Riemann Surfaces

See Arfken & Weber & Harris section 11.10 (mapping) or [1][sections 106-108] for more information.

1 Definition

A *Riemann surface* is a generalization of the complex plane to a surface of more than one sheet such that a multiple-valued function on the original complex plane has only one value at each point of the surface. Briefly, the notion of a Riemann surface is important whenever considering functions with branch cuts.

Perhaps the easiest example is the Riemann surface for $\log z$. Write $z = r\exp(i\theta)$, then $\log z = (\log r) + i\theta$. However, the θ in that expression is not uniquely defined – it is only defined up to a multiple of 2π. As you walk around the origin of the complex plane, the function $\log z$ comes back to itself up to additions of $2\pi i$.

We can construct a cover of the complex plane on which $\log z$ is single-valued by taking a helix over the complex plane – going $360°$ about the origin takes from you from sheet of the cover to another sheet. Put another way, every time you go through the branch cut, you go to a different sheet.

We can construct such a cover as follows. (Our discussion here is verbatim from [1][section 106].) Consider the complex z plane, with the origin deleted, as a sheet R_0 which is cut along the positive half of the real axis. On that sheet, let θ range from 0 to 2π. Let a second sheet R_1 be cut in the same way and placed in front of the sheet R_0. The lower edge of the slit in R_0 is joined to the upper edge of the slit in R_1. On R_1, θ ranges from 2π to 4π.

Proceeding in this fashion, with infinitely many sheets, we succeed in building up a surface on which $\log z$ is single-valued instead of multi-valued.

Another example is obtained from the function $\sqrt{z}$. Following [1][section 107], you can construct a Riemann surface for this function from two sheets, R_0 and R_1, each cut along the positive real axis. The lower edge of the slit in R_0 is joined to the upper edge of the slit in R_1, and the lower edge of the slit in R_1 is joined to the upper edge of the slit in R_0. As a point z starts from the upper edge of the slit in R_0 and walks around the origin in a counterclockwise direction, the angle θ goes from 0 to 2π. The point then passes from sheet R_0 to R_1, where θ increases from 2π to 4π, after which it passes back to R_0, where the values of *theta* can be said to vary either from 4π or 6π or 0 to 2π – the choice does not affect the value of $z^{1/2}$.

The origin is common to both sheets R_0 and R_1, and a curve around the origin must wind around it twice in order to be a closed curve. Such a point is known as a "branch point" – the Riemann surface is a 2-fold cover of the original complex plane, branched over the origin.

A third example is obtained from the function

$$\left(z^2 - 1\right)^{1/2}$$

This function has zeroes at ± 1, and a branch cut running between those two points. A Riemann surface for this function consists of two sheets, R_0 and R_1. Both sheets are cut along the line segment between ± 1. The lower edge of the slit in R_0 is joined to the upper edge of the slit in R_1, and the lower edge in R_1 to the upper edge in R_0.

Write

$$\begin{aligned} z - 1 &= r_1 \exp(i\theta_1) \\ z + 1 &= r_2 \exp(i\theta_2) \end{aligned}$$

On the sheet R_0, let the angles θ_1, θ_2 range from 0 to 2π. If a curve encloses the line segment between ± 1, then both θ_1 and θ_2 change by 2π. The change in $(\theta_1 + \theta_2)/2$ is also 2π.

A curve which starts on the sheet R_0 and passes around $+1$ twice, passing through the branch cut, then it crosses from R_0 onto R_1 and back again, θ_1 changes by 4π, and θ_2 doesn't change at all.

A fourth example is obtained from the function

$$f(z) = \left(z(z^2 - 1)\right)^{1/2}$$

There are three branch points of this function, at $0, \pm 1$. Write

$$\begin{aligned} z &= r \exp(i\theta) \\ z - 1 &= r_1 \exp(i\theta_1) \\ z + 1 &= r_2 \exp(i\theta_2) \end{aligned}$$

A closed curve that encloses all three branch points will have the argument of $f(z)$ change by 3π (after all, for large $|z|$, $f(z) \sim z^{3/2}$, so under $arg(z) \mapsto arg(z) + 2\pi$, $arg f(z) \mapsto arg f(z) + (3/2)(2\pi)$. Thus, there must be a branch cut running from one of the points out to infinity..

Take one branch cut to run between -1 and 0, and the other branch cut to run along the real line from 1 to $+\infty$. Cut two sheets along those two branch cuts, and join them together in the same fashion as before. You can check that the result is a Riemann surface on which $f(z)$ is single-valued.

As a cultural note, the resulting Riemann surface can also be understood as a large open patch on a 2-torus T^2.

2 Compact examples

So far we have discussed covers of the complex plane, branched over some number of points. More generally, by adding a point at infinity, one can "compactify" the complex plane to a 2-sphere, and the Riemann surfaces above become two-dimensional surfaces with handles. There has been a tremendous amount of work done on two-dimensional surfaces with complex structure (known as Riemann surfaces); see for example Farkas-Kra mentioned in the references at the end.

3 Exercises (from [1])

1. Describe a Riemann surface for the triple-valued function $w = (z-1)^{1/3}$, and point out which third of the w plane represents the image of each sheet of that surface.

2. Describe the curve, on a Riemann surface for $z^{1/2}$, whose image is the entire cirlce $|w| = 1$ under the transformation $w = z^{1/2}$.

3. Describe a Riemann surface for the multiple-valued function

$$f(z) \;=\; \left(\frac{z-1}{z}\right)^{1/2}$$

4. Let C describe the positively oriented circle $|z-2| = 1$ on the Riemann surface described above for $z^{1/2}$, where the upper half of the circle lies on the sheet R_0 and the lower half on R_1. Note that, for each point z on C, one can write

$$z^{1/2} = \sqrt{r}\exp(\pm i\theta/2), \quad \text{where } 4\pi - \frac{\pi}{2} < \theta < 4\pi + \frac{\pi}{2}$$

State why it follows that

$$\int_C z^{1/2} dz \;=\; 0$$

Generalize this result to fit the case of other simple closed curves that cross from one sheet to another without enclosing the branch point.

References

[1] R. Churchill, J. Brown, *Complex variables and applications*, 5th edition, McGraw-Hill, 1990.

[2] H. Farkas, I. Kra, *Riemann surfaces*, Springer-Verlag, 1980.