GLSM's, gerbes, and Kuznetsov's homological projective duality

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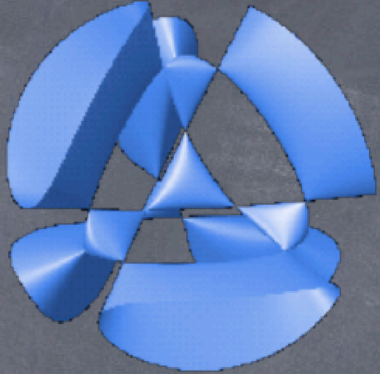
T Pantev, ES, hepth/0502027, 0502044, 0502053
S Hellerman, A Henriques, T Pantev, ES, M Ando, hepth/0606034
R Donagi, ES, arxiv: 0704.1761
A Caldararu, J Distler, S Hellerman, T Pantev, ES, arXiv: 0709.3855
J. Guffin, ES, arXiv: 0801.3836, 0803.3955

Outline

- Basics of string compactifications on stacks
- Cluster decomposition conjecture for strings on gerbes:
  CFT(gerbe) = CFT(disjoint union of spaces)
- Application to GLSM's; realization of Kuznetsov's homological projective duality
- LG models; h.p.d. via matrix factorizations

Stacks



Stacks are a mild generalization of spaces.

One would like to understand strings on stacks:

-- to understand the most general possible string compactifications

-- they often appear physically inside various constructions

Stacks

How to make sense of strings on stacks concretely?

Every* (smooth, Deligne-Mumford) stack can be presented as a global quotient

$$[X/G]$$

for X a space and G a group.

To such a presentation, associate a G-gauged sigma model on X.

(* with minor caveats)

Stacks

If to [X/G] we associate ``G-gauged sigma model,"
then:

$$[\mathbf{C}^2/\mathbf{Z}_2]$$

defines a 2d theory with a symmetry
called conformal invariance

$$=$$

$$\neq$$

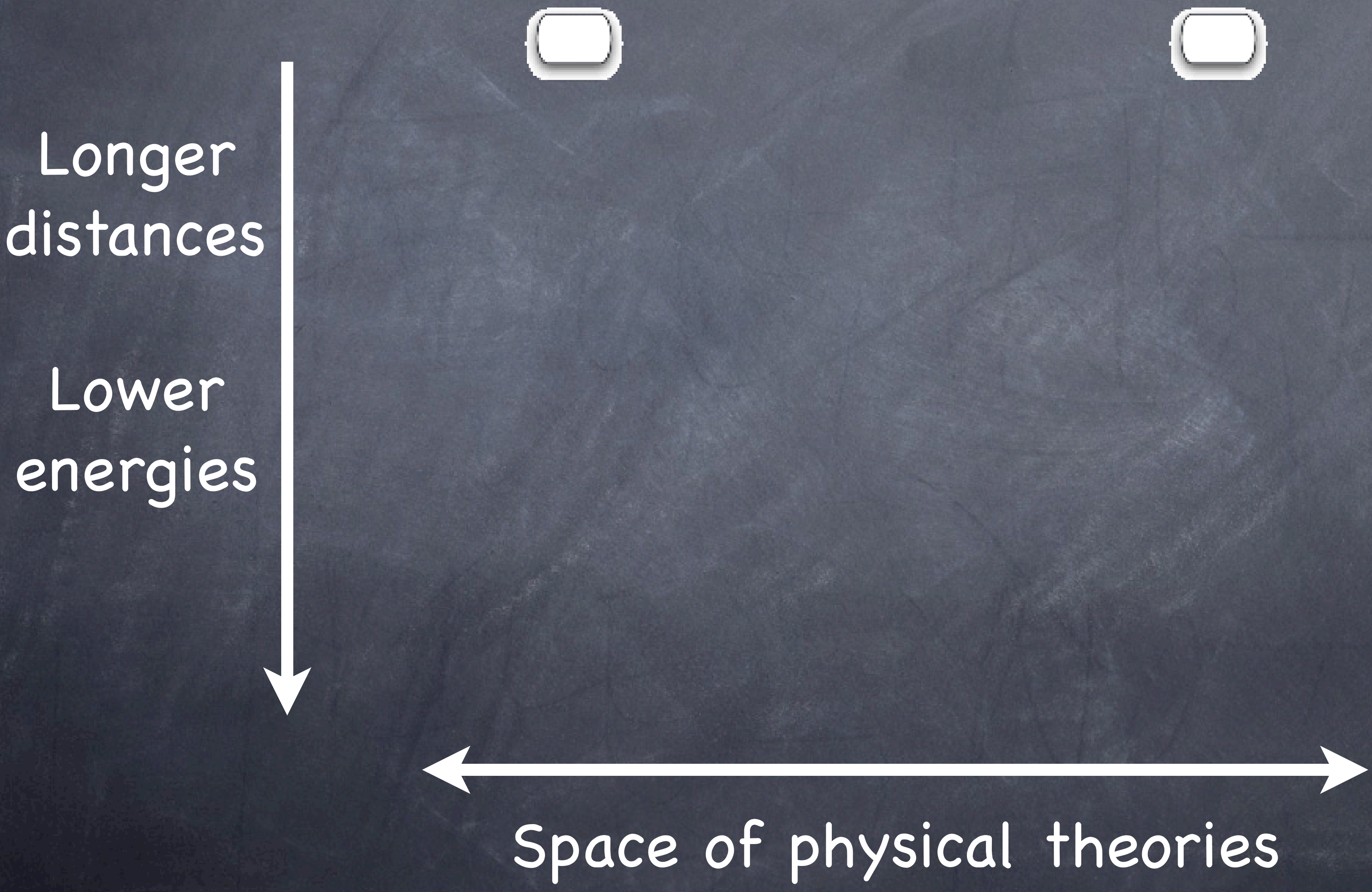
$$[X/\mathbf{C}^\times]$$

$$\left(X = \frac{\mathbf{C}^2 \times \mathbf{C}^\times}{\mathbf{Z}_2} \right)$$

defines a 2d theory
w/o conformal invariance

Potential presentation-dependence problem:
fix with renormalization group flow

Renormalization group



Longer distances

Lower energies

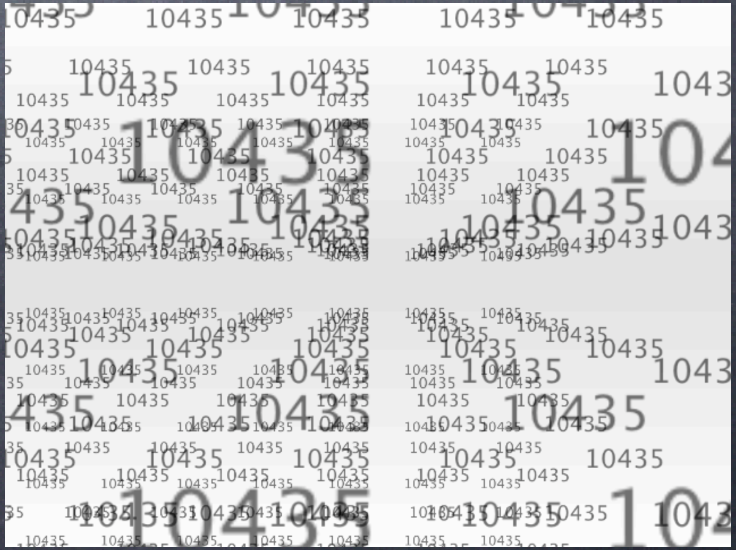
Space of physical theories

Renormalization group

-- is a powerful tool, but unfortunately we really can't follow it completely explicitly in general.

-- can't really prove in any sense that two theories will flow under renormalization group to same point.

Instead, we do lots of calculations,
perform lots of consistency tests,
and if all works out,
then we believe it.



The problems here are analogous to the derived-categories-in-physics program.

There, to a given object in a derived category, one picks a representative with a physical description (as branes/antibranes/tachyons).

Alas, such representatives are not unique.

It is conjectured that different representatives give rise to the same low-energy physics, via boundary renormalization group flow.

Only indirect tests possible, though.

Stacks

Potential problems / reasons to believe that presentation-independence fails:

* Deformations of stacks $\neq$ Deformations of physical theories

* Cluster decomposition issue for gerbes

These potential problems can be fixed. (ES, T Pantev)

Results include: mirror symmetry for stacks, new Landau-Ginzburg models, physical calculations of quantum cohomology for stacks, understanding of noneffective quotients in physics

General decomposition conjecture

Consider $[X/H]$ where

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1$$

and G acts trivially.

We now believe, for (2,2) CFT's,

$$\mathrm{CFT}([X/H]) = \mathrm{CFT}\left(\left[(X \times \hat{G})/K\right]\right)$$

(together with some B field), where $\hat{G}$ is the set of irreps of G

Decomposition conjecture

For banded gerbes, K acts trivially upon $\hat{G}$ so the decomposition conjecture reduces to

$$\mathrm{CFT}(G-\mathrm{gerbe\ on\ }X) \; = \; \mathrm{CFT}\left(\coprod_{\hat{G}} (X,B) \right)$$

where the B field is determined by the image of

$$H^2(X,Z(G)) \xrightarrow{Z(G)\to U(1)} H^2(X,U(1))$$

Banded Example:

Consider $[X/D_4]$ where the center acts trivially.

$$1 \longrightarrow \mathbf{Z}_2 \longrightarrow D_4 \longrightarrow \mathbf{Z}_2 \times \mathbf{Z}_2 \longrightarrow 1$$

The decomposition conjecture predicts

$$\mathrm{CFT}\left([X/D_4]\right) = \mathrm{CFT}\left([X/\mathbf{Z}_2 \times \mathbf{Z}_2] \coprod [X/\mathbf{Z}_2 \times \mathbf{Z}_2] \right)$$

One of the effective orbifolds has vanishing discrete torsion, the other has nonvanishing discrete torsion.

(Using discrete torsion <-> B fields,

ES hepth/0008154, 0008170, 0008184, 0008191, 0302152)

Can check partition functions & more....

A quick check of this example comes from comparing massless spectra:

Spectrum for $[T^6/D_4]$:

$$\begin{array}{ccccccc}
 & & & 2 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 54 & & 0 & \\
2 & & 54 & & 54 & & 2 \\
 & 0 & & 54 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 2 & & &
\end{array}$$

and for each $[T^6/\mathbf{Z}_2 \times \mathbf{Z}_2]$:

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 3 & & 0 & \\
1 & & 51 & & 51 & & 1 \\
 & 0 & & 3 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 51 & & 0 & \\
1 & & 3 & & 3 & & 1 \\
 & 0 & & 51 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Sum matches. ✓

Nonbanded example:

Consider $[X/\mathbf{H}]$ where $\mathbf{H}$ is the eight-element group of quaternions, and a $\mathbf{Z}_4$ acts trivially.

$$1 \longrightarrow < i > (\cong \mathbf{Z}_4) \longrightarrow \mathbf{H} \longrightarrow \mathbf{Z}_2 \longrightarrow 1$$

The decomposition conjecture predicts

$$\mathrm{CFT}([X/\mathbf{H}]) = \mathrm{CFT}\left([X/\mathbf{Z}_2] \coprod [X/\mathbf{Z}_2] \coprod X \right)$$

Straightforward to show that this is true at the level of partition functions, etc.

Another class of examples:
global quotients by nonfinite groups

The banded $\mathbf{Z}_k$ gerbe over $\mathbf{P}^N$
with characteristic class $-1 \bmod k$
can be described mathematically as the quotient

$$\left[\frac{\mathbf{C}^{N+1} - \{0\}}{\mathbf{C}^{\times}} \right]$$

where the $\mathbf{C}^{\times}$ acts as rotations by k times

which physically can be described by a U(1) susy gauge theory with N+1 chiral fields, of charge k

How can this be different from ordinary $\mathbf{P}^N$ model?

The difference lies in nonperturbative effects. (Perturbatively, having nonminimal charges makes no difference.)

Example: Anomalous global U(1)'s

$$\mathbf{P}^{N-1}: \quad U(1)_A \mapsto \mathbf{Z}_{2N}$$

$$\text{Here}: \quad U(1)_A \mapsto \mathbf{Z}_{2kN}$$

Example: A model correlation functions

$$\mathbf{P}^{N-1}: \quad < X^{N(d+1)-1} >= q^d$$

$$\text{Here}: \quad < X^{N(kd+1)-1} >= q^d$$

Example: quantum cohomology

$$\mathbf{P}^{N-1}: \quad \mathbf{C}[x]/(x^N - q)$$

$$\text{Here}: \quad \mathbf{C}[x]/(x^{kN} - q)$$

Different physics

General argument:

Compact worldsheet:

To specify Higgs fields completely, need to specify what bundle they couple to.

If the gauge field $\sim L$
then Φ charge Q implies

$$\Phi \in \Gamma(L^{\otimes Q})$$

Different bundles => different zero modes
=> different anomalies => different physics

For noncpt worldsheets, analogous argument exists.

(Distler, Plesser)

K theory implications

This equivalence of CFT's implies a statement about K theory (thanks to D-branes).

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1$$

If G acts trivially on X

then the ordinary H-equivariant K theory of X

is the same as

twisted K-equivariant K theory of $X \times \hat{G}$

* Can be derived just within K theory

* Provides a check of the decomposition conjecture

D-branes and sheaves

D-branes in the topological B model can be described with sheaves and, more gen'ly, derived categories.

This also is consistent with the decomp' conjecture:

Math fact:

A sheaf on a banded G-gerbe
is the same thing as

a twisted sheaf on the underlying space,
twisted by image of an element of $H^2(X,Z(G))$

which is consistent with the way D-branes should behave according to the conjecture.

D-branes and sheaves

Similarly, massless states between D-branes should be counted by Ext groups between the corresponding sheaves.

Math fact:

Sheaves on a banded G-gerbe decompose according to irrep' of G,
and sheaves associated to distinct irreps have vanishing Ext groups between them.

Consistent w/ idea that sheaves associated to distinct reps should describe D-branes on different components of a disconnected space.

Gromov-Witten prediction

Notice that there is a prediction here for Gromov-Witten theory of gerbes:

GW of $[X/H]$

should match

GW of $\left[(X \times \hat{G})/K\right]$

Works in basic cases:
BG (T Graber), other exs (J Bryan)

Mirrors to stacks

There exist mirror constructions for any model realizable as a 2d abelian gauge theory.

For toric stacks (BCS '04), there is such a description.

Standard mirror constructions now produce character-valued fields, a new effect, which ties into the stacky fan description of (BCS '04).

(ES, T Pantev, '05)

Toda duals

Ex: The ``Toda dual" of **P**^{N} is described by the holomorphic function

$$W \;=\; \exp(-Y_1) + \cdots + \exp(-Y_N) + \exp(Y_1 + \cdots + Y_N)$$

The analogous duals to **Z**$_k$ gerbes over **P**^{N} are described by

$$W \;=\; \exp(-Y_1) + \cdots + \exp(-Y_N) + \Upsilon^n \exp(Y_1 + \cdots + Y_N)$$

where Υ is a character-valued field

(discrete Fourier transform of components in decomp' conjecture)

(ES, T Pantev, '05)

Summary so far:

string compactifications on stacks exist

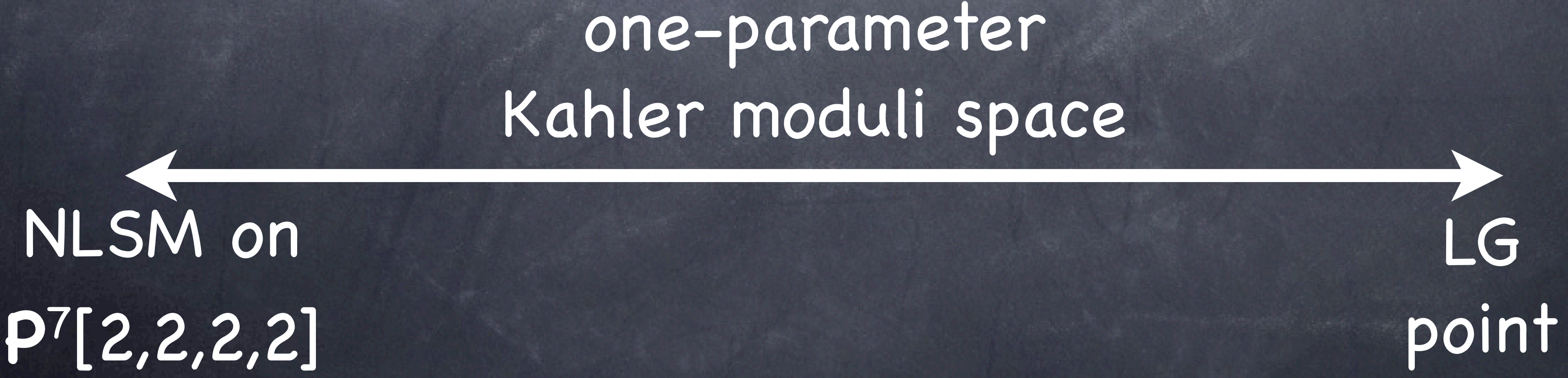
CFT(string on gerbe)
= CFT(string on disjoint union of spaces)

GLSM's

This result can be applied to understand GLSM's.

GLSM's are families of abelian gauge theories that RG flow to families of CFT's.

Example: $\mathbf{P}^7[2,2,2,2]$



one-parameter
Kahler moduli space

NLSM on
$\mathbf{P}^7[2,2,2,2]$

LG
point

GLSM's

Example, cont'd: **P**^{7}[2,2,2,2]

Have 8 fields ϕ_i of charge 1 (homog' coords on **P**^{7}), plus another 4 fields p_a of charge -2.

Superpotential $W = \sum_a p_a G_a(\phi)$

D-terms $D = \sum_i |\phi_i|^2 - 2 \sum_a |p_a|^2 - r$

$r \gg 0$ ==> NLSM on **P**^{7}[2,2,2,2]

GLSM's

Example, cont'd: $\mathbf{P}^7[2,2,2,2]$

At the Landau-Ginzburg point, have superpotential

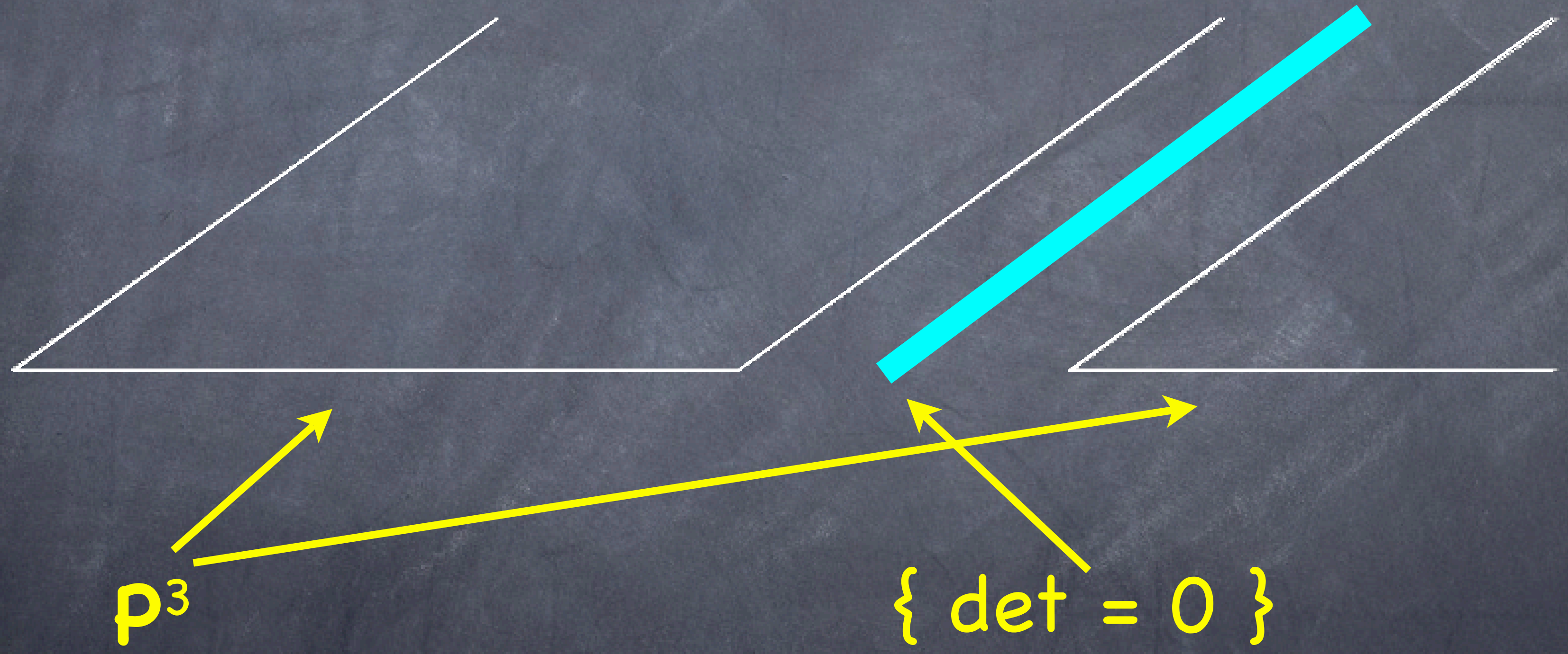
$$\sum_a p_a G_a(\phi) \; = \; \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

* mass terms for the ϕ_i, away from locus $\{\det A = 0\}$.

* leaves just the p fields, of charge -2

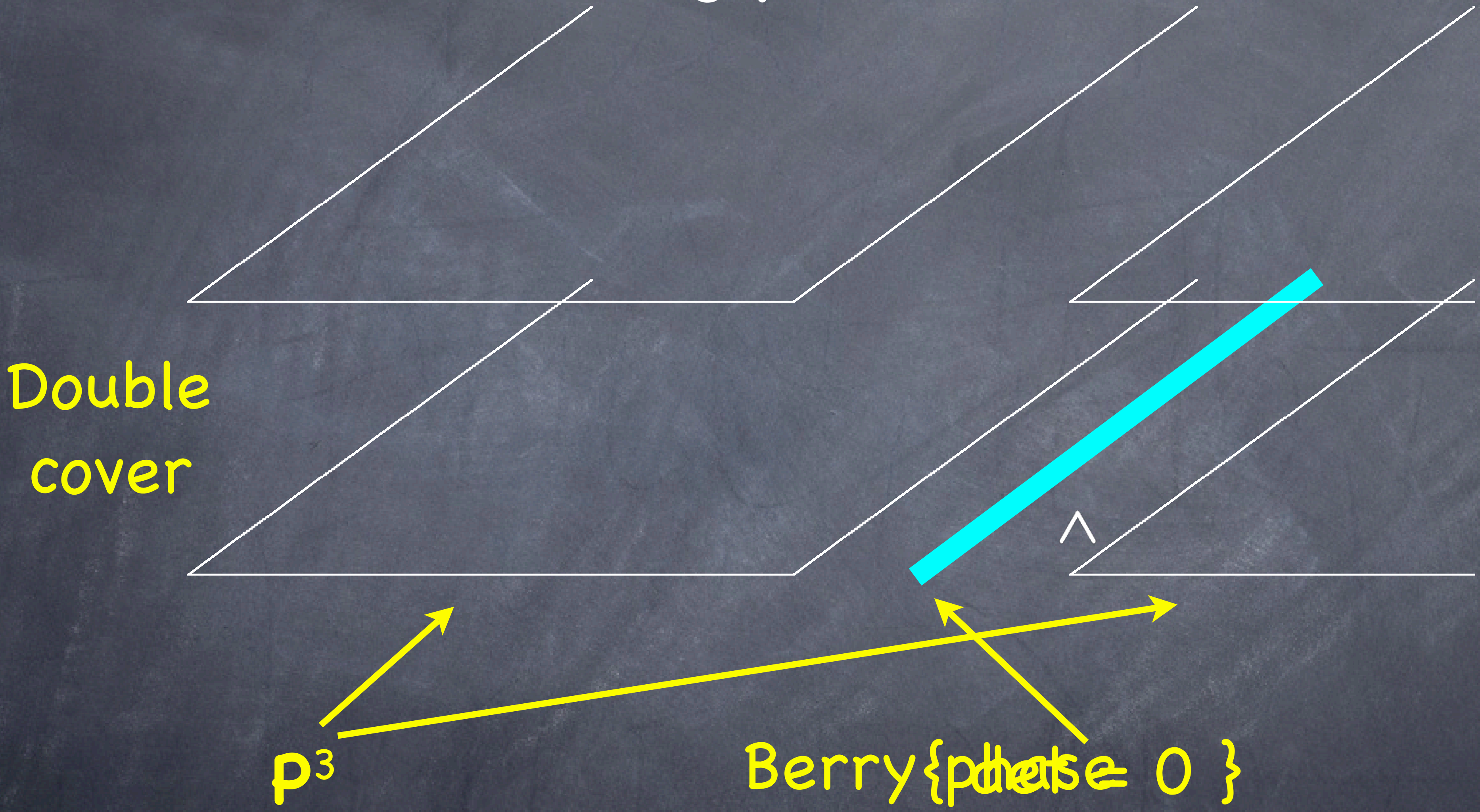
* $\mathbf{Z}_2$ gerbe, hence double cover

The Landau-Ginzburg point:



Because we have a Z_2 gerbe over P^3....

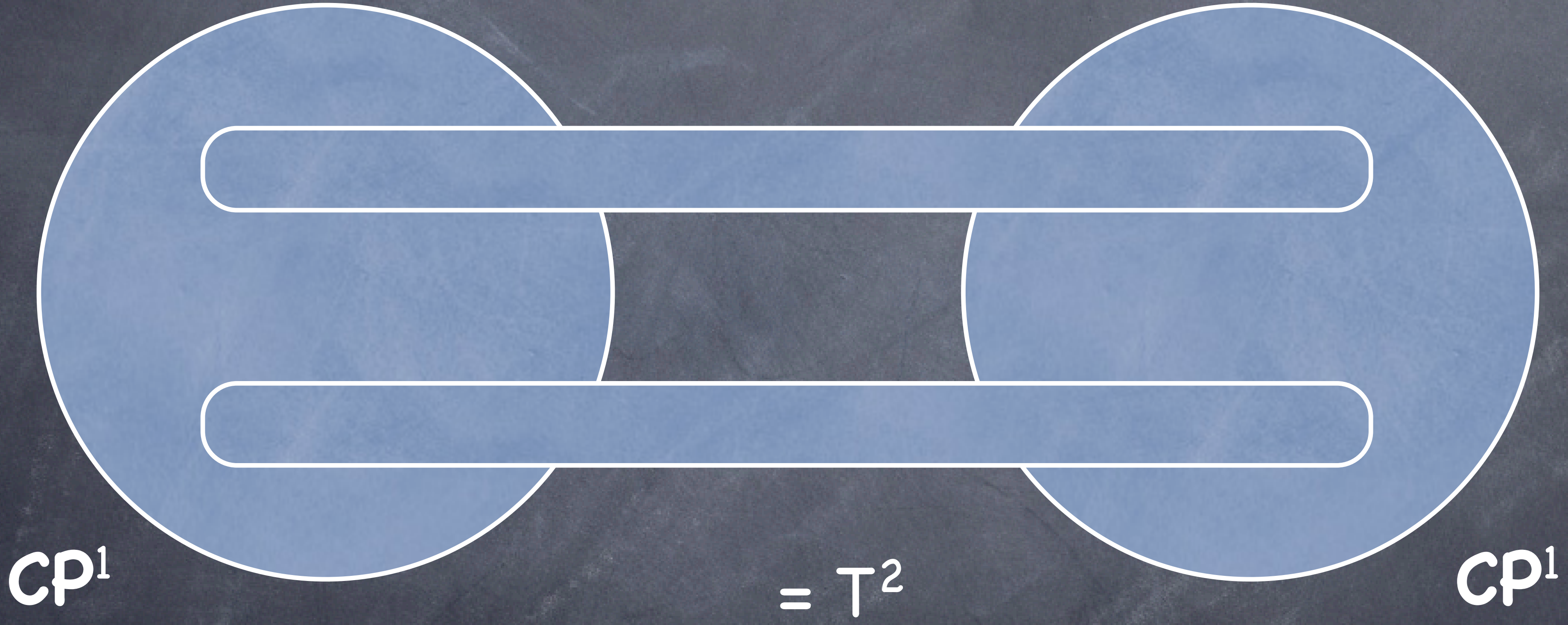
The Landau-Ginzburg point:



Result: branched double cover of $\mathbf{P}^3$

Aside: analogue for GLSM for $\mathbf{P}^3[2,2]$:

Branched double cover of $\mathbf{P}^1$ over deg 4 locus



So a GLSM for $\mathbf{P}^3[2,2]$ relates

$$T^2 \xleftrightarrow{\text{Kahler}} T^2$$

(no surprise)

Back to $\mathbf{P}^7[2,2,2,2]$. Summary so far:

The GLSM realizes:

$\mathbf{P}^7[2,2,2,2]$ $\xleftrightarrow{\text{Kahler}}$ branched double cover of $\mathbf{P}^3$

(Clemens' octic double solid)

where RHS realized at LG point via local $\mathbf{Z}_2$ gerbe structure + Berry phase.

(S. Hellerman, A. Henriques, T. Pantev, ES, M Ando, '06; R Donagi, ES, '07;
A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E.S., arXiv: 0709.3855)

Novel physical realization of geometry

A puzzle:

the branched double cover will be singular,
but the GLSM is smooth at those singularities.

Solution?....

Solution to this puzzle:

We believe the GLSM is actually describing
a `noncommutative resolution' of the branched double
cover worked out by A. Kuznetsov.

Kuznetsov has defined
`homological projective duality'
that relates $\mathbf{P}^7[2,2,2,2]$ to the noncommutative
resolution above,
& we believe the GLSM is physically realizing that
duality.

More detail....

Check that we are seeing K's noncomm' resolution:

K defines a `noncommutative space' via its sheaves
-- so for example, a Landau-Ginzburg model can be a noncommutative space via matrix factorizations.

Here, K's noncomm' res'n is defined by ($\mathbf{P}^3$,B) where B is the sheaf of even parts of Clifford algebras associated with the universal quadric over $\mathbf{P}^3$ defined by the GLSM superpotential.

B plays the role of structure sheaf;
other sheaves are B-modules.

Physics?......

What are the B-branes at the LG point of GLSM?

To answer this, we back up the RG flow to an intermediate point, a Landau-Ginzburg model (ie, integrate out gauge field of GLSM).

$$\begin{tikzcd}[column sep=large]
& \text{GLSM for } \mathbf{P}^n[2,\ldots,2] \arrow[dl, "\text{RG}"] \arrow[dr, "\text{RG}"] & \\
\text{LG model on } \mathrm{Tot}(O(-2)^k \text{ --> } \mathbf{P}^n) \arrow[rr, leftrightarrow, "\text{GLSM}", "\text{Kahler}"'] \arrow[d, "\text{RG}"] & & \text{LG model on } \mathrm{Tot}(O(-1)^{n+1} \text{ --> } \mathbf{P}^{k-1}{}_{[2,\ldots,2]}) \arrow[d, "\text{RG}"] \\
\text{NLSM on } \mathbf{P}^n[2,\ldots,2] \arrow[rr, leftrightarrow, "\text{GLSM}", "\text{Kahler}"'] & & \text{branched double cover of } \mathbf{P}^{k-1}
\end{tikzcd}$$

Then, compute B-branes in LG (= matrix factorizations)

Matrix factorization for a quadratic superpotential: even though the bulk theory is massive, one still has D0-branes with a Clifford algebra structure.

(Kapustin, Li)

Here: a `hybrid LG model' fibered over $\mathbf{P}^3_{[2,2,2]}$, gives sheaves of Clifford algebras (determined by the universal quadric / GLSM superpotential) and modules thereof.

* Physics is clear (= Born-Oppenheimer)
(but math proof still needed)

* **matches** sheaves in K's noncomm' res'n.

Note we have a physical
realization of nontrivial examples of Kontsevich's
`noncommutative spaces'
realized in gauged linear sigma models.

Furthermore, after `backing up' RG flow to
Landau-Ginzburg models,
h.p.d. (on linear sections) becomes an
Orlov/Walcher/Hori-type equivalence of matrix
factorizations in LG models on birational spaces.

(? Kuznetsov = Orlov ?)

Other notes:

* It is now possible in principle to compute GW invariants of a noncommutative resolution
-- compute them in the LG model upstairs,
use the fact that A model is invariant under RG.

(Guffin, ES, 0801.3836, 0801.3955)

* We applied Born-Oppenheimer very briefly here;
it also implies a more general statement,
that matrix factorizations `behave nicely' in families

Summary so far:

The GLSM realizes:

$$\begin{tikzcd} P^7[2,2,2,2] \arrow[r, leftrightarrow, "\text{Kahler}"] & \text{branched double cover of } \mathbf{P}^3 \end{tikzcd}$$

where RHS realized at LG point via local $\mathbf{Z}_2$ gerbe structure + Berry phase.

(A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E.S., arXiv: 0709.3855)

Novel physical realization of geometry

Non-birational twisted derived equivalence

Physical realization of Kuznetsov's homological projective duality

More examples:

CI of 2 quadrics in the total space of

$$\mathbf{P}\left(\mathcal{O}(-1,0)^{\oplus 2} \oplus \mathcal{O}(0,-1)^{\oplus 2}\right) \longrightarrow \mathbf{P}^1 \times \mathbf{P}^1$$

$$\xleftrightarrow{\text{Kahler}}$$

branched double cover of **P**^{1}x**P**^{1}x**P**^{1},
branched over deg (4,4,4) locus

* In fact, the GLSM has 8 Kahler phases,
4 of each of the above.

* Related to an example of Vafa-Witten involving
discrete torsion

(Caldararu, Borisov)

* Believed to be homologically projective dual

A non-CY example:

CI 2 quadrics
in $\mathbf{P}^{2g+1}$

$\xleftrightarrow{\text{Kahler}}$

branched double
cover of $\mathbf{P}^1$,
over deg 2g+2
(= genus g curve)

Homologically projective dual.

Here, r flows under RG -- not a const parameter.

Semiclassically, Kahler moduli space falls apart into 2 chunks.

Positively
curved

Negatively
curved

r flows: $\dashrightarrow$ $\dashrightarrow$ $\dashrightarrow$

Depending upon the cutoff,
can replace branched double cover
by a space with codim 1 orbifolds.

Have double cover outside of cutoff-sized sphere
about the branch locus.

As the cutoff varies,
interpolate between

- * branched double cover
- * codim 1 $\mathbf{Z}_2$ orbifold

$$\longleftarrow \quad \Lambda \quad \longrightarrow$$

Another non-CY example:

CI 2 quadrics
in $\mathbf{P}^4$ $\xleftrightarrow{\text{Kahler}}$ $\mathbf{P}^1$ w/ 5 $\mathbf{Z}_2$ singularities
(= deg 4 del Pezzo)

Why codim 1 sing' instead of a double cover?
Well, no double cover exists, only the other cutoff limit makes sense.

Homologically projective dual

Analogous results for $\mathbf{P}^6[2,2,2]$, $\mathbf{P}^6[2,2,2,2]$

Aside:

One of the lessons of this analysis is that
gerbe structures are commonplace,
even generic,
in the hybrid LG models arising in GLSM's.

To understand the LG points of typical GLSM's,
requires understanding gerbes in physics.

So far we have discussed several GLSM's s.t.:

* the LG point realizes geometry in an unusual way

* the geometric phases are not birational

* instead, related by Kuznetsov's homological projective duality

We conjecture that Kuznetsov's homological projective duality applies much more generally to GLSM's.....

More Kuznetsov duals:

Another class of examples, also realizing Kuznetsov's h.p.d., were realized in GLSM's by Hori-Tong.

$$\begin{tikzcd} G(2,7)[1^7] \arrow[r, leftrightarrow, "\text{Kahler}"] & \text{Pfaffian CY} \end{tikzcd}$$

(Rodland, Kuznetsov, Borisov-Caldararu, Hori-Tong)

* unusual geometric realization
(via strong coupling effects in nonabelian GLSM)

* non-birational

More Kuznetsov duals:

$G(2,N)[1^m]$ (N odd) $\xleftrightarrow{\text{Kahler}}$ vanishing locus in $\mathbf{P}^{m-1}$ of Pfaffians

Check r flow:

$K = O(m-N)$ $K = O(N-m)$

Opp sign, so r flows in same direction,
consistent with GLSM.

r flows: ⇢ ⇢ ⇢

More Kuznetsov duals:

So far we have discussed how Kuznetsov's h.p.d. realizes Kahler phases of several GLSM's with exotic physics.

We conjecture it also applies to ordinary GLSM's.

Ex: flops

Some flops are already known to be related by h.p.d.; K is working on the general case.

Summary

- Basics of string compactifications on stacks
- Cluster decomposition conjecture for strings on gerbes:
  CFT(gerbe) = CFT(disjoint union of spaces)
- Application to GLSM's; realization of Kuznetsov's homological projective duality
- LG models; h.p.d. & matrix factorizations

Mathematics		**Physics**
Geometry: Gromov-Witten Donaldson-Thomas quantum cohomology etc	⇔	Supersymmetric field theories
Homotopy, categories: derived categories, stacks, etc.	⇔	Renormalization group