rightarrow 0 \end{array}$$

Induced bundles

Ex: Cokernel, cont'd

Since $H^1(\mathbf{P}^1, \mathcal{O}) = 0$, this simplifies to

$$0 \longrightarrow \mathcal{O}^{\oplus m} \longrightarrow \oplus_a H^0\left(\mathcal{O}(\vec{n}_a \cdot \vec{d}) \right) \otimes \mathcal{O}(\vec{n}_a) \longrightarrow \mathcal{F} \longrightarrow 0$$

$$\mathcal{F}_1 \cong \oplus_a H^1\left(\mathbf{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d}) \right) \otimes_{\mathbf{C}} \mathcal{O}(\vec{n}_a)$$

Check (2,2) locus

The tangent bundle of a (cpt, smooth) toric variety X can be expressed in the form

$$0 \longrightarrow \mathcal{O}^{\oplus k} \longrightarrow \oplus_i \mathcal{O}(\vec{q}_i) \longrightarrow TX \longrightarrow 0$$

where the $\vec{q}_i$ are the charges of the chiral superfields.

Applying previous ansatz,

$$0 \longrightarrow \mathcal{O}^{\oplus k} \longrightarrow \oplus_i H^0\left(\mathbf{P}^1, \mathcal{O}(\vec{q}_i \cdot \vec{d})\right) \otimes_{\mathbf{C}} \mathcal{O}(\vec{q}_i) \longrightarrow \mathcal{F} \longrightarrow 0$$

$$\mathcal{F}_1 \cong \oplus_i H^1\left(\mathbf{P}^1, \mathcal{O}(\vec{q}_i \cdot \vec{d})\right) \otimes_{\mathbf{C}} \mathcal{O}(\vec{q}_i)$$

but this $\mathcal{F}$ is automatically $T\mathcal{M}$ for $\mathcal{M}$ a LSM moduli space, exactly as desired.

Check (2,2) locus

Also, $\mathcal{F}_1$ =obstruction bundle.

Check:

$$c_{\rm top}(\mathcal{F}_1) \;=\; \prod_{\vec{n}_a \cdot \vec{d} < 0} c_1(\mathcal{O}(\vec{n}_a))^{-\vec{n}_a \cdot d - 1}$$

Induced bundles

Universal form – conjecture:

Target space $V//GL(k)$: (V a vector space)

Let $\mathcal{O}(\rho)$ be a vector bundle over $V//GL(k)$ defined by a representation ρ of $GL(k)$. Corresponding to $\mathcal{O}(\overline{\mathbf{k}})$ is a 'universal subbundle' $S \to \mathcal{O} \otimes_{\mathbf{C}} V$ over $\mathbf{P}^1 \times \mathcal{M}$.

(When restrict to $\mathbf{P}^1 \times \mathrm{Maps}$, S restricts to the pullback of $\mathcal{O}(\overline{\mathbf{k}})$ along the universal instanton.)

Then, build lift of $\mathcal{O}(\rho)$ from S's in the same way that ρ is built from $\overline{\mathbf{k}}$.

Examples....

Induced bundles

Examples:

$$\begin{array}{rcl} 1 & \mapsto & \mathcal{O} \\ \mathbf{k} \otimes \overline{\mathbf{k}} & \mapsto & S^* \otimes S \\ \mathrm{Alt}^2 \mathbf{k} & \mapsto & \mathrm{Alt}^2 S^* \\ \overline{\mathbf{k}} \otimes \mathrm{Sym}^5 \mathbf{k} & \mapsto & S \otimes \mathrm{Sym}^5 S^* \end{array}$$

Re-examine cokernel.....

Induced bundles

Re-examine cokernel: Start with

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^n \longrightarrow T\mathbf{P}^{n-1} \longrightarrow 0$$

on $\mathbf{P}^{n-1}$.

This lifts to

$$0 \longrightarrow \mathcal{O} \longrightarrow (S^*)^n \longrightarrow \mathcal{F}_0 \longrightarrow 0$$

on $\mathbf{P}^1 \times \mathcal{M}$.

Pushforward to $\mathcal{M}$ to get long exact sequence discussed previously. Works for Grassmannians/Quot's.

Similar ideas hold for other bundles appearing in (0,2) GLSM's.

In all cases: so long as the original gauge bundle satisfied GLSM anomaly cancellation, the induced bundles $\mathcal{F}$, $\mathcal{F}_1$ have the desired symmetry properties.

Also, if a given bundle does not satisfy GLSM anomaly cancellation, then the induced bundles $\mathcal{F}$, $\mathcal{F}_1$ often won't have the desired symmetry properties.

Presentation-dependence

Here's an example of what can happen with GLSM anomaly cancellation.
Consider the tangent bundle T of $\mathbf{P}^1 \times \mathbf{P}^1$. This has (at least) 3 presentations:

$$0 \longrightarrow \mathcal{O}^2 \longrightarrow \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow T \longrightarrow 0$$
$$0 \longrightarrow \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,2) \longrightarrow T \longrightarrow 0$$
$$T \cong \mathcal{O}(2,0) \oplus \mathcal{O}(0,2)$$

The same bundle, but only the first presentation satisfies GLSM anomaly cancellation. Next: compute $\mathcal{F}$....

Presentation-dependence

LSM $\mathcal{M} = \mathbf{P}^{2d_1+1} \times \mathbf{P}^{2d_2+1}$
Induced bundles:

$$\mathcal{F} \cong T\mathbf{P}^{2d_1+1} \times \mathbf{P}^{2d_2+1}$$
$$\mathcal{F} \cong \pi_1^* T\mathbf{P}^{2d_1+1} \oplus \oplus_1^{2d_2+1} \mathcal{O}(0,2)$$
$$\mathcal{F} \cong \oplus_1^{2d_1+1} \mathcal{O}(2,0) \oplus \oplus_1^{2d_2+1} \mathcal{O}(0,2)$$

These are isomorphic on the interior of $\mathcal{M}$, on the honest maps, but differ over the compactification.

Only in the first case (which was the only one to satisfy GLSM anomaly cancellation) is $\Lambda^{\rm top} \mathcal{F}^\vee \cong K_{\mathcal{M}}$.

Adams-Basu-Sethi prediction

Adams-Basu-Sethi (2003) studied a massive 2d theory describing $\mathbf{P}^1 \times \mathbf{P}^1$ with a bundle given by a deformation of the tangent bundle.

From analysis of duality in the corresponding massive (0,2) gauged linear sigma model, they made some conjectures for correlation f'ns, which they expressed in terms of a "heterotic quantum cohomology ring,"

really, a **quantum sheaf cohomology ring.**

Chiral rings

The idea of a chiral ring should be familiar from 4d susy gauge theories, *e.g.* Cachazo-Douglas-Seiberg-Witten.

4d $N=1$ pure $SU(N)$ SYM	2d susy $\mathbf{CP}^{N-1}$ model
$S^N = \Lambda^{3N}$	$x^N = q$
$W = S\left(1 + \log(\Lambda^{3N}/S^N)\right)$	$W = \Sigma\left(1 + \log(\Lambda^N/\Sigma^N)\right)$
Konishi	Konishi
etc	*etc*

where for the $\mathbf{CP}^N$ model, the x is identified with a generator of $H^2(\mathbf{CP}^N, \mathbf{Z})$, and so the physical ring relation looks like a modification of the std cohomology ring $\mathbf{C}[x]/(x^N = 0)$, yielding "quantum cohomology" ring $\mathbf{C}[x]/(x^N = q)$.

Quantum cohomology

More concretely, the quantum cohomology ring of $\mathbf{CP}^{N-1}$ tells us that correlation functions are:

$$\langle x^k \rangle = \begin{cases} q^m & \text{if } k = mN + N - 1 \\ 0 & \text{else} \end{cases}$$

Ordinarily use (2,2) worldsheet susy to argue for existence of a quantum cohomology ring.

- Adams-Basu-Sethi ('03) conjectured exist for (0,2)
- ES-Katz ('04) checked correlation f'ns, found ring structure
- Adams-Distler-Ernebjerg ('05) found gen'l argument for (0,2) ring structure
- Guffin, Melnikov, McOrist, Sethi, *etc*

Adams-Basu-Sethi prediction

Adams-Basu-Sethi studied a massive 2d theory describing $\mathbf{P}^1 \times \mathbf{P}^1$ with a bundle given by a deformation of the tangent bundle, with a deformation specified by two parameters ϵ_1, ϵ_2.

From analysis of duality in the corresponding massive (0,2) gauged linear sigma model, they conjectured that the quantum sheaf cohomology ring should be a deformation of the usual ring:

$$\begin{aligned} \tilde{X}^2 &= \exp(it_2) \\ X^2 - (\epsilon_1 - \epsilon_2) X \tilde{X} &= \exp(it_1) \end{aligned}$$

where the t_i are Kähler parameters describing the sizes of the $\mathbf{P}^1$'s, and X, $\tilde{X}$ are the two generators.

Adams-Basu-Sethi prediction

Conjectured relations:

$$\begin{aligned} \tilde{X}^2 &= \exp(it_2) \\ X^2 - (\epsilon_1 - \epsilon_2) X \tilde{X} &= \exp(it_1) \end{aligned}$$

What do those ring relations really mean? For ex:

$$\begin{aligned} \langle \tilde{X}^4 \rangle &= \langle 1 \rangle \exp(2it_2) \; = \; 0 \\ \langle X \tilde{X}^3 \rangle &= \langle (X\tilde{X}) \tilde{X}^2 \rangle \\ &= \langle X \tilde{X} \rangle \exp(it_2) \; = \; \exp(it_2) \\ \langle X^2 \tilde{X}^2 \rangle &= \langle X^2 \rangle \exp(it_2) \; = \; (\epsilon_1 - \epsilon_2) \exp(it_2) \\ \langle X^3 \tilde{X} \rangle &= \exp(it_1) \; + \; (\epsilon_1 - \epsilon_2)^2 \exp(it_2) \\ \langle X^4 \rangle &= 2(\epsilon_1 - \epsilon_2) \exp(it_1) \; + \; (\epsilon_1 - \epsilon_2)^3 \exp(it_2) \end{aligned}$$

Adams-Basu-Sethi prediction

To be brief, using exactly the methods described so far (compute $\mathcal{M}$, $\mathcal{F}$, compute induced sheaf cohomology on $\mathcal{M}$ in terms of Cech reps on toric cover, calculate $\wedge$'s on $\mathcal{M}$ & integrate), we precisely reproduced the results above.

(KS calculated 4-pt interactions, Josh Guffin later did more.)

(0,2) B model

So far I've outlined the (0,2) A model. What about the (0,2) B model?

Recall (2,2) A, B differ by the choice of left twist:

A model	B model
$\psi_-^i \in \Gamma((\phi^* T^{0,1} X)^\vee)$	$\psi_-^i \in \Gamma(K \otimes (\phi^* T^{0,1} X)^\vee)$
$\psi_-^{\overline{\imath}} \in \Gamma(K \otimes (\phi^* T^{0,1} X)^\vee)$	$\psi_-^{\overline{\imath}} \in \Gamma((\phi^* T^{0,1} X)^\vee)$

One can define analogous (0,2) versions:

(0,2) A	(0,2) B
$\lambda_-^a \in \Gamma((\phi^* \overline{\mathcal{E}})^\vee)$	$\lambda_-^a \in \Gamma(K \otimes (\phi^* \overline{\mathcal{E}})^\vee)$
$\lambda_-^{\overline{a}} \in \Gamma(K \otimes \phi^* \overline{\mathcal{E}})$	$\lambda_-^{\overline{a}} \in \Gamma(\phi^* \overline{\mathcal{E}})$

(0,2) B model

(0,2) A	(0,2) B
$\lambda_-^a \in \Gamma((\phi^* \overline{\mathcal{E}})^\vee)$	$\lambda_-^a \in \Gamma(K \otimes (\phi^* \overline{\mathcal{E}})^\vee)$
$\lambda_-^{\overline{a}} \in \Gamma(K \otimes \phi^* \overline{\mathcal{E}})$	$\lambda_-^{\overline{a}} \in \Gamma(\phi^* \overline{\mathcal{E}})$

Note that in the (0,2) version, we can go A $\leftrightarrow$ B by switching $\mathcal{E} \leftrightarrow \mathcal{E}^\vee$.

So, once you know the (0,2) A model, you also know the (0,2) B model – the same model generalizes both simultaneously.

Or, at least, that's the case classically....

(0,2) B model

Corner case: curves in which $\phi^* \mathcal{E} \cong \phi^* \mathcal{E}^\vee$

Better: curves for which $\phi^* TX \cong \phi^* T^* X$. For such curves, the last argument implies that A model contributions = B model contributions.

But there exist examples (eg, fiber of Hirzebruch surface divisor in CY) in which A model gets quantum corrections, but B model remains classical.

How to resolve puzzle?

(0,2) B model

Resolution of puzzle in corner cases:

For *honest* maps $\phi : \Sigma \rightarrow X$, if $\phi^* TX \cong \phi^* T^* X$, the contributions to A, B models are same:

$$\int_{\mathcal{M}} \alpha$$

But extensions over compactification divisor differ:

1. B model: α becomes exact form on $\mathcal{M}$
2. A model: α is closed but not exact

Two different regularizations of the same theory.

Consistency conditions on (2,2) locus

Recall our (0,2) A model had the following constraint on the gauge bundle $\mathcal{E}$:

$$\Lambda^{\rm top} \mathcal{E}^{\vee} \cong K_X$$

To get the B model analogue, we replace $\mathcal{E}$ with $\mathcal{E}^{\vee}$, and so have another constraint:

$$\Lambda^{\rm top} \mathcal{E} \cong K_X$$

Put together, these constraints imply

$$K_X^{\vee} \cong K_X \Longrightarrow K_X^2 \cong \mathcal{O}$$

Consistency conditions on (2,2) locus

We're used to saying the closed string B model is well-defined only on Calabi-Yau's ($K_X \cong \mathcal{O}$), but we just derived instead the condition that $K_X^2 \cong \mathcal{O}$.

And, in fact, it's an obscure fact that consistency of the closed string B model merely requires $K_X^2 \cong \mathcal{O}$, not actually $K_X \cong \mathcal{O}$.

Consistency conditions on (2,2) locus

- Implicit in old expressions for B model correlation f'ns, Kodaira-Spencer.
- Loop calculation yields ambiguous result, as it cannot sense torsion, and $K_X^2 \cong \mathcal{O} \Rightarrow c_1$ torsion
- Careful analysis of anomaly cancellation yields the desired result
- Check: Serre duality correctly maps massless spectrum into itself
- Exs: Enriques surfaces, hyperelliptic surfaces
- Open string B model requires stronger condition: $K \cong \mathcal{O}$

Summary

- review A model TFT, (0,2) A model
- review correlation f'n computations in A model, describe analogue for (0,2) models
  - formal structure similar; (0,2) generalizes A model
  - compactification issues; not only $\mathcal{M}$, but bundles on $\mathcal{M}$
- apply GLSM's; not only naturally compactify $\mathcal{M}$, but also naturally extend the bundles
- Adams-Basu-Sethi prediction
- Analogue for B model
- Consistency conditions in closed string B model

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