A-twisted Landau-Ginzburg models, gerbes, and Kuznetsov’s homological projective duality

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Outline:

* A, B topological twists of Landau-Ginzburg models on nontrivial spaces

* Stacks in physics: how to build the QFT, puzzles and problems w/ new string compactifications

* Strings on gerbes: decomposition conjecture

* Application of decomposition conj’ to LG: physical realization of Kuznetsov’s homological projective duality, LG for K’s noncommutative resolutions
A Landau-Ginzburg model is a nonlinear sigma model on a space or stack X plus a "superpotential" W.

\[ S = \int \Sigma d^2 x \left( g_{ij} \partial \phi^i \overline{\partial} \phi^j + ig_{ij} \psi^j D_z \psi^i + ig_{ij} \psi^j D_z \psi^i - i g_{ij} \psi^j D_z \psi^i - \cdot \cdot \cdot + g^{ij} \partial_i W \partial_j \overline{W} + \psi_+ \psi_- D_i \partial_j W + \psi_+ \psi_- D_i \partial_j \overline{W} \right) \]

The superpotential \( W : X \rightarrow \mathbb{C} \) is holomorphic, (so LG models are only interesting when X is noncompact).

There are analogues of the A, B model TFTs for Landau-Ginzburg models.....
For nonlinear sigma models (i.e., LG w/ W=0), there are 2 topological twists: the A, B models.

1) A model

\[ \psi^i_+ \in \Gamma(\phi^*(T^{1,0}X)) \rightarrow \chi^i \quad \psi^i_- \in \Gamma(\phi^*(T^{0,1}X)) \rightarrow \chi^i \]

\[ Q \cdot \phi^i = \chi^i, \quad Q \cdot \phi^i_\bar{i} = \chi^i_\bar{i}, \quad Q \cdot \chi = 0, \quad Q^2 = 0 \]

Identify \[ \chi^\mu \sim dx^\mu \quad Q \sim d \]

States \[ b_{\mu \ldots \nu} \chi^\mu \cdots \chi^\nu \leftrightarrow H^{\cdot \cdot}(X) \]
2) B model

\[ \psi_\pm \in \Gamma(\phi^*(T^{0,1}X)) \]

\[ \eta\bar{i} = \psi_+ + \psi_- \]

\[ \theta_i = g_{i\bar{j}} \left( \psi_\bar{j} - \psi_\bar{\bar{j}} \right) \]

\[ Q \cdot \phi^i = 0, \quad Q \cdot \phi_i = \eta_i, \quad Q \cdot \eta_i = 0, \quad Q \cdot \theta_j = 0, \quad Q^2 = 0 \]

Identify \[ \eta_i \leftrightarrow d\bar{z}^i \quad \theta_j \leftrightarrow \frac{\partial}{\partial z^j} \quad Q \leftrightarrow \bar{\partial} \]

States:

\[ b_{i_1 \ldots i_n}^{j_1 \ldots j_m} \eta^{i_1} \ldots \eta^{i_n} \theta_{j_1} \ldots \theta_{j_m} \leftrightarrow H^n(X, \Lambda^m T^* X) \]

We can also talk about A, B twists of LG models over nontrivial spaces....
LG B model:

The states of the theory are Q-closed (mod Q-exact) products of the form

\[ b(\phi)_{j_1 \cdots j_m} \eta^{i_1} \cdots \eta^{i_n} \theta_{j_1} \cdots \theta_{j_m} \]

where \( \eta, \theta \) are linear comb's of \( \psi \)

\[ Q \cdot \phi^i = 0, \quad Q \cdot \phi^{\bar{i}} = \eta^{\bar{i}}, \quad Q \cdot \eta^{\bar{i}} = 0, \quad Q \cdot \theta_j = \partial_j W, \quad Q^2 = 0 \]

Identify \( \eta^{\bar{i}} \leftrightarrow d\bar{z}^{\bar{i}}, \quad \theta_j \leftrightarrow \frac{\partial}{\partial z^j}, \quad Q \leftrightarrow \bar{\partial} \)

so the states are hypercohomology

\[ H \left( X, \cdots \longrightarrow \Lambda^2 TX \xrightarrow{dW} TX \xrightarrow{dW} O_X \right) \]
Quick checks:

1) $W=0$, standard $B$-twisted NLSM

$$H \cdot \left( X, \cdots \rightarrow \Lambda^2 TX \xrightarrow{dW} TX \xrightarrow{dW} O_X \right)$$

$$\mapsto H \cdot (X, \Lambda \cdot TX)$$

2) $X=\mathbb{C}^n$, $W = \text{quasihomogeneous polynomial}$

Seq’ above resolves fat point $\{dW=0\}$, so

$$H \cdot \left( X, \cdots \rightarrow \Lambda^2 TX \xrightarrow{dW} TX \xrightarrow{dW} O_X \right)$$

$$\mapsto \mathbb{C}[x_1, \cdots, x_n]/(dW)$$
LG A model:

Defining the A twist of a LG model is more interesting. 

(An: Jarvis, Ruan) (Ito; J Guffin, ES)

Producing a TFT from a NLSM involves changing what bundles the $\psi$ couple to, e.g.

$$\psi \in \Gamma(\Sigma, \sqrt{K_\Sigma} \otimes \phi^*TX) \mapsto \Gamma(\Sigma, \phi^*TX), \Gamma(\Sigma, K_\Sigma \otimes \phi^*TX)$$

The two inequivalent possibilities are the A, B twists. To be consistent, the action must remain well-defined after the twist.

True for A, B NLSM's & B LG, but not A LG....
LG A model:

The problem is terms in the action of the form

$$\psi^i \psi^j D_i \partial_j W$$

If do the standard A NLSM twist, this becomes a 1-form on $\Sigma$, which can't integrate over $\Sigma$.

Fix: modify the A twist.
LG A model:

There are several ways to fix the A twist, and hence, several different notions of a LG A model.

One way: multiply offending terms in the action by another 1-form.

Another way: use a different prescription for modifying bundles.

The second is advantageous for physics, so I’ll use it, but,

disadvantage: not all LG models admit A twist in this prescription.
To twist, need a $U(1)$ isometry on $X$ w.r.t. which the superpotential is quasi-homogeneous.

Twist by "$R$-symmetry + isometry"

Let $Q(\psi_i)$ be such that

$$W(\lambda Q(\psi_i) \phi_i) = \lambda W(\phi_i)$$

then twist:

$$\psi \mapsto \Gamma \left( \text{original} \otimes K_{\Sigma}^{-(1/2)Q_R} \otimes \bar{K}_{\Sigma}^{-(1/2)Q_L} \right)$$

where $Q_{R,L}(\psi) = Q(\psi) + \begin{cases} 1 & \psi = \psi^i_+, R \\ 1 & \psi = \psi^i_-, L \\ 0 & \text{else} \end{cases}$
Example: \( X = \mathbb{C}^n, W \) quasi-homog' polynomial

Here, to twist, need to make sense of e.g. \( \frac{K}{\Sigma}^{1/r} \)

where \( r = 2 \) (degree)

Options: * couple to top' gravity (FJR)

* don't couple to top' grav' (GS)
  -- but then usually can't make sense of \( \frac{K}{\Sigma}^{1/r} \)

I'll work with the latter case.
LG A model:

A twistable example:

LG model on \( X = \text{Tot}(\mathcal{E}^\vee \xrightarrow{\pi} B) \)

with \( W = p\pi^* s, s \in \Gamma(B, \mathcal{E}) \)

\( U(1) \) action acts as phases on fibers

Turns out that correlation functions in this theory match those in a NLSM on \( \{s = 0\} \subset B \).
Correlation functions:

B-twist:

Integrate over $X$, weight by

$$\exp\left(-|dW|^2 + \text{fermionic}\right)$$

and then perform transverse Gaussian, to get the standard expression.

A-twist:

Similar: integrate over $M_X$

and weight as above.
Witten equ’n in A-twist:

BRST: \[ \delta \psi^i_\_ = -\alpha \left( \bar{\partial} \phi^i - i g^{i\bar{j}} \partial_{\bar{j}} \bar{W} \right) \]

implies localization on sol’ns of

\[ \bar{\partial} \phi^i - i g^{i\bar{j}} \partial_{\bar{j}} \bar{W} = 0 \] ("Witten equ’n")

On complex Kahler mflds, there are 2 independent BRST operators:

\[ \delta \psi^i_\_ = -\alpha + \bar{\partial} \phi^i + \alpha - i g^{i\bar{j}} \partial_{\bar{j}} \bar{W} \]

which implies localization on sol’ns of

\[ \bar{\partial} \phi^i = 0 \]
\[ g^{i\bar{j}} \partial_{\bar{j}} \bar{W} = 0 \] which is what we’re using.
Sol’ns of Witten equ’n:

\[ \int \Sigma |\bar{\partial} \phi^i - ig^{ij} \partial_j W|^2 = \int \Sigma \left( |\bar{\partial} \phi^i|^2 + |\partial_i W|^2 \right) \]

\[ \text{LHS} = 0 \quad \text{iff} \quad \text{RHS} = 0 \]

hence sol’ns of Witten equ’n

same as the moduli space we’re looking at.
LG A model, cont’d

In prototypical cases,

\[ \langle O_1 \cdots O_n \rangle = \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_n \int d\chi^p d\bar{\chi}^\bar{p} \exp \left( -|s|^2 - \chi^p dz^i D_i s - \text{c.c.} - F_{ij} dz^i \bar{d}z^\bar{j} \chi^p \chi^{\bar{p}} \right) \]

The MQ form rep’s a Thom class, so

\[ \langle O_1 \cdots O_n \rangle = \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_n \wedge \text{Eul}(N_{\{s=0\}}/\mathcal{M}) \]
\[ = \int_{\{s=0\}} \omega_1 \wedge \cdots \wedge \omega_n \]

-- same as A twisted NLSM on \{s=0\}

Not a coincidence, as we shall see shortly....
Renormalization (semi)group flow

Constructs a series of theories that are approximations to the previous ones, valid at longer and longer distance scales.

The effect is much like starting with a picture and then standing further and further away from it, to get successive approximations; final result might look very different from start.

Problem: cannot follow it explicitly.
Furthermore, RG preserves TFT’s.

If two physical theories are related by RG, then, correlation functions in a top’ twist of one = correlation functions in corresponding twist of other.
Example:

\[ \text{LG model on } X = \text{Tot}( \mathcal{E}^\vee \xrightarrow{\pi} B ) \]
with \( W = p s \)

NLSM on \( \{ s = 0 \} \subset B \)
where \( s \in \Gamma(\mathcal{E}) \)

This is why correlation functions match.
Another way to associate LG models to NLSM.

S’pose, for ex, the NLSM has target space
= hypersurface \{G=0\} in \mathbb{P}^n of degree \(d\)

Associate LG model on \([C^{n+1}/\mathbb{Z}_d]\)
with \(W = G\)

* Not related by RG flow

* But, related by Kahler moduli,
  so have same B model
LG model on \( \text{Tot}(O(-5) \rightarrow \mathbb{P}^4) \)
with \( W = p s \)

(RG flow)

(Same TFT)

Relations between LG models

NLSM on \( \{s=0\} \subset \mathbb{P}^4 \)

(Kahler)

(Only B twist same)

LG model on \( \mathbb{C}^5/\mathbb{Z}_5 \)
with \( W = s \)
Elliptic genera:

Elliptic genus of LG model on $X = \text{Tot}(\mathcal{E}^\vee \overset{\pi}{\to} B)$

$$\int_B \text{Td}(TB) \wedge \text{ch} \left( \Lambda_{-1}(TB) \otimes \Lambda_{-1}(\mathcal{E}^\vee) \right)$$

$$\bigotimes_{n=1,2,3,\ldots} S_{q^n}((TB)^C) \bigotimes_{n=0,1,2,\ldots} S_{q^n}((\mathcal{E}^\vee)^C)$$

$$\bigotimes_{n=1,2,3,\ldots} \Lambda_{-q^n}((TB)^C) \bigotimes_{n=1,2,3,\ldots} \Lambda_{q^n}((\mathcal{E}^\vee)^C)$$

matches Witten genus of $\{s = 0\} \subset B$

by virtue of a Thom class computation.

(M Ando, ES, '09)
RG flow interpretation:

In the case of the A-twisted correlation f’ns, we got a Mathai-Quillen rep of a Thom form.

Something analogous happens in elliptic genera: elliptic genera of the LG & NLSM models are related by Thom forms.

Suggests: RG flow interpretation in twisted theories as Thom class.

(possibly from underlying Atiyah-Jeffrey, Baulieu-Singer description)
Next:

* decomposition conjecture for strings on gerbes

* LG duals to gerbes

* application of gerbes to LG’s & GLSM’s as, physical realization of Kuznetsov’s homological projective duality

To do this, need to review how stacks appear in physics....
String compactifications on stacks

First, motivation:

-- new string compactifications

-- better understand certain existing string compactifications

Next: how to construct QFT’s for strings propagating on stacks?
Stacks

How to make sense of strings on stacks concretely?

Most (smooth, Deligne-Mumford) stacks can be presented as a global quotient

\[ \frac{X}{G} \]

for \( X \) a space and \( G \) a group.

(G need not be finite; need not act effectively.)

To such a presentation, associate a "\( G \)-gauged sigma model on \( X \)."

Problem: such presentations not unique
If to $[X/G]$ we associate "G-gauged sigma model," then:

$[C^2/Z_2]$ defines a 2d theory with a symmetry called conformal invariance

$[X/C^\times]$ defines a 2d theory w/o conformal invariance

Same stack, different physics!

Potential presentation-dependence problem: fix with renormalization group flow (Can’t be checked explicitly, though.)
The problems here are analogous to the derived-categories-in-physics program.

There, to a given object in a derived category, one picks a representative with a physical description (as branes/antibranes/tachyons).

Alas, such representatives are not unique.

It is conjectured that different representatives give rise to the same low-energy physics, via boundary renormalization group flow.

Only indirect tests possible, though.
Stacks

Other issues: deformation theory
massless spectra

To justify application of stacks to physics, need to conduct tests of presentation-dependence, understand issues above.
This was the subject of several papers.

For the rest of today’s talk, I want to focus on special kinds of stacks, namely, gerbes.

(= quotient by noneffectively-acting group)
Gerbes

Gerbes have add’l problems when viewed from this physical perspective.

Example: The naive massless spectrum calculation contains multiple dimension zero operators, which manifestly violates cluster decomposition, one of the foundational axioms of quantum field theory.

There is a single known loophole: if the target space is disconnected. We think that’s what’s going on....
Decomposition conjecture

Consider \([X/H]\) where

\[
1 \rightarrow G \rightarrow H \rightarrow K \rightarrow 1
\]

and \(G\) acts trivially.

Claim

\[
\text{CFT}([X/H]) = \text{CFT} \left( \left\lfloor (X \times \hat{G})/K \right\rfloor \right)
\]

(together with some B field), where \(\hat{G}\) is the set of irreps of \(G\)
Decomposition conjecture

For banded gerbes, $K$ acts trivially upon $\hat{G}$ so the decomposition conjecture reduces to

$$CFT(G - \text{gerbe on } X) = CFT \left( \bigsqcup_{\hat{G}} (X, B) \right)$$

where the B field is determined by the image of

$$H^2(X, Z(G)) \xrightarrow{Z(G) \rightarrow U(1)} H^2(X, U(1))$$
Checks:

* For global quotients by finite groups, can compute partition f’ns exactly at arb’ genus

* Implies \( K_H(X) = \text{twisted } K_K(X \times \hat{G}) \) which can be checked independently

* Implies known facts about sheaf theory on gerbes

* Implications for Gromov-Witten theory

(Andreini, Jiang, Tseng, 0812.4477, 0905.2258, 0907.2087, and to appear)
In more detail:

global quotients by nonfinite groups

The banded $\mathbb{Z}_k$ gerbe over $\mathbb{P}^N$
with characteristic class $-1 \ mod \ k$
can be described mathematically as the quotient

$$\left[ \frac{\mathbb{C}^{N+1} - \{0\}}{\mathbb{C}^\times} \right]$$

where the $\mathbb{C}^\times$ acts as rotations by k times

which physically can be described by a U(1) susy
gauge theory with N+1 chiral fields, of charge k

How can this be different from ordinary $\mathbb{P}^N$ model?
The difference lies in nonperturbative effects. (Perturbatively, having nonminimal charges makes no difference.)

To specify Higgs fields completely, need to specify what bundle they couple to.

If the gauge field $\sim L$
then $\Phi$ charge $Q$ implies
$\Phi \in \Gamma(L \otimes Q)$

Different bundles $\Rightarrow$ different zero modes
$\Rightarrow$ different anomalies $\Rightarrow$ different physics

(Noncompact worldsheet - theta angle -- J Distler, R Plesser)
Return to the example

\[
\begin{bmatrix}
\mathbb{C}^{N+1} - \{0\} \\
\mathbb{C}^\times
\end{bmatrix}
\]

Example: Anomalous global U(1)’s

\[
P^{N-1} : \ U(1)_A \mapsto \mathbb{Z}_{2N}
\]
Here: \( U(1)_A \mapsto \mathbb{Z}_{2kN} \)

Example: A model correlation functions

\[
P^{N-1} : \ \langle X^{N(d+1)-1} \rangle = q^d
\]
Here: \( \langle X^{N(kd+1)-1} \rangle = q^d \)

Example: quantum cohomology

\[
P^{N-1} : \ \mathbb{C}[x]/(x^N - q)
\]
Here: \( \mathbb{C}[x]/(x^{kN} - q) \)

Different physics
Quantum cohomology

We can see the decomposition conjecture in the quantum cohomology rings of toric stacks.

Ex: Q.c. ring of a $\mathbb{Z}_k$ gerbe on $\mathbb{P}^N$ is given by

$$\mathbb{C}[x,y]/(y^k - q, x^{N+1} - y^n q)$$

In this ring, the $y$'s index copies of the quantum cohomology ring of $\mathbb{P}^N$ with variable $q$'s.

The gerbe is banded, so this is exactly what we expect -- copies of $\mathbb{P}^N$, variable B field.
Mirrors to stacks

There exist mirror constructions for any model realizable as a 2d abelian gauge theory.

For toric stacks (BCS ’04), there is such a description.

Standard mirror constructions now produce character-valued fields, a new effect, which ties into the stacky fan description of (BCS ‘04).

(ES, T Pantev, ‘05)
Toda duals

Ex: The LG mirror of $\mathbb{P}^N$ is described by the holomorphic function

$$W = \exp(-Y_1) + \cdots + \exp(-Y_N) + \exp(Y_1 + \cdots + Y_N)$$

The analogous duals to $\mathbb{Z}_k$ gerbes over $\mathbb{P}^N$ are described by

$$W = \exp(-Y_1) + \cdots + \exp(-Y_N) + \Upsilon^n \exp(Y_1 + \cdots + Y_N)$$

where $\Upsilon$ is a character-valued field
(discrete Fourier transform of components in decomp' conjecture)

(ES, T Pantev, '05; E Mann, '06)
GLSM’s

These are families of QFT’s that RG flow to families of CFT’s.

Ex:

- Large radius
- NLSM on $\mathbb{P}^7[2,2,2,2]$
- GLSM for $\mathbb{P}^7[2,2,2,2]$
- RG
- LG model on $\text{Tot}(O(-2)^4 \rightarrow \mathbb{P}^7)$
- GLSM Kahler
- LG model on $\text{Tot}(O(-1)^8 \rightarrow \mathbb{P}^3_{[2,2,2,2]}$)
- RG
- $r \ll 0$
- ??????
Let's apply decomposition conjecture.

At $r \ll 0$ limit, $X = \text{Tot}(O(-1)^8 \rightarrow \mathbb{P}^3_{[2,2,2,2]})$, have superpotential

$$\sum_a p_a G_a(\phi) = \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

* mass terms for the $\phi_i$, away from locus $\{\det A = 0\}$.

* leaves just the $p$ fields, of charge 2

* $\mathbb{Z}_2$ gerbe, hence double cover
The $r \ll 0$ limit:

Because we have a $\mathbb{Z}_2$ gerbe over $\mathbb{P}^3$ - det....
The $r \ll 0$ limit:

Double cover

$\{ \det = 0 \}$

$P^3$

Berry phase

Result looks like branched double cover of $P^3$
So far:

The GLSM seems to realize:

\[ \mathbb{P}^7[2,2,2,2] \]

branched double cover of \( \mathbb{P}^3 \)

(Clemens’ octic double solid)

where RHS realized at LG point via local \( \mathbb{Z}_2 \) gerbe structure + Berry phase.


Non-birational twisted derived equivalence

Unusual physical realization of geometry
Rewrite:

GLSM for $\mathbb{P}^7[2,2,2,2]$

\[ r \ll 0 \]

RG

Large radius

LG model on

$\text{Tot}(O(-2)^4 \rightarrow \mathbb{P}^7)$

GLSM

Kahler

RG

NLSM on

$\mathbb{P}^7[2,2,2,2]$

RG

NLSM on branched double cover of $\mathbb{P}^3$, branched over deg 8 locus

LG model on

$\text{Tot}(O(-1)^8 \rightarrow \mathbb{P}^3[2,2,2,2])$
Puzzle:

the branched double cover will be singular, but the GLSM is smooth at those singularities.

Solution?....

We believe the GLSM is actually describing a `noncommutative resolution’ of the branched double cover worked out by Kuznetsov.

Kuznetsov has defined `homological projective duality’ that relates $\mathbb{P}^7[2,2,2,2]$ to the noncommutative resolution above.
Check that we are seeing K's noncomm' resolution:

K defines a `noncommutative space' via its sheaves -- so for example, a Landau-Ginzburg model can be a noncommutative space via matrix factorizations.

Here, K's noncomm' res'n = \((\mathbb{P}^3, B)\)
where B is the sheaf of even parts of Clifford algebras associated with the universal quadric over \(\mathbb{P}^3\) defined by the GLSM superpotential.

B ~ structure sheaf; other sheaves ~ B-modules.
Physics:

B-branes in the RG limit theory
= B-branes in the intermediate LG theory.

Claim: matrix factorizations in intermediate LG
= Kuznetsov's B-modules

K has a rigorous proof of this;
B-branes = Kuznetsov's nc res'n sheaves.

Intuition....
Local picture:

Matrix factorization for a quadratic superpotential: even though the bulk theory is massive, one still has DO-branes with a Clifford algebra structure. (Kapustin, Li)

Here: a `hybrid LG model' fibered over $\mathbb{P}^3$, gives sheaves of Clifford algebras (determined by the universal quadric / GLSM superpotential) and modules thereof.

So: open string sector duplicates Kuznetsoy's def'n.
Summary so far:

The GLSM realizes:

\[ P^7[2,2,2,2] \leftrightarrow \text{Kahler} \leftrightarrow \text{nc res'n of branched double cover of } P^3 \]

where RHS realized at \( r \ll 0 \) limit via local \( \mathbb{Z}_2 \) gerbe structure + Berry phase.

(A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E.S., '07)

Non-birational twisted derived equivalence

Unusual physical realization of geometry

Physical realization of Kuznetsov’s homological projective duality
More examples:

CI of \(n\) quadrics in \(\mathbb{P}^{2n-1}\)

branched double cover of \(\mathbb{P}^{n-1}\), branched over deg \(2n\) locus

Both sides CY

Homologically projective dual
Rewrite with Landau-Ginzburg models:

\[ \text{LG model on } \text{Tot}( O(-2)^k \to \mathbb{P}^n) \quad \text{GLSM} \quad \text{Kahler} \quad \text{LG model on } \text{Tot}( O(-1)^{n+1} \to \mathbb{P}^{k-1}[2,\ldots,2]) \]

\[ \text{NLSM on } \mathbb{P}^n[2,\ldots,2] \quad \text{Kuznetsov's h.p.d.} \quad \text{NLSM on n.c. res'n of branched double cover of } \mathbb{P}^{k-1}, \text{ branched over deg } n+1 \text{ locus} \]
A math conjecture:

Kuznetsov defines his h.p.d. in terms of coherent sheaves. In the physics language

\[
\text{LG model on } \text{Tot}( O(-2)^k \rightarrow \mathbb{P}^n ) \quad \text{GLSM} \quad \text{Kahler} \quad \text{LG model on } \text{Tot}( O(-1)^{n+1} \rightarrow \mathbb{P}^{k-1}_{[2,\ldots,2]} )
\]

Kuznetsov's h.p.d. becomes a statement about matrix factorizations, analogous to those in Orlov's work.

**Math conjecture:** Kuznetsov's h.p.d. has an alternative (and hopefully easier) description in terms of matrix factorizations between LG models on birational spaces.
More examples:

CI of 2 quadrics in the total space of
\[ \mathbb{P} \left( \mathcal{O}(-1, 0)^{\oplus 2} \oplus \mathcal{O}(0, -1)^{\oplus 2} \right) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \]

branched double cover of \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \),
branched over deg (4,4,4) locus

* In fact, the GLSM has 8 Kahler phases,
  4 of each of the above.

* Related to an example of Vafa-Witten involving
discrete torsion
  (Caldararu, Borisov)

* Believed to be homologically projective dual
A non-CY example:

CI 2 quadrics in $\mathbb{P}^{2g+1}$

branched double cover of $\mathbb{P}^1$, over deg $2g+2$

(= genus $g$ curve)

Homologically projective dual.

Here, $r$ flows -- not a parameter.

Semiclassically, Kahler moduli space falls apart into 2 chunks.

Positively curved

Negatively curved

$r$ flows: \[ \cdots \rightarrow \]
More examples:

Hori-Tong 0609032 found closely related phenomena in nonabelian GLSMs:

\[ G(2,7)[1^7] \rightarrow \text{Pfaffian CY} \]

Also:
* novel realization of geometry
* nonbirational
* Kuznetsov’s h.p.d.

Further nonabelian examples:
Donagi, ES, 0704.1761
So far we have discussed several GLSM's s.t.:

* the LG point realizes geometry in an unusual way
* the geometric phases are not birational
* instead, related by Kuznetsov's homological projective duality

Conjecture: all phases of GLSM's are related by Kuznetsov's h.p.d.
Summary:

* A, B topological twists of Landau-Ginzburg models on non-trivial spaces

* Stacks in physics: how to build the QFT, puzzles and problems with new string compactifications

* Strings on gerbes: decomposition conjecture

* Application of decomposition conj’ to LG & GLSM’s: physical realization of Kuznetsov’s homological projective duality, GLSM’s for K’s noncommutative resolutions
Mathematics

Geometry:
- Gromov-Witten
- Donaldson-Thomas
- quantum cohomology
- etc

Homotopy, categories:
- derived categories,
- stacks, etc.

Physics

Supersymmetric field theories

Renormalization group