Moduli of heterotic string compactifications

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Introduction

This talk will concern compactifications of heterotic strings.

Recall: in 10D, a heterotic string is specified by metric + nonabelian gauge field, so to compactify, we specify not only a space, but also a bundle over that space.
Introduction, cont’d

Let $X$ be a space, let $\mathcal{E}$ be a bundle over that space.

Constraints:

\[ \text{ch}_2(\mathcal{E}) = \text{ch}_2(TX) \quad \text{(Green–Schwarz anom’ canc’)} \]

\[ F_{ij} = 0 = F_{\overline{i}\overline{j}} \quad \text{(\mathcal{E} holomorphic)} \]

\[ g^{i\overline{j}} F_{\overline{i}j} = 0 \quad \text{(Donaldson–Uhlenbeck–Yau equ’n; stability, D-terms)} \]
Often, we want $X$ to be a Calabi-Yau manifold. -- complex, Kahler, nowhere-zero hol' top-form

In this case, we often divide possible bundles into 2 classes, corresponding to amount of worldsheet susy.

* (2,2) susy: $\mathcal{E} = TX$ "standard embedding"

* (0,2) susy: $\mathcal{E} \neq TX$
Other times, we want $X$ to be non-Kahler:

-- complex, non-Kahler, nowhere-zero hol’ top-form

(Strominger, ’86)

In this case, there is a nonzero background $H$ flux:

$$\omega = i g_{i\bar{j}} \, dz^i \wedge d\bar{z}^\bar{j}$$

$$H = i (\bar{\partial} - \partial) \omega$$

-- vanishes for Kahler metric, nonzero for non-Kahler
Introduction, cont’d

So far I’ve just reviewed various heterotic compactifications.

In this talk, I’m interested in the possible deformations, the `moduli,' of such heterotic compactifications.

Broadly speaking, the moduli are:
* metric moduli
* bundle moduli

and because of conditions such as $F_{ij} = 0 = F_{ij}$, these can be intertangled.
Introduction, cont’d

In Calabi-Yau compactifications, these moduli have been identified with complex, Kahler, bundle moduli for many years (though updated recently, and worldsheet description missing).

In non-Kahler heterotic compactifications, moduli have been a mystery.

Today, I’ll present an expression for moduli of heterotic non-Kahler compactifications (which will also give a worldsheet description of recent updates to Calabi-Yau story).
Introduction, cont’d

Outline of the rest of the talk:

* Describe what is known about moduli.

  * Present the result:
    infinitesimal moduli,
    expressed as cocycles + coboundaries

* Derivation of result.
Moduli

As mentioned earlier, there are 2 sources of moduli:

* metric moduli
* bundle moduli

which are required to obey constraints such as

$$F_{ij} = 0 = F_{\bar{i}\bar{j}}, \quad g^{i\bar{j}} F_{i\bar{j}} = 0$$

(so the metric & bundle moduli are linked in general).
Metric moduli:

On Calabi-Yau manifolds, from Yau’s theorem, metric moduli decompose into 2 types:

* Deformations of the complex structure, counted by $H^1(X, TX)$

$$N(J^\nu_\mu + \delta J^\nu_\mu) = 0 \implies \overline{\partial} \delta J^j_i = 0$$

* Deformations of the Kahler structure, counted by $H^1(X, T^*X)$

$$\overline{\partial} (\omega + \delta \omega) = 0 \implies \overline{\partial} \delta \omega_{ij} = 0$$
Metric moduli:

On non-Kahler mflds, no analogue of Yau’s thm known, which is part of why moduli of non-Kahler compactifications are mysterious. We’ll present a proposal, later....
Bundle moduli:

Consider an infinitesimal deformation of the gauge field:

\[ A^a_\mu + \delta A^a_\mu \]

If we demand \( F^i_j = 0 \) for both the original gauge field and the deformation, then the deformation must satisfy

\[ \bar{\partial} \delta A = 0 \]

Interpretation: \( \delta A \in H^1(\text{End} \ E) \)

These are the bundle moduli.
So far, I’ve described the metric & bundle moduli separately. However, b/c of the conditions

\[ F_{ij} = 0 = F_{i\bar{j}}, \quad g^{i\bar{j}} F_{i\bar{j}} = 0 \]

the metric & bundle moduli are linked, and mix.

For the simplest cases ( (2,2) Calabi-Yau compactifications), there’s no mixing, but in gen’l (0,2) Calabi-Yau cases these do mix. (Recent result: Anderson, Gray, Lukas, Ovrut)

We’ll see the details as I proceed....
I’ll systematically review what’s known about moduli in the following three cases:

- (2,2) susy worldsheet (Calabi-Yau)
- (0,2) susy worldsheet (Calabi-Yau)
- Non-Kahler heterotic compactification

and then I’ll present our result.
(2,2) susy worldsheet:

This is the `standard embedding,’ in which gauge bundle = tangent bundle.

Here, the allowed moduli are:

* Complex moduli: \( Z_i \in H^1(X, TX) \)
\[ \overline{\partial} Z = 0 \]

* Kahler moduli: \( Y_{i\bar{j}} \in H^1(X, T^* X) \)
\[ \overline{\partial} Y = 0 \]

* Bundle moduli: \( \Lambda^\alpha_{\beta\bar{r}} \in H^1(\text{End} \mathcal{E}) \)
\[ \overline{\partial} \Lambda = 0 \]
(2,2) susy worldsheet:

For later reference, let us express the moduli in local coordinates, as cocycles mod coboundaries:

Cocycles:

\[ Z^i_{\bar{k}, \bar{\imath}} - Z^i_{k, \bar{\imath}} = 0 \]
\[ Y^i_{\bar{k}, \bar{\imath}} - Y^i_{k, \bar{\imath}} = 0 \]
\[ \Lambda^{\alpha}_{\beta \bar{k}, \bar{\imath}} - \Lambda^{\alpha}_{\beta k, \bar{\imath}} = 0 \]

Coboundaries:

\[ Z^i_{\bar{k}} \sim Z^i_{\bar{k}} + \zeta^i_{\bar{\imath}} \]
\[ Y^i_{\bar{\imath}} \sim Y^i_{\bar{\imath}} + \mu_{\bar{\imath}} \]
\[ \Lambda^{\alpha}_{\beta \bar{k}} \sim \Lambda^{\alpha}_{\beta \bar{k}} + \lambda^{\alpha}_{\beta, \bar{\imath}} \]
(0,2) susy worldsheets (Calabi-Yau):

In this case, the gauge bundle \( \neq \) tangent bundle.

It was thought for many years that one still had the same complex, Kahler, bundle moduli in this case.

However, a year ago, L Anderson, J Gray, A Lukas, B Ovrut argued that the complex & bundle moduli mix, so that infinitesimally, one has a subset of complex & bundle moduli.
(0,2) susy worldsheet (Calabi-Yau):

One only has a **subset** of complex + bundle moduli, ultimately because of the constraints

$$F_{ij} = 0 = F_{ji}.$$  

These tie together the bundle and complex moduli, so that they are no longer independent.

Correct replacement for complex+bundle is

$$H^1(X, Q)$$

where

$$0 \rightarrow \mathcal{E}^* \otimes \mathcal{E} \rightarrow Q \rightarrow TX \rightarrow 0$$

(Atiyah sequence)

(extension determined by F)
(0,2) susy worldsheets (Calabi-Yau):

From Atiyah sequence

\[ 0 \rightarrow \mathcal{E}^* \otimes \mathcal{E} \rightarrow Q \xrightarrow{\pi} TX \rightarrow 0 \]

we get

\[ 0 \rightarrow H^1(X, \mathcal{E}^* \otimes \mathcal{E}) \rightarrow H^1(X, Q) \xrightarrow{d\pi} H^1(TX) \xrightarrow{F} H^2(X, \mathcal{E}^* \otimes \mathcal{E}) \]

Interpretation:

If a complex structure modulus in $H^1(X, TX)$ is in the image of $d\pi$, then it came from an element of $H^1(X, Q)$ and survives.

However, a complex structure modulus not in the image of $d\pi$ is lifted by the bundle.

We’ll see more details shortly....
Examples are discussed in e.g. Anderson, Gray, Lukas, Ovrut, 1010.0255.

Gen'l form:

Start with a deg \((2,2,3)\) hypersurface in \(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2\).

Define a stable indecomposable rk 2 bundle over an 58-dim'l sublocus of cpx moduli space (not holomorphic elsewhere).

If the complex structure modulus takes one out of that sublocus, then, hol' structure on bundle broken.
Aside: An analogue in D-branes.

If I wrap a D-brane on a cpx submfd S, and describe the Chan-Paton factors with a holomorphic vector bundle $\mathcal{E}$, then naively have moduli

$$H^0(S, \mathcal{E}^* \otimes \mathcal{E} \otimes N_{S/X})$$ -- brane motions
$$\bigoplus H^1(S, \mathcal{E}^* \otimes \mathcal{E})$$ -- bundle moduli

However, in general some motions of S can destroy holomorphic structure on bundle; only want a subset of brane motions.

Correct moduli: $$\text{Ext}^1_X(i_*\mathcal{E}, i_*\mathcal{E})$$

(Katz, ES, hepth/0208104)
(0,2) susy worldsheet (Calabi-Yau):

For later reference, let us express the moduli in local coordinates, as cocycles mod coboundaries:

Cocycles:
\[ Z^{i}_{i,k} - Z^{i}_{k,i} = 0 \]
\[ Y^{i}_{ii,k} - Y^{i}_{ik,i} = 0 \]
\[ \Lambda^{\alpha}_{\beta i,k} - \Lambda^{\alpha}_{\beta k,i} = F^{\alpha}_{\beta ki} Z^{i}_{i} - F^{\alpha}_{\beta vi} Z^{i}_{k} \]

Coboundaries:
\[ Z^{i}_{i} \sim Z^{i}_{i} + \zeta^{i}_{i} \]
\[ Y^{i}_{ii} \sim Y^{i}_{ii} + \mu_{i,i} \]
\[ \Lambda^{\alpha}_{\beta i} \sim \Lambda^{\alpha}_{\beta i} + \lambda^{\alpha}_{\beta,i} - F^{\alpha}_{\beta vi} \zeta^{i} \]

(Anderson et al, 1107.5076, equ’n (3.8))
Check that this obstructs (some) complex moduli:

\[
\Lambda^{\alpha}_{\beta \bar{i}, k} - \Lambda^{\alpha}_{\beta k, \bar{i}} = F^{\alpha}_{\beta \bar{k} \bar{i}} Z^{i}_{\bar{i}} - F^{\alpha}_{\beta \bar{i} \bar{i}} Z^{i}_{\bar{k}} \\
\in H^2(X, \mathcal{E}^* \otimes \mathcal{E})
\]

\(Z^{i}_{\bar{i}}\) is a complex structure modulus.

The right-hand side above is the image of the map \(F\) in

\[
0 \longrightarrow H^1(X, \mathcal{E}^* \otimes \mathcal{E}) \longrightarrow H^1(X, Q) \xrightarrow{d\pi} H^1(TX) \xrightarrow{F} H^2(X, \mathcal{E}^* \otimes \mathcal{E})
\]

The left-hand side is cohomologically trivial.

Only if the right-hand side is cohomologically trivial can there be a solution.
Check that this reduces correctly on (2,2) locus:

\[
\Lambda_{\beta \bar{i}, k}^{\alpha} - \Lambda_{\beta k, \bar{i}}^{\alpha} = F_{\beta k i}^{\alpha} Z_i^\bar{i} - F_{\beta \bar{i} i}^{\alpha} Z_k^i
\]

On the (2,2) locus, those F's aren't zero, so I need to explain how this reduces to

\[
\Lambda_{\beta \bar{i}, k}^{\alpha} - \Lambda_{\beta k, \bar{i}}^{\alpha} = 0
\]

Briefly, on the (2,2) locus, 
\( F = \text{Riemann curvature tensor} \, R \), and in that case, one can redefine \( \Lambda_{\beta \bar{i}}^{\alpha} \) with a translation to absorb the \( R \) terms. 

Details next....
Check that this reduces correctly on (2,2) locus:

Cocycle condition:

\[
\Lambda^m_{ni,k} - \Lambda^m_{nk,i} = R^m_{nk i} Z^i_k - R^m_{nii} Z^i_k
\]

Define

\[
\tilde{\Lambda}^m_{ni} = \Lambda^m_{ni} - \nabla_n Z^m_i
\]

then

\[
\tilde{\Lambda}^m_{ni,k} - \tilde{\Lambda}^m_{nk,i} = \Lambda^m_{ni,k} - \Lambda^m_{nk,i} - \nabla_k \nabla_n Z^m_i + \nabla_i \nabla_n Z^m_k
\]

\[
= R^m_{nk i} Z^i_k - R^m_{nii} Z^i_k - [\nabla_k, \nabla_n] Z^m_i + [\nabla_i, \nabla_n] Z^m_k
\]

\[
= 0 + \nabla_n (-Z^m_{i,k} + Z^m_{k,i}) = 0
\]

Cocycle condition:

problematic
Check that this reduces correctly on (2,2) locus:

We can treat coboundaries similarly:

Recall

\[ \tilde{\Lambda}_{\frac{m}{n_i}} = \Lambda_{\frac{m}{n_i}} - \nabla_n Z_{\frac{m}{i}} \]

where

\[ \Lambda_{\frac{m}{n_i}} \sim \Lambda_{\frac{m}{n_i}} + \lambda_{\frac{m}{n, i}} - R_{\frac{m}{nii}} \zeta^i, \quad Z_{\frac{i}{i}} \sim Z_{\frac{i}{i}} + \zeta_{\frac{i}{i}} \]

hence

\[ \tilde{\Lambda}_{\frac{m}{n_i}} \sim \tilde{\Lambda}_{\frac{m}{n_i}} + \lambda_{\frac{m}{n, i}} - R_{\frac{m}{nii}} \zeta^i - \nabla_n \nabla_{\frac{i}{i}} \zeta^i \]

Define

\[ \tilde{\lambda}_{\frac{m}{n}} = \lambda_{\frac{m}{n}} - \nabla_n \zeta^m \]

Then

\[ \tilde{\Lambda}_{\frac{m}{n_i}} \sim \tilde{\Lambda}_{\frac{m}{n_i}} + \tilde{\lambda}_{\frac{m}{n, i}} + [\nabla_{\frac{i}{i}}, \nabla_n] \zeta^m - R_{\frac{m}{nii}} \zeta^i \]

\[ = 0 \]
Check that this reduces correctly on (2,2) locus:

Started with

\[ \Lambda_{n\bar{i},k}^m - \Lambda_{n\bar{k},\bar{i}}^m = R_{n\bar{k}i}^m Z_i^i - R_{n\bar{i}i}^m Z_k^i \]

\[ \Lambda_{n\bar{i}}^m \sim \Lambda_{n\bar{i}}^m + \lambda_{n,i}^m - R_{n\bar{i}i}^m \zeta^i \]

but we can absorb the R terms into redef’ns:

\[ \tilde{\Lambda}_{n\bar{i},k}^m - \tilde{\Lambda}_{n\bar{k},\bar{i}}^m = 0 \]

\[ \tilde{\Lambda}_{n\bar{i}}^m \sim \tilde{\Lambda}_{n\bar{i}}^m + \tilde{\lambda}_{n,i}^m \]

and so we recover the cocycles, coboundaries for the (2,2) locus.
Non-Kahler heterotic compactifications:

Here, almost nothing is known about moduli.

Nearly the only thing known is that the `breathing mode', which rescales the entire metric of a CY, is absent.

* Since $H$ appears in multiple places,

$$dH = \alpha' \left( \text{tr } R_H \wedge R_H - \text{tr } F \wedge F \right)$$

is nonlinear in $\alpha'$ so values of $\alpha'$ are isolated

* $H$ is quantized (mod anom') but $H \propto (\bar{\partial} - \partial)\omega$
Non-Kahler heterotic compactifications:

Because that `breathing mode' is absent, one cannot smoothly deform a non-Kahler compactification to a weak coupling, `large radius' limit.

Hence, any claims about rel'ns to geometry, are necessarily somewhat formal.
Non-Kahler heterotic compactifications:

Aside from the absence of the breathing mode, there’s no systematic understanding of the moduli of heterotic non-Kahler compactifications.

That said, there are a few special cases where something is known. Ex: Adams/Lapan compute spectra at LG points in their torsion LSMs, but, interpretation & rel’n to geometry are unclear.

Next: proposal for the answer....
Briefly, our proposal for infinitesimal moduli:

Cocycles:

\[ Z_{i,k}^i - Z_{k,i}^i = 0 \]

\[ Y_{i,k}^i - Y_{k,i}^i = Z_{k}^j H_{j}^i - Z_{i}^j H_{j,k}^i \]

\[ \Lambda_{\beta,i,k}^\alpha - \Lambda_{\beta,k,i}^\alpha = F_{\beta,i,k}^\alpha Z_{i}^i - F_{\beta,i,k}^\alpha Z_{k}^i \]

Coboundaries:

\[ Z_{i}^i 
\sim Z_{i}^i + (\zeta^i + g^{ij} \xi_j),_i + g_{i,k}^i \left( \xi_{k,i}^i - \xi_{k,i}^i \right) \]

\[ Y_{i,i}^i 
\sim Y_{i,i}^i + \mu_{i,i}^i + \xi_{i,i}^i + H_{i,i}^j \left( \zeta^j + g^{ij} \xi_j \right) \]

\[ \Lambda_{\beta,i}^\alpha 
\sim \Lambda_{\beta,i}^\alpha + \lambda_{\beta,i}^\alpha - F_{\beta,i,i}^\alpha \left( \zeta^i + g^{i,j} \xi_j \right) \]
Check:

For (0,2) Calabi-Yau compactifications, \( H = 0 \), so cocycles reduce to

\[
Z^{i}_{\bar{i}, k} - Z^{i}_{k, \bar{i}} = 0
\]

\[
Y^{\bar{i}i, k} - Y^{\bar{i}k, i} = 0
\]

\[
\Lambda^{\alpha}_{\beta \bar{i}, k} - \Lambda^{\alpha}_{\beta \bar{k}, i} = F^{\alpha}_{\beta \bar{ki}} Z^{i}_{\bar{i}} - F^{\alpha}_{\beta \bar{i}i} Z^{i}_{\bar{k}}
\]

which is what we described earlier.
Check:

For heterotic non-Kahler compactifications, where $H$ is not zero, about the only thing we know is that the Kahler `breathing mode' is obstructed.

$$Y_{i\bar{i},k} - Y_{i\bar{k},i} = Z_{\bar{j}}^{\bar{i}} H_{j\bar{i}\bar{i}} - Z_{\bar{i}}^{\bar{i}} H_{j\bar{i}k}$$

If we take $Z=0$, and take $Y_{i\bar{i}} \propto g_{i\bar{i}}$ (so as to describe the breathing mode), then since the space is non–Kahler, $\bar{\partial}Y \neq 0$, and so we see the breathing mode is obstructed.
Mathematical interpretation

How to interpret the structure mathematically as some sort of cohomology theory?

Begin with the pure metric part:

\[
\frac{Z^i_{i,k}}{Z^i_{k,i}} - \frac{Z^i_{j,k}}{Z^j_{j,i}} = 0
\]

\[
\frac{Y_{ii,k}}{Y_{ik,i}} - \frac{Y_{ji}}{Y_{ji}} = \frac{Z^j_{j,k}}{Z^j_{i,k}} H_{ji} - \frac{Z^j_{j,i}}{Z^j_{i,j}} H_{ji}
\]

At tree level, we can interpret this using an analogue of the Atiyah sequence from earlier....
Mathematical interpretation

Tree level: Since \( dhH = 0 = \overline{\partial}(H_{(2,1)}) \)
the (2,1) part of \( H \), at tree level,
defines an element of
\[ H^1(\bigwedge^2 T^*X) \subseteq H^1(T^*X \otimes T^*X) \]
and hence an extension
\[
0 \longrightarrow T^*X \longrightarrow Q \longrightarrow TX \longrightarrow 0
\]
The metric moduli are then elements of \( H^1(Q) \).

When \( H=0 \), then \( Q = T^*X \oplus TX \)
and \( H^1(Q) = H^1(T^*X) + H^1(TX) \)
-- standard Calabi-Yau result
Similarly, for the full heterotic moduli, F+H defines an extension
\[ 0 \longrightarrow T^*X \bigoplus \text{End} \mathcal{E} \longrightarrow Q \longrightarrow TX \longrightarrow 0 \]
and the heterotic moduli are then elements of
\[ H^1(Q) \]
Mathematical interpretation

Examples? In progress.

Interpretation for nonzero $\alpha'$? Unknown.

Derivation directly from study of metric moduli? Desired, not known.

Any rel'n to Hitchin's generalized complex geometry? Unknown at present.
Physics: why has this been missed in spectrum computations?

After all, people have computed heterotic massless spectra for a quarter century now....

Answer: We usually assume that at large radius, we can reduce to free fields and compute zero energy spectrum.

However,

$$T_L \propto g_{ij} \partial \phi^i \partial \phi^j + \overline{\gamma}_\beta \partial \gamma^\beta + A_{\beta j} \partial \phi^j \overline{\gamma}_\alpha \gamma^\beta$$

not quadratic

hence difficult to pick out massless part of spectrum.
Physical intuition:

Why is the worldsheet BRST cohomology changing?

-- perturbative corrections to OPE's.

On the worldsheet, BRST operator $Q \propto g^i_j \partial \phi^i \bar{\psi}^\jmath$

Bdle modulus $\Lambda^{\alpha}_{\beta_i} \gamma^{\alpha} \gamma^{\beta} \psi^i$, Cpx modulus $Z^i_{\bar{i}} \bar{\partial} \phi^i \bar{\psi}^\bar{i}$

$Q \cdot \text{(moduli)} = (g^i_j \bar{\partial} \phi^i \bar{\psi}^\jmath) \cdot (\Lambda^{\alpha}_{\beta_i} \gamma^{\alpha} \gamma^{\beta} \psi^i)$

$+ (g^i_j \bar{\partial} \phi^i \bar{\psi}^\jmath) (Z^i_{\bar{i}} \bar{\partial} \phi^i \bar{\psi}^\bar{i}) \left( \int d^2 z F^\alpha_{\beta k m} \gamma^{\alpha} \gamma^{\beta} \psi^k \bar{\psi}^m \right)$

Compare

$\Lambda^{\alpha}_{\beta_i, k} - \Lambda^{\alpha}_{\beta k, i} = F^\alpha_{\beta k i} Z^i_{\bar{i}} - F^\alpha_{\beta i i} Z^i_{\bar{k}}$
Physical intuition:

Why is the worldsheet BRST cohomology changing?

Similarly, for complex moduli,

Cpx modulus $\bar{Z}_i^i \bar{\partial} \phi^i \psi^i$, Kahler modulus $Y_{i\bar{j}} \bar{\partial} \phi^i \psi^i$

$$Q \cdot (\text{moduli}) = (g_{i\bar{j}} \bar{\partial} \phi^{i} \psi^{\bar{j}}) \cdot (Y_{k\bar{m}} \bar{\partial} \phi^{k} \psi^{m})$$

$$+ (g_{i\bar{j}} \bar{\partial} \phi^{i} \psi^{\bar{j}}) \left( \bar{Z}_k^k \bar{\partial} \phi^k \psi^k \right) \left( \int d^2 z H_{m\bar{n}j} \partial \phi^m \psi^\bar{n} \psi^j \right)$$

Compare

$Y_{i\bar{j},k} - Y_{i\bar{k},j} = \bar{Z}_k^k H_{j\bar{i}i} - \bar{Z}_i^i H_{j\bar{i}k}$
So far I have described the result, checked that the result is consistent, and given some intuition for why it arises.

In principle, I could push the OPE computations I outlined further to give a derivation, but instead I’ll use a different approach.

I’ll classify marginal operators we could add to a 2d UV theory, in the spirit of Beasley-Witten’s analysis of 4d SQCD. (an idea I must attribute to my collaborator, I Melnikov)

To that end, a brief review of (0,2) superspace....
(0,2) superspace:

Coordinates \((z, \bar{z}, \theta, \bar{\theta})\)

\[ D = \frac{\partial}{\partial \theta} + \theta \bar{\partial} \]
\[ \bar{D} = \frac{\partial}{\partial \bar{\theta}} + \bar{\theta} \partial \]

\[ Q = -\frac{\partial}{\partial \theta} + \bar{\theta} \partial \]
\[ \bar{Q} = -\frac{\partial}{\partial \bar{\theta}} + \theta \partial \]

\[ \{D, \bar{D}\} = +2\bar{\partial} \]
\[ \{Q, \bar{Q}\} = -2\bar{\partial} \]

\(D, Q\) have \(U(1)_R\) charge -1
\(\bar{D}, \bar{Q}\) have \(U(1)_R\) charge +1
(0,2) superfields:

Chiral superfields:

$$\Phi = \phi + \sqrt{2} \theta \psi + \theta \bar{\theta} \bar{\partial} \phi$$

Fermi superfields:

$$\Gamma = \gamma + \sqrt{2} \theta G + \theta \bar{\theta} \bar{\partial} \gamma$$

$$\bar{D} \Phi = 0 = \bar{D} \Gamma$$
(0,2) Lagrangian: 

\[
D \overline{D} \left[ \frac{1}{2} \left( K_i(\Phi, \Phi) \partial \Phi^i - \overline{K}_i(\Phi, \Phi) \overline{\partial} \Phi^i \right) - H_{\beta\alpha}(\Phi, \Phi) \Gamma^\alpha \Gamma^\beta \right]
\]

The \( K_i \) is a potential for the metric, just as, in (2,2) cases, the Kahler potential determines the metric.

Since the metric is not Kahler in gen’l here, 

\[
g_{i\overline{j}} \neq \partial_i \partial_{\overline{j}} K \quad \text{for any} \quad K
\]

Instead, we’ll see next 

\[
g_{i\overline{j}} = \frac{1}{2} (K_{i,\overline{j}} + \overline{K}_{\overline{j},i})
\]

(Dine–Seiberg PLB 180 ’86)
(0,2) Lagrangian:

\[ \omega = ig_{ij} \, dz^i \wedge d\bar{z}^j \quad H = dB = i(\bar{\partial} - \partial)\omega \]

At leading order, \( dH = 0 \) so \( \partial\bar{\partial}\omega = 0 \)

Implies \( g_{i[j,k]}m = g_{m[j,k]}i \)

Implies \( g_{i[j,k]} = \partial_i \bar{W}_{\bar{j}k} \) for some \( \bar{W} \) s.t. \( \partial\bar{W} = 0 \)

\( \partial\bar{W} = 0 \) implies locally \( \bar{W}_{\bar{j}k} = (1/2)(\bar{K}_{j,k} - \bar{K}_{k,j}) \) for some \( \bar{K}_j^i \)

Combining with complex conjugate, we see

\[ g_{ij} = \frac{1}{2} \left(K_{i,j} + \bar{K}_{j,i}\right) \]
(0,2) Lagrangian: 

\[
D \bar{D} \left[ \frac{1}{2} \left( K_i(\Phi, \bar{\Phi}) \partial \Phi^i - \bar{K}_i(\Phi, \bar{\Phi}) \partial \bar{\Phi}^i \right) - H_{\beta\bar{\alpha}}(\Phi, \bar{\Phi}) \bar{\Gamma}^\alpha \Gamma^\beta \right]
\]

fiber metric

\[
g_{i\bar{j}} \left( \partial \phi^i \partial \bar{\phi}^\bar{j} + \partial \bar{\phi}^\bar{j} \partial \phi^i \right) + B_{i\bar{j}} \left( \partial \phi^i \partial \bar{\phi}^\bar{j} - \partial \bar{\phi}^\bar{j} \partial \phi^i \right) \\
+ 2g_{i\bar{j}} \bar{\psi}^\bar{j} \partial \psi^i + 2\bar{\psi}^\bar{i} \left( \partial \phi^k \Omega_{\bar{i}k\bar{j}} + \partial \bar{\phi}^k \bar{\Omega}_{\bar{i}k\bar{j}} \right) \psi^j \\
+ \bar{\gamma}_\alpha \left( \partial \gamma^\alpha + \partial \phi^j A^\alpha_{\beta j} \gamma^\beta \right) + \bar{\gamma}_\alpha F^\alpha_{\beta j k} \gamma^\beta \psi^k \psi^\bar{j}
\]

where

\[
g_{i\bar{j}} = \frac{1}{2} \left( K_{i\bar{j}} + \bar{K}_{\bar{j}i} \right) \quad B_{i\bar{j}} = \frac{1}{2} \left( K_{i\bar{j}} - \bar{K}_{\bar{j}i} \right)
\]

\[
\Omega_{\bar{i}k\bar{j}} = \Gamma_{\bar{i}k\bar{j}} - \frac{1}{2} H_{\bar{i}k\bar{j}} \quad H = dB = i(\bar{\partial} - \partial)\omega
\]

\[(\text{Dine-Seiberg PLB 180 '86)}\]
In this language, a susy marginal operator should be of the form $DX$

where $X$ is a (0,2) chiral superfield with classical dim 1, $U(1)_R$ charge +1.

Check:

Under a susy transformation,

$$\int d^2 z DX \mapsto \int d^2 z D(-\xi Q - \bar{\xi}\bar{Q})X$$

Up to total derivatives, $Q = -D, \bar{Q} = -\bar{D}$

$$\int d^2 z DX \mapsto \int d^2 z D(\xi D + \bar{\xi}\bar{D})X = 0$$

((2,2): Selberg et al, 1005.3546; (0,2): Adam, M, Plesser, unpub)
In this language, a susy marginal operator should be of the form $DX$

where $X$ is a (0,2) chiral superfield with classical dim 1, $U(1)_R$ charge +1.

Most general possibility:

$$X = \left[ \Gamma_\alpha \Gamma^\beta \Lambda^\alpha_{\beta \bar{\tau}}(\Phi, \bar{\Phi}) + \partial \Phi^i Y_{i \bar{\tau}}(\Phi, \bar{\Phi}) + \partial \bar{\Phi}^j g_{ij} Z^i_{\bar{\tau}}(\Phi, \bar{\Phi}) \right] \overline{D\Phi}^\bar{\tau}$$

This defines $Z^i_{\bar{\tau}}, Y_{i \bar{\tau}}, \Lambda^\alpha_{\beta \bar{\tau}}$
Susy marginal operators on worldsheet:

\[ X = \left[ \Gamma_\alpha \Gamma^\beta \Lambda^\alpha_{\beta \overline{k}}(\Phi, \overline{\Phi}) + \partial \Phi^i Y_{i \overline{i}}(\Phi, \overline{\Phi}) + \partial \overline{\Phi}^j g_{ij} Z^i_{\overline{i}}(\Phi, \overline{\Phi}) \right] \overline{D \Phi}^\overline{i} \]

Demand \( X \) be chiral on-shell, ie, \( \overline{D} X = 0 \).

This gives the cocycle conditions:

\[ Z^i_{\overline{i}, k} - Z^i_{k, \overline{i}} = 0 \]

\[ Y_{i \overline{i}, k} - Y_{i k, \overline{i}} = Z^j_{k \overline{j}} H_{j \overline{i} \overline{\overline{i}}} - Z^j_{i \overline{j}} H_{j \overline{i} \overline{k}} \]

\[ \Lambda^\alpha_{\beta \overline{i}, k} - \Lambda^\alpha_{\beta k, \overline{i}} = F^\alpha_{\beta ki} Z^i_{\overline{i}} - F^\alpha_{\beta \overline{i} i} Z^i_{\overline{k}} \]
Not all solutions/cocycles correspond to distinct infinitesimal moduli.

For example, in SCFT, two marginal operators that differ by a superspace derivative, define the same deformation (though the action itself can change).

**Ex:** in (2,2) theory, changing the Kahler form by an exact 2-form does change the action, but does not deform the SCFT.
In the present case, if two marginal operators $X, X'$ differ by $\bar{D}Y$, for some superfield $\ Y$, then they define the same SCFT deformation.

If $X$ is chiral, then $\bar{D}X = 0$

Note $\bar{D}X' = \bar{D}X + \bar{D}^2Y = 0$

so $X' = X + \bar{D}Y$ is also chiral

Lagrangian changes by $D X' = DX + D\bar{D}Y$

but SCFT unchanged.
Coboundaries resulting from $X \leftrightarrow X + \overline{DY}$:

Dimensions and symmetries require

$$Y = \Gamma_\alpha \Gamma^\beta \lambda^\alpha_\beta + \partial \Phi^i \mu_i + \partial \Phi^i \bar{g}_{\bar{i}i} \zeta^i$$

for some $\lambda^\alpha_\beta, \mu_i, \zeta^i$

Resulting shifts:

$$Z^i_{\bar{i}} \mapsto Z^i_{\bar{i}} + \zeta^i_{\bar{i}}$$

$$Y_{\bar{i}i} \mapsto Y_{\bar{i}i} + \mu_{i\bar{i}} + H_{\bar{i}i\bar{j}} \zeta^j$$

$$\Lambda^\alpha_{\beta \bar{i}} \mapsto \Lambda^\alpha_{\beta \bar{i}} + \lambda^\alpha_{\beta \bar{i}} - F^\alpha_{\beta \bar{i}i} \zeta^i$$
Another source of coboundaries:

\[ X \mapsto X + \partial Y' \quad \text{for chiral } Y' \]

Lagrangian changes by total derivative:

\[ DX \mapsto DX + \partial DY' \]

Dimensional analysis, symmetries imply

\[ Y' = \overline{D\Phi^i}\xi_i \]

Combine with previous action to get full coboundaries:

\[
Z^i_i \sim Z^i_i + (\zeta^i + g^{ij}\xi_j)\xi_i + g^{ik}(\xi_i, k - \xi_k, i) \\
Y_{i\bar{i}} \sim Y_{i\bar{i}} + \mu_{i, \bar{i}} + \xi_{i, \bar{i}} + H_{i\bar{j}j} (\zeta^j + g^{jj}\xi_j) \\
\Lambda_{\alpha\beta\bar{i}} \sim \Lambda_{\alpha\beta\bar{i}} + \lambda_{\beta, \bar{i}} - F_{\alpha\beta\bar{i}i} (\zeta^i + g^{ij}\xi_j)
\]
So far, I’ve discussed moduli.

What about charged matter?

In a heterotic CY compactification, charged matter believed to be counted by

\[ H^* (X, \Lambda^* \mathcal{E}) \]

Is this modified?
Is the spectrum of charged matter modified?

On the one hand, there is a PDE one could write down which would mix states:

\[
    h_{\alpha_1 \cdots \alpha_m}^{[\bar{v}_1 \cdots \bar{v}_n, \bar{v}_{n+1}]} + h'_{\bar{v}_1 \cdots \bar{v}_n \bar{j}} [\alpha_1 \cdots \alpha_{m-1} | \beta | F_{\frac{\alpha_m}{\beta \bar{v}_{n+1}}} \bar{j} = 0
\]

However, the different elements of \( H^*(X, \Lambda^* E) \) typically correspond to different representations of the low-energy gauge group.

Thus, we currently believe no modification to charged matter spectrum, only to singlet matter.
Summary:

* Discussed moduli in heterotic string compactifications.

Issues:

-- what are moduli in non-Kahler cases?
-- worldsheet understanding of recent results of Anderson, Ovrut, et al ?

* Presented solutions:
  a proposal for infinitesimal moduli of all compactifications, including non-Kahler cases.

Thank you for your time!