GLSM's, gerbes, and Kuznetsov's homological projective duality

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T Pantev, ES, hepth/0502027, 0502044, 0502053

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R Donagi, ES, arxiv: 0704.1761

A Caldararu, J Distler, S Hellerman, T Pantev, ES, arXiv: 0709.3855

Outline

- Basics of GLSM's, & an interesting example
- Cluster decomposition conjecture for strings on gerbes:
  CFT(gerbe) = CFT(disjoint union of spaces)
- Application to GLSM's; realization of Kuznetsov's homological projective duality

GLSM's

Today: gauged linear sigma models (GLSM's).

These are two-dimensional gauge theories, generalizing the susy $\mathbf{CP}^N$ model.

We add a superpotential to the $\mathbf{CP}^N$ model, and the resulting theory flows in the IR to e.g. a nonlinear sigma model on a hypersurface in $\mathbf{CP}^N$.

Standard lore about GLSM's:

* Only (complete intersections of) hypersurfaces in $\mathbf{CP}^N$ and other toric varieties can be described with 2d abelian gauge theory.

* Geometries arising in different limits of Kahler moduli space are `birational' to one another.

Today we'll learn that's all wrong.

Let's first briefly review a simple example of a GLSM and its interpretation.

Build an abelian gauge theory that flows in the IR to a NLSM on a quintic hypersurface in $\mathbf{CP}^4$.

Start with susy $\mathbf{CP}^4$ model:

5 chiral superfields $\Phi = (\phi, \psi, F)$

(one for each homogeneous coordinate on $\mathbf{CP}^4$)

Each has charge 1 under a gauged U(1)

D-terms:

Have bosonic potential $|D|^2$

where $D = \sum_i |\phi_i|^2 - r$

In the IR, susy vacua satisfy D=0

Classical moduli space =

$$S^5/U(1) = \mathbf{CP}^4$$

How to describe the hypersurface $\{G = 0\} \subset \mathbf{CP}^4$
where G is a degree 5 homogeneous poly?

First guess: Add a superpotential W = G.

This fails:

* superpotential must be gauge-invariant.

* Wrong F terms: get bosonic potential

$$\sum_i |\partial_i G|^2$$

which wants to flow to dG = 0 locus, not G=0 locus.

Correct method:

First, add a new chiral superfield $P = (p, \psi_p, F_p)$

of charge -5.

Then, W = pG

* W is gauge-invariant.

* F-terms are

$$|G|^2 + |p|^2 \sum_i |\partial_i G|^2$$

and so have susy vacua at G=0=p, exactly the desired quintic!

A little more carefully:

D-terms: $$\left|\sum_i |\phi_i|^2 \;-\; 5|p|^2 \;-\; r\right|^2$$

F-terms: $$|G|^2 \;+\; |p|^2 \sum_i |\partial_i G|^2$$

$r \gg 0:$

ϕ_i not all zero

$G = p = 0$

NLSM on quintic

$r \ll 0:$

$p \neq 0$

$< \phi_i >$ all vanish

$\mathbf{Z}_5$ orb' of LG model

A more interesting example:

Describe complete intersection of 2 deg 2 hypersurfaces in **CP**^{3}. (=T^2)

Have 4 chiral superfields $\Phi_i = (\phi_i, \psi_i, F_i)$
(one for each homog' coord' on **CP**^{3})
each of charge 1

Add 2 chiral superfields $P_a = (p_a, \psi_{pa}, F_{pa})$
(one for each of the $\{G_a = 0\}$)

D-terms: $$\left|\sum_i |\phi_i|^2 - 2\sum_a |p_a|^2 - r\right|^2$$

F-terms: $$\sum_i \sum_a |G_a|^2 + \sum_i \sum_a |p_a|^2 \, |\partial_i G_a|^2$$

$r \gg 0:$

ϕ_i not all zero

$p_a = G_a = 0$

NLSM on CY CI

The other limit is more interesting....

D-terms: $$\left| \sum_i |\phi_i|^2 \, - \, 2 \sum_a |p_a|^2 \, - \, r \right|^2$$

$$W \, = \, \sum_a p_a G_a(\phi) \, = \, \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

$$r \ll 0 :$$

p_a not all zero

ϕ_i massive (since deg 2)

NLSM on $\mathbf{P}^1$????

The correct analysis of the $r \ll 0$ limit is more subtle.

One subtlety is that the ϕ_i are not massive everywhere.

Write $$W = \sum_a p_a G_a(\phi) = \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

then they are only massive away from the locus

$$\{\det A = 0\} \subset \mathbf{P}^1$$

But that just makes things more confusing....

A more important subtlety is the fact that the p's have nonminimal charge,

so over most of the $\mathbf{P}^1$ of p vevs,

we have a nonminimally-charged abelian gauge theory,

meaning massless fields have charge -2,

instead of 1 or -1.

Why should this matter?

Nonperturbative effects

General argument:

Compact worldsheet:

To specify Higgs fields completely, need to specify what bundle they couple to.

If the gauge field $\sim L$
then Φ charge Q implies

$$\Phi \in \Gamma(L^{\otimes Q})$$

Different bundles => different zero modes
=> different anomalies => different physics

For noncpt worldsheets, analogous argument exists.

(Distler, Plesser)

To illustrate, imagine an analogue of the $\mathbf{CP}^{N-1}$ model but in which all chiral superfields have charge k instead of charge 1.

Example: Anomalous global U(1)'s

$$\mathbf{P}^{N-1}: \quad U(1)_A \mapsto \mathbf{Z}_{2N}$$
$$\text{Here}: \quad U(1)_A \mapsto \mathbf{Z}_{2kN}$$

Example: A model correlation functions

$$\mathbf{P}^{N-1}: \quad < X^{N(d+1)-1} > = q^d$$
$$\text{Here}: \quad < X^{N(kd+1)-1} > = q^d$$

Example: quantum cohomology

$$\mathbf{P}^{N-1}: \quad \mathbf{C}[x]/(x^N - q)$$
$$\text{Here}: \quad \mathbf{C}[x]/(x^{kN} - q)$$

Different physics

This variation of the **CP**^{N} model is how we describe strings propagating on certain **Z**$_k$ gerbes over **CP**^{N}.

More generally, we make sense of strings propagating on stacks as follows:

Every* (smooth, Deligne-Mumford) stack can be presented as a global quotient

$$[X/G]$$

for X a space and G a group.

To such a presentation, associate a G-gauged sigma model on X.

(* with minor caveats)

When some subgroup of G acts trivially,
the result is mathematically a gerbe.

Physically, we see that strings on gerbes
are different from
strings on spaces.

The difference is nonperturbative effects
-- a sigma model on a gerbe
looks like a sigma model on a space,
but with fewer nonperturbative sectors.

There is a cluster decomposition issue, solved as follows....

General decomposition conjecture

Consider $[X/H]$ where

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1$$

and G acts trivially.

We now believe, for (2,2) CFT's,

$$\mathrm{CFT}([X/H]) \, = \, \mathrm{CFT}\left(\left[(X \times \hat{G})/K \right] \right)$$

(together with some B field), where $\hat{G}$ is the set of irreps of G

Decomposition conjecture

For banded gerbes, K acts trivially upon $\hat{G}$ so the decomposition conjecture reduces to

$$\mathrm{CFT}(G - \mathrm{gerbe\ on\ } Y) \;=\; \mathrm{CFT}\left(\coprod_{\hat{G}} (Y, B) \right)$$

$$(Y \;=\; [X/K])$$

where the B field is determined by the image of

$$H^2(Y, Z(G)) \stackrel{Z(G) \rightarrow U(1)}{\longrightarrow} H^2(Y, U(1))$$

Basic point:

Maps into $\mathbf{Z}_k$ gerbe over X
= maps into X of degree divisible by k

Path integral into disjoint union of k copies of X,
with variable B fields:

* if degree not divisible by k,
then proportional to sum over kth roots of unity
= 0 -- cancel out

* if degree is divisible by k,
then add instead of cancelling out

Result is same as path integral on gerbe.

Banded Example:

Consider $[X/D_4]$ where the center acts trivially.

$$1 \longrightarrow \mathbf{Z}_2 \longrightarrow D_4 \longrightarrow \mathbf{Z}_2 \times \mathbf{Z}_2 \longrightarrow 1$$

The decomposition conjecture predicts

$$\mathrm{CFT}\left([X/D_4]\right) = \mathrm{CFT}\left([X/\mathbf{Z}_2 \times \mathbf{Z}_2] \coprod [X/\mathbf{Z}_2 \times \mathbf{Z}_2] \right)$$

One of the effective orbifolds has vanishing discrete torsion, the other has nonvanishing discrete torsion.

(Using the relationship between discrete torsion and B fields first worked out by ES, c. 2000.)

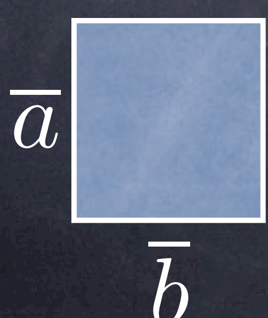
Check genus one partition functions:

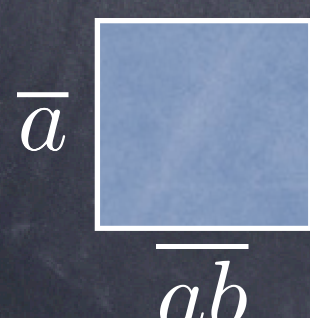
$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

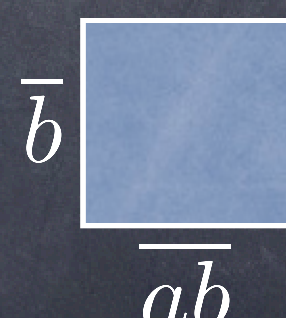
$$\mathbf{Z}_2 \times \mathbf{Z}_2 = \{1, \overline{a}, \overline{b}, \overline{ab}\}$$

$$Z(D_4) = \frac{1}{|D_4|} \sum_{g,h \in D_4, gh=hg} Z_{g,h}$$

Each of the $Z_{g,h}$ twisted sectors that appears, is the same as a $\mathbf{Z}_2 \times \mathbf{Z}_2$ sector, appearing with multiplicity $|\mathbf{Z}_2|^2 = 4$ except for the







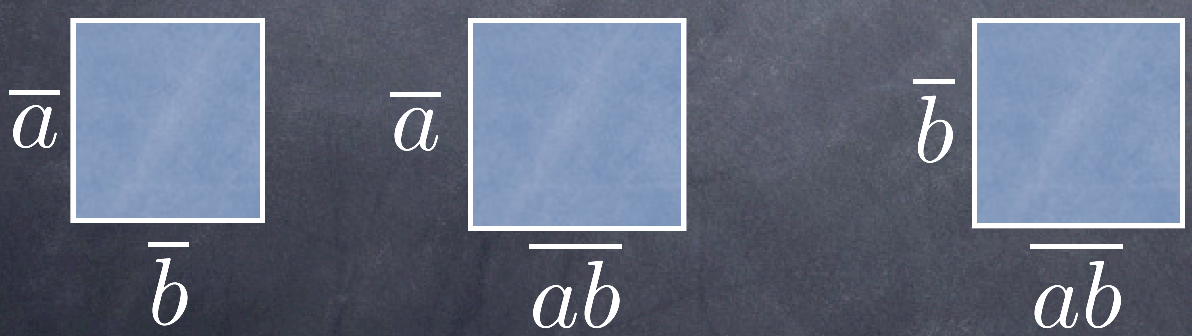
sectors.

Partition functions, cont'd

$$\begin{aligned} Z(D_4) &= \frac{|\mathbf{Z}_2 \times \mathbf{Z}_2|}{|D_4|} |\mathbf{Z}_2|^2 \left(Z(\mathbf{Z}_2 \times \mathbf{Z}_2) \ - \ (\text{some twisted sectors}) \right) \\ &= 2 \left(Z(\mathbf{Z}_2 \times \mathbf{Z}_2) \ - \ (\text{some twisted sectors}) \right) \end{aligned}$$

(In ordinary QFT, ignore multiplicative factors, but string theory is a 2d QFT coupled to gravity, and so numerical factors are important.)

Discrete torsion acts as a sign on the



twisted sectors

so we see that $Z([X/D_4]) = Z\left([X/\mathbf{Z}_2 \times \mathbf{Z}_2] \coprod [X/\mathbf{Z}_2 \times \mathbf{Z}_2] \right)$

with discrete torsion in one component.

A quick check of this example comes from comparing massless spectra:

Spectrum for $[T^6/D_4]$:

$$
\begin{array}{ccccccc}
 & & & 2 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 54 & & 0 & \\
2 & & 54 & & 54 & & 2 \\
 & 0 & & 54 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 2 & & &
\end{array}
$$

and for each $[T^6/\mathbf{Z}_2 \times \mathbf{Z}_2]$:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 3 & & 0 & \\
1 & & 51 & & 51 & & 1 \\
 & 0 & & 3 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 51 & & 0 & \\
1 & & 3 & & 3 & & 1 \\
 & 0 & & 51 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Sum matches. ✓

Nonbanded example:

Consider $[X/\mathbf{H}]$ where $\mathbf{H}$ is the eight-element group of quaternions, and a $\mathbf{Z}_4$ acts trivially.

$$1 \longrightarrow < i > (\cong \mathbf{Z}_4) \longrightarrow \mathbf{H} \longrightarrow \mathbf{Z}_2 \longrightarrow 1$$

The decomposition conjecture predicts

$$\mathrm{CFT}([X/\mathbf{H}]) = \mathrm{CFT}\left([X/\mathbf{Z}_2] \coprod [X/\mathbf{Z}_2] \coprod X \right)$$

Straightforward to show that this is true at the level of partition functions, as before.

K theory implications

This equivalence of CFT's implies a statement about K theory (thanks to D-branes).

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1$$

If G acts trivially on X

then the ordinary H-equivariant K theory of X

is the same as

twisted K-equivariant K theory of $X \times \hat{G}$

* Can be derived just within K theory
* Provides a check of the decomposition conjecture

D-branes and sheaves

D-branes in the topological B model can be described with sheaves and, more gen'ly, derived categories.

This also is consistent with the decomp' conjecture:

Math fact:

A sheaf on a banded G-gerbe

is the same thing as

a twisted sheaf on the underlying space,
twisted by image of an element of $H^2(X,Z(G))$

which is consistent with the way D-branes should behave according to the conjecture.

D-branes and sheaves

Similarly, massless states between D-branes should be counted by Ext groups between the corresponding sheaves.

Math fact:

Sheaves on a banded G-gerbe decompose according to irrep' of G,
and sheaves associated to distinct irreps have vanishing Ext groups between them.

Consistent w/ idea that sheaves associated to distinct reps should describe D-branes on different components of a disconnected space.

Gromov-Witten prediction

Notice that there is a prediction here for Gromov-Witten theory of gerbes:

GW of $[X/H]$

should match

GW of $\left[(X \times \hat{G})/K \right]$

Works in basic cases:
BG (T Graber), other exs (J Bryan)

Mirrors to stacks

There exist mirror constructions for any model realizable as a 2d abelian gauge theory.

For toric stacks (BCS '04), there is such a description.

Standard mirror constructions now produce character-valued fields, a new effect, which ties into the stacky fan description of (BCS '04).

(ES, T Pantev, '05)

Toda duals

Ex: The "Toda dual" of $\mathbf{CP}^N$ is described by the holomorphic function

$$W = \exp(-Y_1) + \cdots + \exp(-Y_N) + \exp(Y_1 + \cdots + Y_N)$$

The analogous duals to $\mathbf{Z}_k$ gerbes over $\mathbf{CP}^N$ are described by

$$W = \exp(-Y_1) + \cdots + \exp(-Y_N) + \Upsilon^n \exp(Y_1 + \cdots + Y_N)$$

where Υ is a character-valued field

(discrete Fourier transform of components in decomp' conjecture)

(ES, T Pantev, '05)

GLSM's

Let's now return to our analysis of GLSM's.

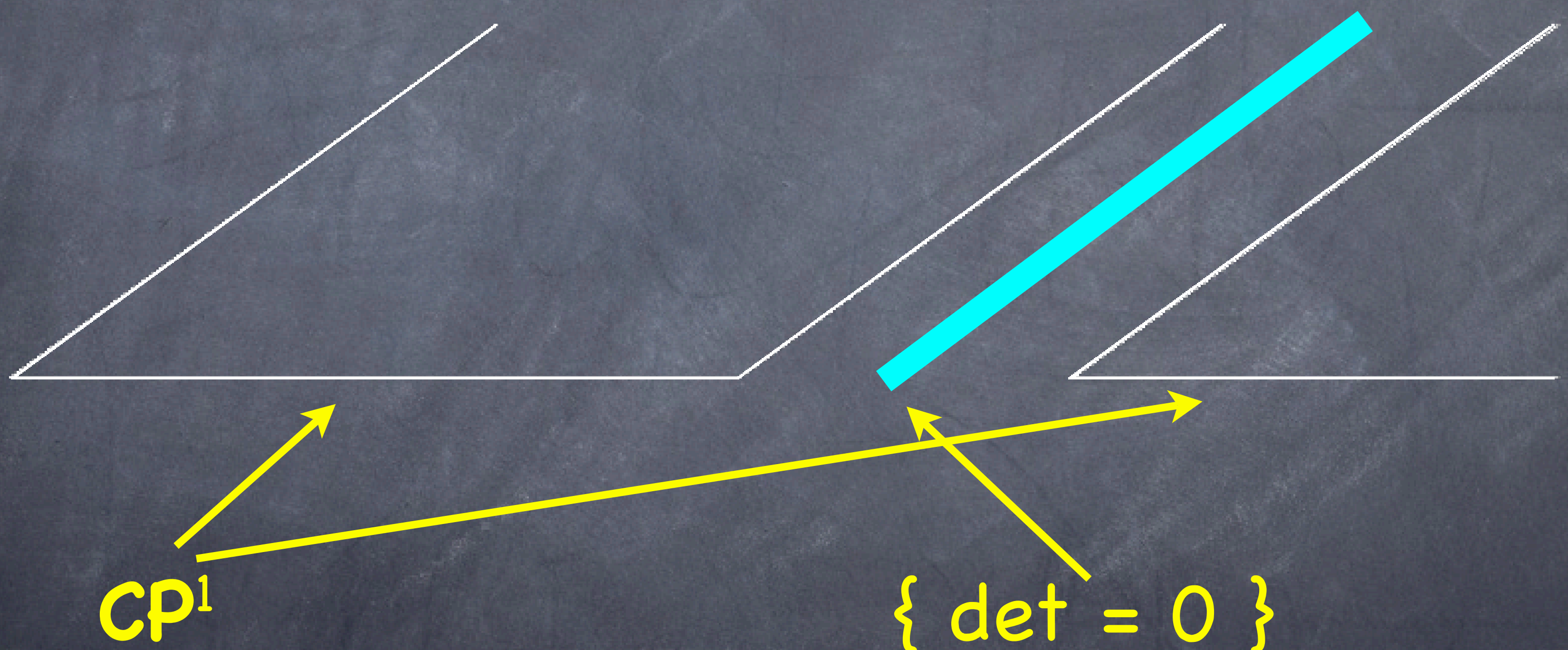
Example: **CP**^{3}[2,2]

Superpotential: $$\sum_a p_a G_a(\phi) = \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

$r \ll 0:$

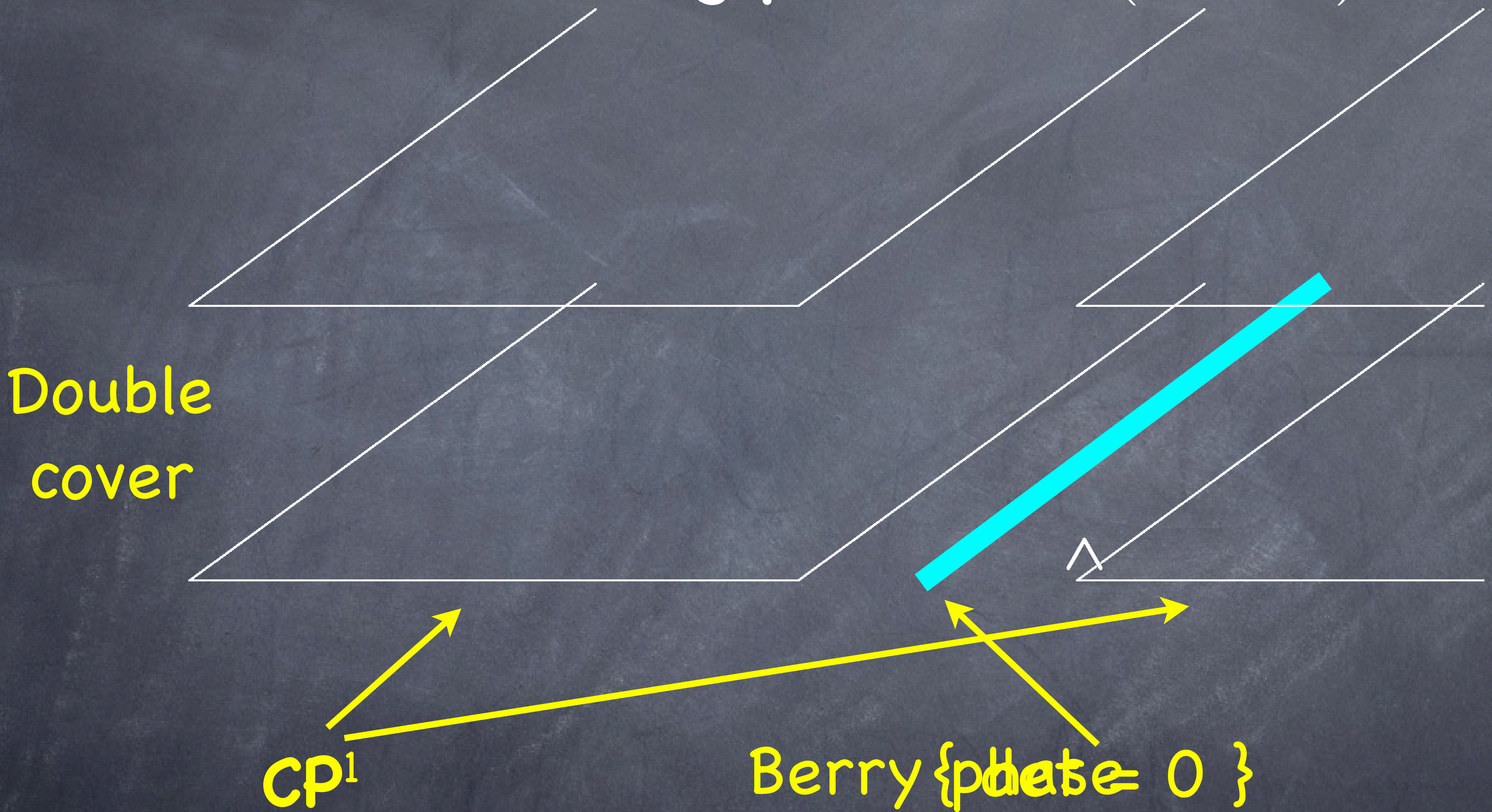
* mass terms for the ϕ_i, away from locus $\{\det A = 0\}$.
  * leaves just the p fields, of charge -2
  * $\mathbf{Z}_2$ gerbe, hence double cover

The Landau-Ginzburg point: $(r \ll 0)$



Because we have a $\mathbf{Z}_2$ gerbe over $\mathbf{CP}^1$....

The Landau-Ginzburg point: $(r \ll 0)$



Result: branched double cover of $\mathbf{CP}^1$

So far:

The GLSM realizes:

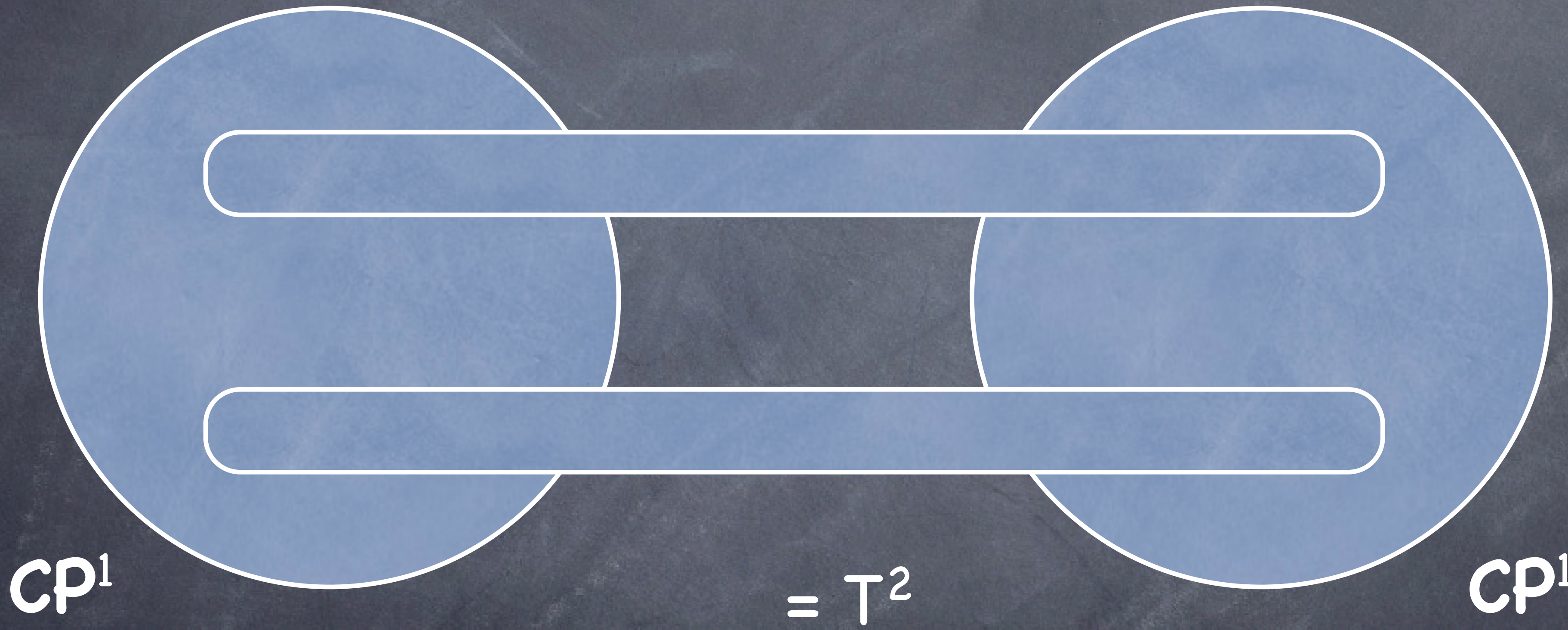
$\mathbf{CP}^3[2,2]$ $\xleftrightarrow{\text{Kahler}}$ branched double cover of $\mathbf{CP}^1$

where RHS realized at LG point via local $\mathbf{Z}_2$ gerbe structure + Berry phase.

(S. Hellerman, A. Henriques, T. Pantev, ES, M Ando, '06; R Donagi, ES, '07; A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E.S., '07)

* novel realization of geometry (as something other than CI)

Branched double cover of $\mathbf{CP}^1$ over deg 4 locus



So our GLSM for $\mathbf{CP}^3[2,2]$ relates

$T^2 \xleftrightarrow{\text{Kahler}} T^2$ (no surprise)

Next simplest example:

GLSM for $\mathbf{CP}^5[2,2,2]$ = K3

At LG point, have a branched double cover of $\mathbf{CP}^2$, branched over a degree 6 locus
--- another K3

$$\mathrm{K3} \xleftrightarrow{\text{Kahler}} \mathrm{K3}$$

(no surprise)

A more interesting example:

GLSM for $\mathbf{CP}^7[2,2,2,2]$ = CY 3-fold

At LG point,

get branched double cover of $\mathbf{CP}^3$,
branched over degree 8 locus.

-- another CY
(Clemens' octic double solid)

Here, **different** CY's;
so different, they're not even birational !

We'll see same pattern in more examples
-- complete intersections of quadrics are
related to branched double covers.

This particular example is more interesting,
but, let's pause a moment.

* novel realization of geometry
(as something other than CI)

* limits of Kahler moduli space not birational

Violates std lore on GLSMs.

If the limits aren't birational,
then how are they related?

They are related by Kuznetsov's
``homological projective duality"

More gen'ly, we conjecture that all Kahler phases of GLSM's are related by h.p.d.

First, let's return to the $\mathbf{CP}^7[2,2,2,2]$ example,
to uncover more details,
then we'll see more examples.

There's more going on in this particular example.

A puzzle:

* the branched double cover will be singular,
but the GLSM is smooth at those singularities.

Solution?....

Solution:

We believe the GLSM is actually describing a `noncommutative resolution' of the branched double cover worked out by Kuznetsov.

Kuznetsov has defined `homological projective duality' that relates $\mathbf{CP}^7[2,2,2,2]$ to the noncommutative resolution above.

Check that we are seeing K's noncomm' resolution:

K defines a `noncommutative space' via its sheaves -- so for example, a Landau-Ginzburg model can be a noncommutative space via matrix factorizations.

Here, K's noncomm' res'n is defined by $(\mathbf{P}^3, B)$ where B is the sheaf of even parts of Clifford algebras associated with the universal quadric over $\mathbf{P}^3$ defined by the GLSM superpotential.

B plays the role of structure sheaf; other sheaves are B-modules.

Physics?......

Physics picture of K's noncomm' space:

Matrix factorization for a quadratic superpotential: even though the bulk theory is massive, one still has D0-branes with a Clifford algebra structure.

(Kapustin, Li)

Here: a `hybrid LG model' fibered over $\mathbf{P}^3$, gives sheaves of Clifford algebras (determined by the universal quadric / GLSM superpotential) and modules thereof.

So: open string sector duplicates Kuznetsov's def'n.

Note we have a physical
realization of nontrivial examples of Kontsevich’s
`noncommutative spaces’
realized in gauged linear sigma models.

Summary so far:

The GLSM realizes:

$\mathbf{CP}^7[2,2,2,2]$ $\xleftrightarrow{\text{Kahler}}$ branched double cover of $\mathbf{CP}^3$

where RHS realized at LG point via local $\mathbf{Z}_2$ gerbe structure + Berry phase.

(A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E.S., '07)

Non-birational twisted derived equivalence

Physical realization of Kuznetsov's homological projective duality

More examples:

CI of n quadrics in $\mathbf{P}^{2n-1}$ $\xleftrightarrow{\text{Kahler}}$ branched double cover of $\mathbf{P}^{n-1}$, branched over deg 2n locus

Both sides CY

Homologically projective dual

More examples:

CI of 2 quadrics in the total space of

$$\mathbf{P}\left(\mathcal{O}(-1,0)^{\oplus 2} \oplus \mathcal{O}(0,-1)^{\oplus 2}\right) \longrightarrow \mathbf{P}^1 \times \mathbf{P}^1$$

$\xleftrightarrow{\text{Kahler}}$

branched double cover of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$,
branched over deg (4,4,4) locus

* In fact, the GLSM has 8 Kahler phases,
4 of each of the above.

* Related to an example of Vafa-Witten involving discrete torsion

(Caldararu, Borisov)

* Believed to be homologically projective dual

A non-CY example:

CI 2 quadrics in $\mathbf{P}^{2g+1}$ $\overset{\text{Kahler}}{\longleftrightarrow}$ branched double cover of $\mathbf{P}^1$, over deg 2g+2 (= genus g curve)

Homologically projective dual.

Here, r flows -- not a parameter.

Semiclassically, Kahler moduli space falls apart into 2 chunks.

Positively curved | Negatively curved

r flows: ·······➤ ·······➤ ·······➤

Depending upon the cutoff,
can replace branched double cover
by a space with codim 1 orbifolds.

Have double cover outside of cutoff-sized sphere
about the branch locus.

As the cutoff varies,
interpolate between
* branched double cover
* codim 1 $\mathbf{Z}_2$ orbifold

$\longleftarrow \quad \Lambda \quad \longrightarrow$

Another non-CY example:

CI 2 quadrics in $\mathbf{P}^4$ (= deg 4 del Pezzo) $\xleftrightarrow{\text{Kahler}}$ $\mathbf{P}^1$ w/ 5 $\mathbf{Z}_2$ singularities

Why codim 1 sing' instead of a double cover?
Well, no double cover exists, only the other cutoff limit makes sense.

Homologically projective dual

Analogous results for $\mathbf{P}^6[2,2,2]$, $\mathbf{P}^6[2,2,2,2]$

So far, we have only considered complete intersections of **quadrics**.

However, part of the analysis applies more generally.

Ex: $\mathbf{P}^5[3,3]$

The LG point of the GLSM is a hybrid LG model, with base a $\mathbf{Z}_3$ gerbe over $\mathbf{P}^1$, and fibers LG models for K3's.

Matches Kuznetsov's homological projective duality.

Aside:

One of the lessons of this analysis is that
gerbe structures are commonplace,
even generic,
in the hybrid LG models arising in GLSM's.

To understand the LG points of typical GLSM's,
requires understanding gerbes in physics.

So far we have discussed several GLSM's s.t.:

* the LG point realizes geometry in an unusual way

* the geometric phases are not birational

* instead, related by Kuznetsov's homological projective duality

We conjecture that Kuznetsov's homological projective duality applies much more generally to GLSM's.....

More Kuznetsov duals:

Another class of examples, also realizing Kuznetsov's h.p.d., were realized in GLSM's by Hori-Tong.

$G(2,7)[1^7]$ $\xleftrightarrow{\text{Kahler}}$ Pfaffian CY

(Rodland, Kuznetsov, Borisov-Caldararu, Hori-Tong)

* unusual geometric realization (via strong coupling effects in nonabelian GLSM)

* non-birational

More Kuznetsov duals:

$G(2,5)[1^4]$ (= deg 5 del Pezzo) ⟷ (Kahler) Vanishing locus in $\mathbf{P}^3$ of Pfaffians

=   =

Vanishing locus in $\mathbf{P}^5$ of Pfaffians ⟷ (Kahler) $G(2,5)[1^6]$

Positively curved   Negatively curved

r flows: ⇢ ⇢ ⇢

More Kuznetsov duals:

$G(2,N)[1^m]$ (N odd) $\overset{\text{Kahler}}{\longleftrightarrow}$ vanishing locus in P^{m-1} of Pfaffians

Check r flow:

$K = O(m-N)$ $\quad\quad$ $K = O(N-m)$

Opp sign, as desired,
so all flows in same direction.

More Kuznetsov duals:

So far we have discussed how Kuznetsov's h.p.d. realizes Kahler phases of several GLSM's with exotic physics.

We conjecture it also applies to ordinary GLSM's.

Ex: flops

Some flops are already known to be related by h.p.d.; K is working on the general case.

Summary

- Setup of a GLSM with an interesting limit
- Cluster decomposition conjecture for strings on gerbes:
  CFT(gerbe) = CFT(disjoint union of spaces)
- Application to GLSM's; realization of Kuznetsov's homological projective duality
- Future directions