GLSM's, gerbes, and Kuznetsov's homological projective duality

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Based on work with:
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Outline:

* noneffective group actions (gerbes)

* decomposition conjecture

* Application of decomposition conjecture to GLSM's:
physical realization of Kuznetsov's
``homological projective duality,''
and new string compactifications:
strings on nc resolutions

Noneffective orbifolds

This talk is going to concern applications of noneffective orbifolds to physics & geometry.

What is a noneffective orbifold?

It's [X/G] where a subgroup of G, call it K, acts trivially on X.

(= gerbe)

Why isn't that the same as [X / (G/K)] ?

Why isn't that the same as [X / (G/K)] ?

Example:

Consider $[X/D_4]$ where the center acts trivially.

$$1 \longrightarrow \mathbf{Z}_2 \longrightarrow D_4 \longrightarrow \mathbf{Z}_2 \times \mathbf{Z}_2 \longrightarrow 1$$

(Center = $\mathbf{Z}_2$)

We'll show that the T^2 partition function of $[X/D_4]$ is very different from the partition function of $[X / \mathbf{Z}_2 \times \mathbf{Z}_2]$.

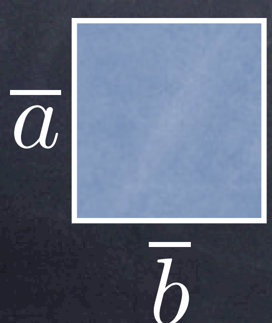
Check genus one partition functions:

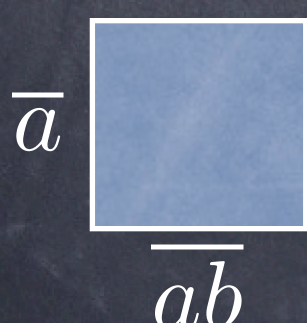
$$D_4 \;=\; \{1, z, a, b, az, bz, ab, ba = abz\}$$

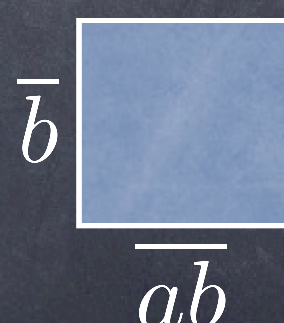
$$\mathbf{Z}_2 \times \mathbf{Z}_2 \;=\; \{1, \overline{a}, \overline{b}, \overline{ab}\}$$

$$Z(D_4) \;=\; \frac{1}{|D_4|} \sum_{g,h \in D_4, gh=hg} Z_{g,h}$$

Each of the $Z_{g,h}$ twisted sectors that appears, is the same as a $\mathbf{Z}_2 \times \mathbf{Z}_2$ sector, appearing with multiplicity $|\mathbf{Z}_2|^2 = 4$ except for the







sectors.

Partition functions, cont'd

$$
\begin{aligned}
Z(D_4) &= \frac{|\mathbf{Z}_2 \times \mathbf{Z}_2|}{|D_4|} |\mathbf{Z}_2|^2 \left(Z(\mathbf{Z}_2 \times \mathbf{Z}_2) \; - \; \text{(some twisted sectors)} \right) \\
&= 2 \left(Z(\mathbf{Z}_2 \times \mathbf{Z}_2) \; - \; \text{(some twisted sectors)} \right)
\end{aligned}
$$

Discrete torsion acts as a sign on the

twisted sectors

so we see that $Z([X/D_4]) = Z\left([X/\mathbf{Z}_2 \times \mathbf{Z}_2] \coprod [X/\mathbf{Z}_2 \times \mathbf{Z}_2] \right)$

with discrete torsion in one component.

Thus: physics knows about even trivial gp actions.

The same issue exists in 2d gauge theories,
where it manifests as a question of whether
e.g. an abelian gauge theory with matter of charge 2
is the same as if matter is charge 1.

Perturbatively, the same.

Nonperturbatively, different.

Example: $\mathbf{P}^{N-1}$ model, vs with fields of charge k

Example: Anomalous global U(1)'s

$$\mathbf{P}^{N-1}: \quad U(1)_A \mapsto \mathbf{Z}_{2N}$$

$$\text{Here}: \quad U(1)_A \mapsto \mathbf{Z}_{2kN}$$

Example: A model correlation functions

$$\mathbf{P}^{N-1}: \quad < X^{N(d+1)-1} > = q^d$$

$$\text{Here}: \quad < X^{N(kd+1)-1} > = q^d$$

Example: quantum cohomology

$$\mathbf{P}^{N-1}: \quad \mathbf{C}[x]/(x^N - q)$$

$$\text{Here}: \quad \mathbf{C}[x]/(x^{kN} - q)$$

Different physics

General argument:

Compact worldsheet:

To specify Higgs fields completely, need to specify what bundle they couple to.

If the gauge field $\sim L$
then Φ charge Q implies

$$\Phi \in \Gamma(L^{\otimes Q})$$

Different bundles => different zero modes
=> different anomalies => different physics

For noncpt worldsheets, analogous argument exists.

(Distler, Plesser)

4d analogues

* SU(n) vs SU(n)/$\mathbf{Z}_n$, Spin(n) vs SO(n) gauge theories

N=1:

Spin(n) gauge theory w/ massive spinors

Seiberg dual to

SO(n) gauge theories w/ $\mathbf{Z}_2$ monopoles

(M Strassler, hepth/9709081; P Pouliot, 9507018; etc)

N=4:

Crucial for Kapustin-Witten geom' Langlands; work here gives a bit of insight into behavior of 2d compactification

Back to 2d.....

Decomposition conjecture

Consider $[X/H]$ where

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1$$

and G acts trivially.

Claim

$$\mathrm{CFT}([X/H]) \, = \, \mathrm{CFT}\left(\left[(X \times \hat{G})/K\right]\right)$$

(together with some B field), where $\hat{G}$ is the set of irreps of G

Decomposition conjecture

When K acts trivially upon $\hat{G}$

the decomposition conjecture reduces to

$$\mathrm{CFT}([X/H]) \;=\; \mathrm{CFT}\left(\coprod_{\hat{G}} (X, B) \right)$$

where the B field is determined by the image of

$$H^2(X, Z(G)) \stackrel{Z(G) \rightarrow U(1)}{\longrightarrow} H^2(X, U(1))$$

Checks:

* For global quotients by finite groups, can check partition f'ns exactly at arb' genus

* Implies $K_H(X) \, = \, \text{twisted} \, K_K(X \times \hat{G})$ which can be checked independently

* Consistent with results on sheaves on gerbes

* Implications for Gromov-Witten theory (Andreini, Jiang, Tseng, 0812.4477, 0905.2258, 0907.2087, and to appear)

* Toda mirrors to Fano toric stacks computed (same results independently obtained later by E Mann)

Apply to GLSM's: Describe $\mathbf{P}^7[2,2,2,2]$

* 8 chiral superfields ϕ_i, charge 1 (homog' coord's $\mathbf{P}^7$)

* 4 chiral superfields p_a of charge -2

$$W = \sum_a p_a G_a(\phi)$$

D-terms: $$\left| \sum_i |\phi_i|^2 - 2 \sum_a |p_a|^2 - r \right|^2$$

$r \gg 0$:

ϕ_i not all zero

$$p_a = G_a = 0$$

NLSM on CY CI

The other limit is more interesting....

D-terms: $$\left|\sum_i |\phi_i|^2 - 2\sum_a |p_a|^2 - r\right|^2$$

$$W = \sum_a p_a G_a(\phi) = \sum_{ij} \phi_i A^{ij}(p)\phi_j$$

$$r \ll 0 :$$

p_a not all zero

ϕ_i massive (since deg 2)

NLSM on $\mathbf{P}^3$????

The correct analysis of the $r \ll 0$ limit is more subtle.

One subtlety is that the ϕ_i are not massive everywhere.

Write $$W = \sum_a p_a G_a(\phi) = \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

then they are only massive away from the locus

$$\{\det A = 0\} \subset \mathbf{P}^3$$

But that just makes things more confusing....

A more important subtlety is the fact that the p's have nonminimal charge,

so over most of the $\mathbf{P}^3$ of p vevs,

we have a nonminimally-charged abelian gauge theory,

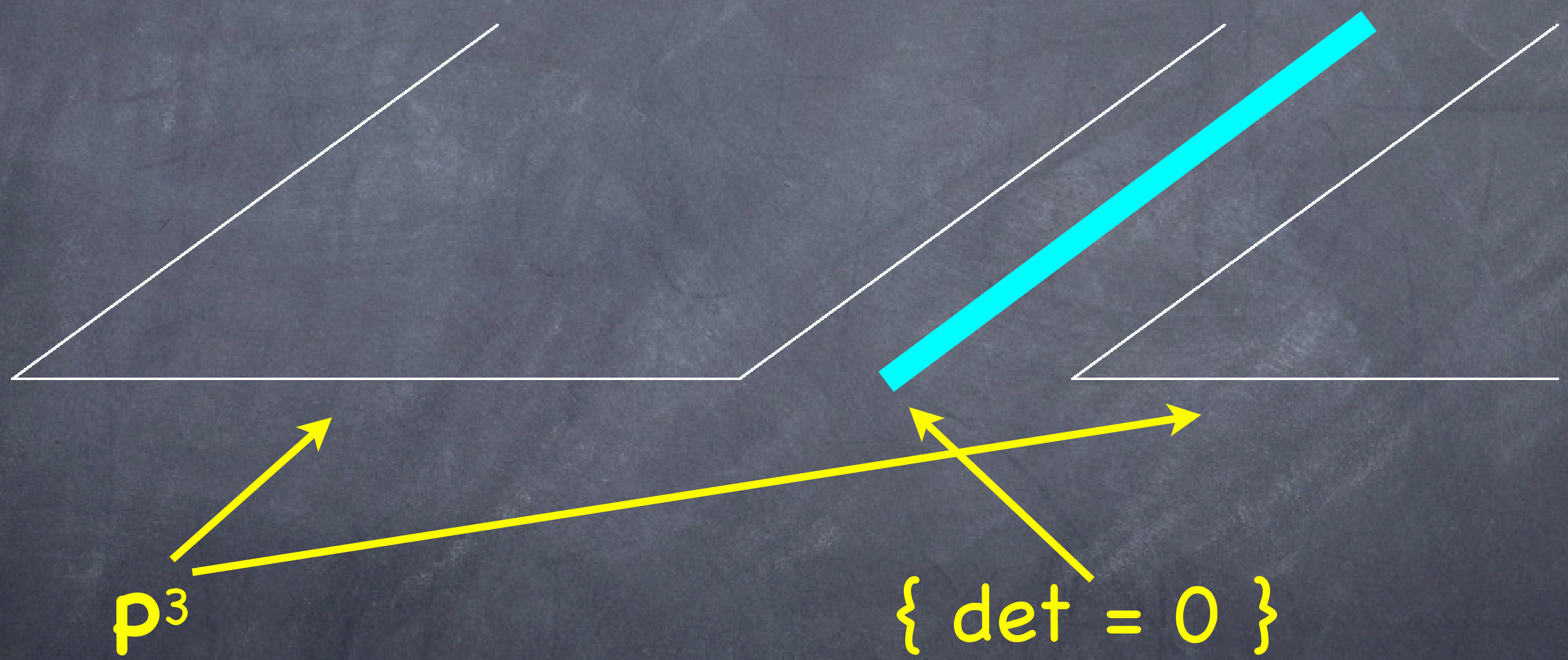
meaning massless fields have charge -2,

instead of 1 or -1.

-- local noneffective $\mathbf{Z}_2$ orbifold

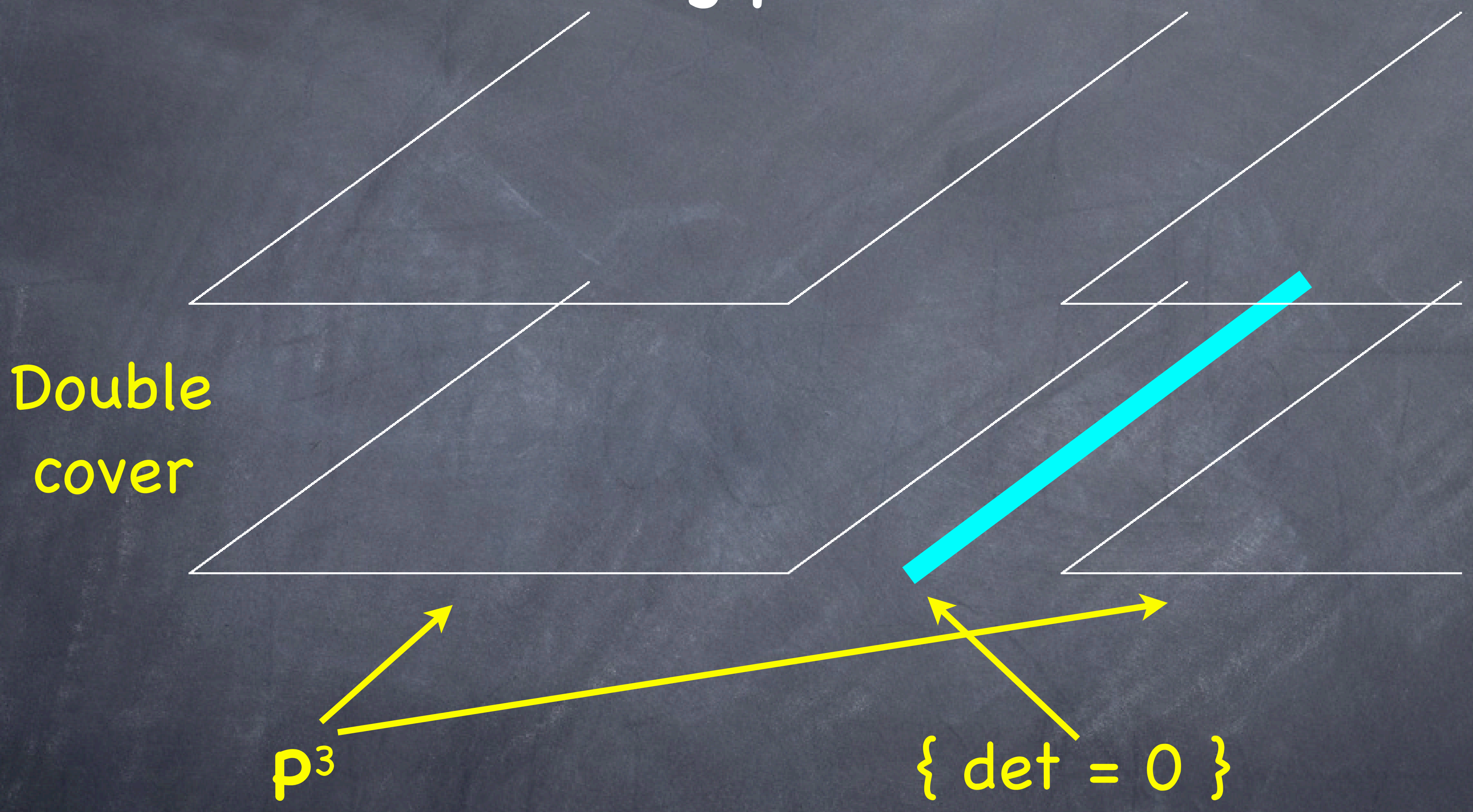
($\mathbf{Z}_2$ gerbe)

The Landau-Ginzburg model:

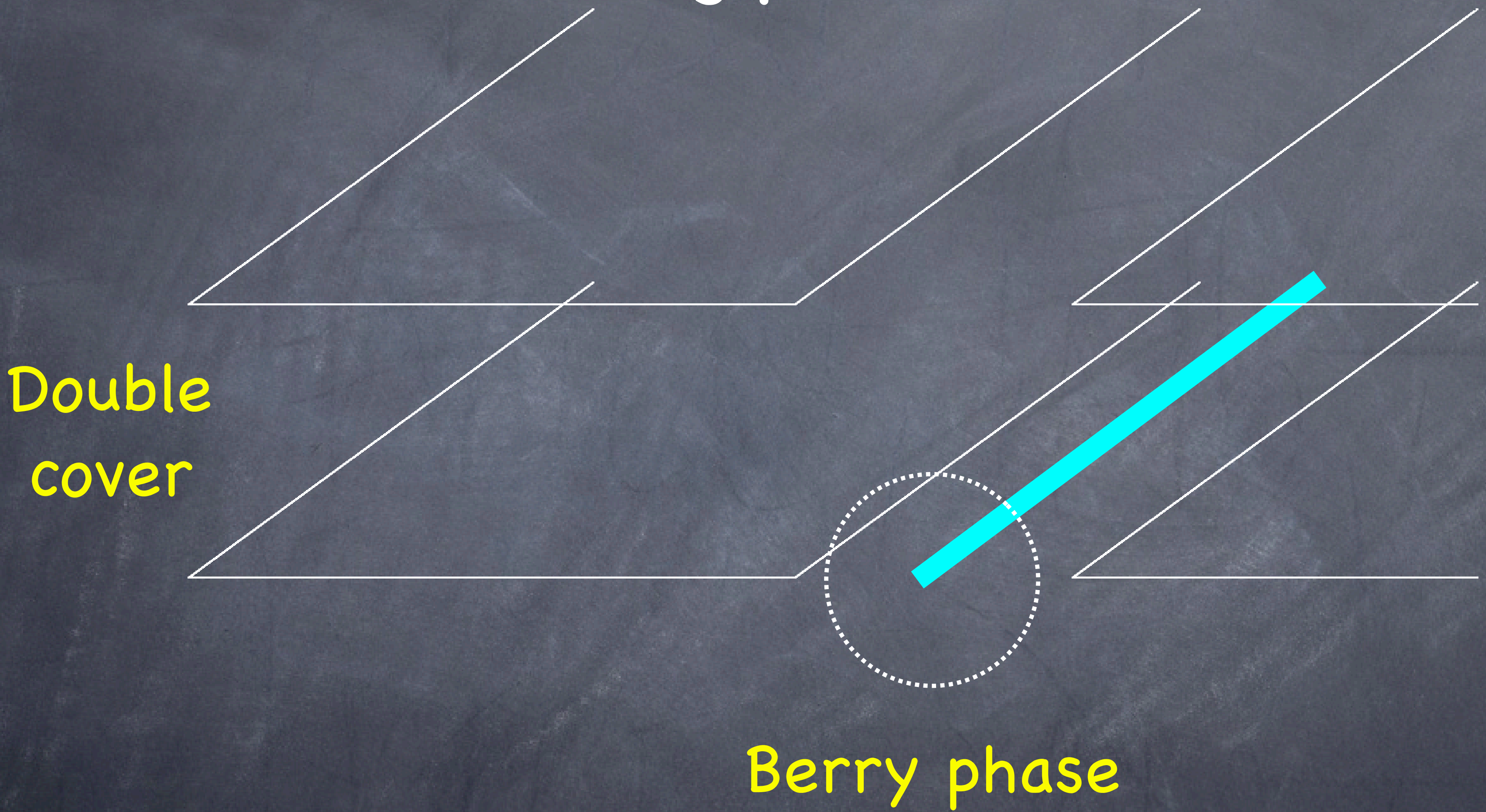


Because we have a Z_2 gerbe over P^3 - det....

The Landau-Ginzburg point:



The Landau-Ginzburg point:



Result: branched double cover of $\mathbf{P}^3$

So far:

The LG realizes:

branched double cover

of $\mathbf{P}^3$

(Clemens' octic double solid)

realized via

local $\mathbf{Z}_2$ gerbe structure + Berry phase.

(S. Hellerman, A. Henriques, T. Pantev, ES, M Ando, '06; R Donagi, ES, '07;
A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E.S., '07)

Unusual physical realization of geometry

Non-birational: violates GLSM lore

Puzzle:

the branched double cover will be singular, but the physics behaves as if smooth at those singularities.

Solution?....

We believe the LG is actually describing a `noncommutative resolution' of the branched double cover worked out by Kuznetsov.

Check that we are seeing K's noncomm' resolution:

K (+Kontsevich, Kapranov, Costello, van den Bergh,..) define a \`noncommutative space' via its sheaves

Here, K's noncomm' res'n = ($\mathbf{P}^3$,B)
where B is the sheaf of even parts of Clifford algebras associated with the universal quadric over $\mathbf{P}^3$ defined by the LG superpotential.

B ~ structure sheaf; other sheaves ~ B-modules.

Physics?......

Physics:

Claim: D-branes (``matrix factorizations”) in LG
= Kuznetsov’s B-modules

K has a rigorous proof of this;
D-branes = Kuznetsov’s nc res’n sheaves.

Intuition....

Local picture:

Matrix factorization for a quadratic superpotential: even though the bulk theory is massive, one still has D0-branes with a Clifford algebra structure.

(Kapustin, Li)

Here: a LG model fibered over $\mathbf{P}^3$, gives sheaves of Clifford algebras (determined by the universal quadric / superpotential) and modules thereof.

So: D-branes duplicate Kuznetsov's def'n.

Summary so far:

The LG realizes:

nc res'n of

branched double cover

of $\mathbf{P}^3$

realized via

local $\mathbf{Z}_2$ gerbe structure + Berry phase.

(A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E.S., '07)

Unusual physical realization of geometry

Non-birational: violates GLSM lore

+ physical realization of nc res'n

Topology change:

The GLSM links $\mathbb{P}^7[2,2,2,2]$
to nc res'n of a branched double cover

-- Kuznetsov's ``homological projective duality"

Many more examples exist, all also h.p.d.

We conjecture all GLSM phases are related by h.p.d.

D-brane moduli spaces:

The moduli space of D-branes propagating on this nc resolution, is a non-Kahler small resolution of the singular space.

(N Addington '09 & work in progress)

-- non-Kahler OK b/c it's open string moduli space, not where closed strings propagate.

Another example where closed string target different from open string space: orbifolds. (D-branes see res'n, closed strings see quot' stack)

Outline:

* noneffective group actions (gerbes)

* decomposition conjecture

* Application of decomposition conjecture to GLSM's: physical realization of Kuznetsov's homological projective duality, and strings on nc resolutions

Mathematics		**Physics**
Geometry: Gromov-Witten, Donaldson-Thomas, quantum cohomology, etc	⇔	Supersymmetric field theories
Homotopy, categories: derived categories, stacks, etc.	⇔	Renormalization group