Analytic Continuation

See Arfken & Weber pp 432-434 (in section 6.5 on Laurent expansions) for some of the material below. Our description here will closely follow [1].

1 Definition

The intersection of two domains (regions in the complex plane) $D_1$, $D_2$, denoted $D_1 \cap D_2$, is the set of all points common to both $D_1$ and $D_2$. The union of two domains $D_1$, $D_2$, denoted $D_1 \cup D_2$, is the set of all points in either $D_1$ or $D_2$.

Now, suppose you have two domains $D_1$ and $D_2$, such that the intersection is nonempty and connected, and a function $f_1$ that is analytic over the domain $D_1$. If there exists a function $f_2$ that is analytic over the domain $D_2$ and such that $f_1 = f_2$ on the intersection $D_1 \cap D_2$, then we say $f_2$ is an analytic continuation of $f_1$ into domain $D_2$.

Now, whenever an analytic continuation exists, it is unique. The reason for this is a basic mathematical result from the theory of complex variables:

A function that is analytic in a domain $D$ is uniquely determined over $D$ by its values over a domain, or along an arc, interior to $D$.

Define the function $F(z)$, analytic over the union $D_1 \cup D_2$, as

$$F(z) = \begin{cases} f_1(z) & \text{when } z \text{ is in } D_1 \\ f_2(z) & \text{when } z \text{ is in } D_2 \end{cases}$$

In other words, $F$ is given by $f_1$ over $D_1$ and by $f_2$ over $D_2$, and since $f_1 = f_2$ over the intersection of $D_1$ and $D_2$, this is a well-defined, holomorphic function. By the mathematical result quoted above, since $F$ is analytic in $D_1 \cup D_2$, it is uniquely determined by $f_1$ on $D_1$. (For that matter, it is also uniquely determined by $f_2$ on $D_2$.) In other words, there is only one possible holomorphic function on $D_1 \cup D_2$ that matches $f_1$ on $D_1$.

In this case, the function $F(z)$ is said to be the analytic continuation over $D_1 \cup D_2$ of either $f_1$ or $f_2$.

Example: Consider first the function

$$f_1(z) = \sum_{n=0}^{\infty} z^n$$

This power series converges when $|z| < 1$ to $1/(1 - z)$, and is not defined when $|z| \geq 1$. (In particular, this is just a geometric series, so we can sum it as a geometric series, so long as we’re in the region of convergence.)
On the other hand, the function 

$$f_2(z) = \frac{1}{1 - z}$$

is defined and analytic everywhere except $z = 1$.

Since $f_1 = f_2$ on the disk $|z| < 1$, we can view $f_2$ as the analytic continuation of $f_1$ to the rest of the complex plane (minus the point $z = 1$).

**Example:** Consider the function

$$f_1(z) = \int_0^\infty \exp(-zt)\,dt$$

This integral exists only when $\text{Re } z > 0$, and for such $z$, this integral has value $1/z$.

Since the function $1/z$ matches $f_1$ on the domain $\text{Re } z > 0$, the function $1/z$ is the analytic continuation of $f_1$ to nonzero complex numbers.

While we're at it, define

$$f_2(z) = i \sum_{n=0}^\infty \left(\frac{z + i}{i}\right)^n$$

This series converges for $|z + i| < 1$, and so $f_2$ is defined only within that disk centered on $-i$. Within that unit disk, one can show that $f_2(z) = 1/z$, using the fact that the series is a geometric series.

Since $f_2$ matches $1/z$ on a disk, we can view $1/z$ as the analytic continuation of $f_2$ to nonzero complex numbers.

Also, we can view $f_2$ as the analytic continuation of $f_1$ to the disk $|z + i| < 1$.

**Example:** The Gamma function.

Recall the second definition of the Gamma function,

$$\Gamma(z) = \int_0^\infty \exp(-t)t^{z-1}\,dt$$

is valid for $\text{Re } z > 0$. Other definitions, such as the Weierstrass form

$$\frac{1}{\Gamma(z)} = z\exp(\gamma z) \prod_{n=1}^\infty \left(1 + \frac{z}{n}\right)\exp(-z/n)$$

are valid more generally. Thus, we can view the Weierstrass form as an analytic continuation of the Euler integral form.
## 2 Exercises (taken from [1])

1. Show that the holomorphic function

\[ f_2(z) = \frac{1}{z^2 + 1} \]

\((z \neq \pm i)\) is the analytic continuation of the function

\[ f_1(z) = \sum_{n=0}^{\infty} (-)^n z^{2n} \]

\((|z| < 1)\) into the domain consisting of all points in the \(z\) plane except \(z = \pm i\).

2. Show that the function \(f_2(z) = 1/z^2\) \((z \neq 0)\) is the analytic continuation of the function

\[ f_1(z) = \sum_{n=0}^{\infty} (n + 1)(z + 1)^n \]

\((|z + 1| < 1)\) into the domain consisting of all points in the \(z\) plane except \(z = 0\).

3. Find the analytic continuation of the function

\[ f(z) = \int_0^{\infty} t \exp(-zt)dt \]

\((\text{Re} \: z > 0)\) into the domain consisting of all points in the \(z\) plane except the origin.

4. Show that the function \(1/(z^2 + 1)\) is the analytic continuation of the function

\[ f(z) = \int_0^{\infty} \exp(-zt)(\sin t) \: dt \]

\((\text{Re} \: z > 0)\) into the domain consisting of all points in the \(z\) plane except \(z = \pm i\).

## References