Appendix D

Asymptotic series

Asymptotic series play a crucial role in understanding quantum field theory, as Feynman diagram expansions are typically asymptotic series expansions. As I will occasionally refer to asymptotic series, I have included in this appendix some basic information on the subject.

See [AW] sections 5.9, 5.10, 7.3 (7.4 in 5th edition), 8.3 (10.3 in 5th edition) for some of the material below.

D.1 Definition

By now as graduate students you have seen infinite series appear many times. However, in most of those appearances, you have probably made the assumption that the series converged, or that the series is only useful when convergent.

Asymptotic series are non-convergent series, that nevertheless can be made useful, and play an important role in physics. The infinite series one gets in quantum field theory by summing Feynman diagrams, for example, are asymptotic series.

To be precise, consider a function $f(z)$ with an expansion as

$$f(z) \;=\; A_0 \,+\, \frac{A_1}{z} \,+\, \frac{A_2}{z^2} \,+\, \cdots$$

where the A_i are numbers. We can think of the series $\sum A_i/z^i$ as approximating $f(z)/\varphi(z)$ for large values of z.

We say that the series $\sum A_i/z^i$ represents $f(z)$ *asymptotically* in direction $e^{i\phi}$ if, for a given n, the first n terms of the series may be made as close as desired to $f(z)$ by making $|z|$ large enough with $\arg z$ fixed to ϕ, *i.e.*

$$\lim_{|z|\rightarrow\infty} z^n \left[f(z) \,-\, \sum_{p=0}^{n} \frac{A_p}{z^p} \right] \;=\; 0$$

(Put another way, write $z = re^{i\phi}$, then take the limit as $r \to \infty$ but hold ϕ fixed.) We shall see later that as one varies the direction $e^{i\phi}$, one can get different asymptotic series expansions for the same function – this is known as Stokes' phenomenon, and we shall study it in section *** CITE ***.

Asymptotic series need not converge; in fact, in typical cases of interest, an asymptotic series will never converge. (Nonconvergence is sometimes added to the definition of asymptotic series, so that, in that alternate definition, an asymptotic series can never converge. In our definition here, convergence is allowed, albeit it is unusual.)

It is important to note that asymptotic series are distinct from convergent series: a convergent series need not be asymptotic. For example, consider the Taylor series for $\exp(z)$. This is a convergent power series, but the same power series does *not* define an asymptotic series for $\exp(z)$. After all,

$$\lim_{z \to \infty} z^n \left[\exp(z) \; - \; \sum_{p=0}^{n} \frac{z^p}{n!} \right] \; \to \; \infty$$

and so the series is not asymptotic to $\exp(z)$, though it does converge to $\exp(z)$.

Not all functions have an asymptotic expansion; $\exp(z)$ is one such function. If a function does have an asymptotic expansion, then that asymptotic expansion is unique. However, several different functions can have the same asymptotic expansion; the map from functions to asymptotic expansions is many-to-one, when it is well-defined.

Example: Consider the function

$$-\mathrm{Ei}(-x) \; = \; E_1(x) \; = \; \int_x^\infty \frac{\exp(-t)}{t} dt$$

We can generate a series approximating this function by a series of integrations by parts:

$$\begin{aligned} E_1(x) \; &= \; \frac{\exp(-x)}{x} \; - \; \int_x^\infty \frac{\exp(-t)}{t^2} dt \\ &= \; \frac{\exp(-x)}{x} \left[1 \; - \; \frac{1}{x} \; + \; \frac{2!}{x^2} \; - \; \frac{3!}{x^3} \; + \; \cdots \; \frac{(-)^n n!}{x^n} \right] \; + \; (-)^{n+1} (n+1)! \int_x^\infty \frac{\exp(-t)}{t^{n+2}} dt \end{aligned}$$

The series

$$\sum_{n=0}^{\infty} (-)^n \frac{n!}{x^n}$$

is not convergent, in any standard sense. For example, when $x = 1$, this is the series

$$1 \; - \; 2! \; + \; 3! \; - \; 4! \; + \; \cdots$$

More generally, for any fixed x the magnitude of the terms *increases* as n grows, so this alternating series necessarily diverges.

However, although the series diverges, it is asymptotic to $E_1(x) x \exp(x)$. To show this, we must prove that for fixed n,

$$\lim_{x\to\infty} x^n \left[E_1(x) \; - \; \frac{\exp(-x)}{x} \sum_{p=0}^{n} \frac{(-)^n n!}{x^n} \right] \; = \; 0$$

Using our expansion from successive integrations by parts, we have that the limit is given by

$$\lim_{x\to\infty} x^n \left[(-)^{n+1} (n+1)! \int_x^\infty \frac{\exp(-t)}{t^{n+2}} dt \right]$$

We can evaluate this limit using the fact that

$$\int_x^\infty \frac{\exp(-t)}{t^{n+2}} dt \; < \; \frac{1}{x^{n+2}} \int_x^\infty \exp(-t) dt \; = \; \frac{\exp(-x)}{x^{n+2}}$$

Since

$$\lim_{x\to\infty} \frac{x^n (n+1)! \exp(-x)}{x^{n+2}} \; = \; 0$$

we see that the series is asymptotic.

Example: Consider the ordinary differential equation

$$y' \; + \; y \; = \; \frac{1}{x}$$

The solutions of this ODE have an asymptotic expansion, as we shall now verify.

To begin, assume that the solutions have a power series expansion of the form

$$y(x) \; = \; \sum_{n=0}^{\infty} \frac{a_n}{x^n}$$

for some constants a_n. Plugging this ansatz into the differential equation above and solving for the coefficients, we find

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_2 &= a_1 \; = \; 1 \\ a_3 &= 2a_2 \; = \; 2 \\ a_4 &= 3a_3 \; = \; 3! \\ a_5 &= 4a_4 \; = \; 4! \end{aligned}$$

and so forth, leading to the expression

$$y(x) \; = \; \sum_{n=1}^{\infty} \frac{(n-1)!}{x^n}$$

First, let us check convergence of this series. Apply the ratio test to find

$$\lim_{n \to \infty} \frac{n!/x^{n+1}}{(n-1)!/x^n} \; = \; \lim_{n \to \infty} \frac{n}{x} \; \longrightarrow \; \infty$$

In particular, by the ratio test, this series diverges for all x (strictly speaking, all x for which it is well-defined, *i.e.* all $x \neq 0$).

We can derive this asymptotic series in an alternate fashion, which will explain its close resemblance to the previous example. Recall the method of variation of parameters for solving inhomogeneous equations: first find the solutions of the associated homogeneous equations, then make an ansatz that the solution to the inhomogeneous equation is given by multiplying the solutions to the homogeneous solutions by functions of x. In the present case, the associated homogeneous equation is given by

$$y' \, + \, y \; = \; 0$$

which has solution $y(x) \propto \exp(-x)$. Following the method of variation of parameters, we make the ansatz

$$y(x) \; = \; A(x) \exp(-x)$$

for some function $A(x)$, and plug back into the (inhomogeneous) differential equation to solve for $A(x)$. In the present case, that yields

$$A' \exp(-x) \; = \; \frac{1}{x}$$

which we can solve as

$$A(x) \; = \; \int_{-\infty}^{x} \frac{\exp(t)}{t} dt$$

(Note that I am implicitly setting a value of the integration constant by setting a lower limit of integration. Also note that the integral above is ill-defined if x is positive, a matter I will gloss over for the purposes of this discussion.) Thus, the solution to the inhomogeneous equation is given by

$$y(x) \; = \; \exp(-x) \int_{-\infty}^{x} \frac{\exp(t)}{t} dt$$

whose resemblance to the previous example should now be obvious.

D.2 The gamma function and the Stirling series

**** See also 0711.4412, which claims to have a very simple proof of Stirling's formula.

An important example of an asymptotic series is the asymptotic series for the gamma function, known as the Stirling series. The gamma function is a meromorphic function on the complex plane that generalizes the factorial function. Denoted $\Gamma(z)$, it has the properties

$$\begin{aligned} \Gamma(z+1) &= z\Gamma(z) \\ \Gamma(1/2) &= \sqrt{\pi} \\ \Gamma(1) &= 1 \\ \Gamma(n+1) &= n! \text{ for } n \text{ a positive integer} \end{aligned}$$

Because of that last property, because the gamma function generalizes the factorial function, people sometimes define $z! \equiv \Gamma(z+1)$ for any complex number z. The gamma function has simple poles at $z = 0, -1, -2, -3, \cdots$. It also obeys numerous curious identities, including

$$\Gamma(z)\Gamma(1-z) \;=\; \frac{\pi}{\sin \pi z}$$

and "Legendre's duplication formula"

$$\Gamma(1+z)\Gamma(z+1/2) \;=\; 2^{-2z}\sqrt{\pi}\Gamma(2z+1)$$

The gamma function has several equivalent definitions. It can be expressed as an integral, using an expression due to Euler:

$$\Gamma(z) \;=\; \int_0^\infty \exp(-t) t^{z-1} dt, \quad \Re z > 0$$

It can also be expressed as a limit, using another expression also due to Euler:

$$\Gamma(z) \;=\; \lim_{n\to\infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{z(z+1)(z+2)\cdots(z+n)} n^z, \quad n \neq 0, -1, -2, -3, \cdots$$

It can also be expressed as an infinite product, using an expression due to Weierstrass:

$$\frac{1}{\Gamma(z)} \;=\; z \exp(\gamma z) \prod_{n=1}^{\infty} \left(1 \;+\; \frac{z}{n}\right) \exp(-z/n)$$

where γ is the Euler-Mascheroni constant

$$\begin{aligned} \gamma \;&\equiv\; \lim_{n\to\infty} \left(\sum_{m=1}^{n} \frac{1}{m} \;-\; \log n \right) \\ &\approx\; 0.577215661901 \cdots \end{aligned}$$

where we use log to denote the natural logarithm. (The regions of validity of each definition are slightly different; analytic continuation defines the function globally.)

We can also define the digamma and polygamma functions, which are various derivatives of the gamma function. The digamma function, denoted either ψ or F, is defined by

$$\psi(z+1) \;\equiv\; F(z) \;\equiv\; \frac{d}{dz} \log \Gamma(z+1)$$

It can be shown that

$$\psi(z+1) \;=\; -\gamma \;+\; \sum_{n=1}^{\infty} \frac{z}{n(n+z)}$$

The polygamma function $\psi^{(n)}$ is defined to be a higher-order derivative:

$$\psi^{(n)}(z+1) \;\equiv\; F^{(n)}(z) \;\equiv\; \frac{d^{n+1}}{dz^{n+1}} \log \Gamma(z+1)$$

The Stirling series for the gamma function is derived using the Euler-Maclaurin integration formula, which we shall digress briefly to explain. (See section 5.9 of [AW] for this background, and section 8.3 on the resulting asymptotic series, known as the Stirling series.)

First, we derive the Euler-Maclaurin integration formula, which we will use to derive an asymptotic series for the gamma function. Consider the integral

$$\int_0^1 f(x)dx \;=\; \int_0^1 f(x)B_0(x)dx$$

where $B_0(x)$ is a Bernoulli function (**** SEE section ****). The idea is to use the relation $B_n'(x) = nB_{n-1}(x)$ between the Bernoulli functions and successive integrations by parts to generate an infinite series. At the first step, use $B_1'(x) = B_0(x)$ and integrate by parts to get

$$\begin{aligned}
\int_0^1 f(x)dx &= \int_0^1 f(x)B_1'(x)dx \\
&= f(x)B_1(x)]_0^1 \;-\; \int_0^1 f'(x)B_1(x)dx \\
&= \frac{1}{2}\left(f(1)+f(0)\right) \;-\; \frac{1}{2}\int_0^1 f'(x)B_2'(x)dx \\
&= \frac{1}{2}\left(f(1)+f(0)\right) \;-\; \frac{1}{2}\left[f'(x)B_2(x)\right]_0^1 \;+\; \frac{1}{2}\int_0^1 f''(x)B_2(x)dx
\end{aligned}$$

and so forth. Continuing this process, we find

$$\begin{aligned}
\int_0^1 f(x)dx &= \frac{1}{2}\left[f(1)+f(0)\right] \;-\; \sum_{p=1}^{q}\frac{B_{2p}}{(2p)!}\left(f^{(2p-1)}(1) - f^{(2p-1)}(0)\right) \\
&\quad + \frac{1}{(2q)!}\int_0^1 f^{(2q)}(x)B_{2q}(x)dx
\end{aligned}$$

where the B_{2p} in the sum above are the Bernoulli numbers, not the functions, and where we have used the relations

$$\begin{aligned}
B_{2n}(1) \;&=\; B_{2n}(0) \;=\; B_{2n} \\
B_{2n+1}(1) \;&=\; B_{2n+1}(0) \;=\; 0
\end{aligned}$$

(Compare equation (5.168a) in [AW].)

***** Move that to earlier?

By replacing $f(x)$ with $f(x+1)$ we can shift the integration region from $[0,1]$ to $[1,2]$, and by adding up results for different regions, we finally get the Euler-Maclaurin integration formula:

$$\begin{aligned}
\int_0^n f(x)dx &= \frac{1}{2}f(0) \;+\; f(1) \;+\; f(2) \;+\; \cdots \;+\; f(n-1) \;+\; \frac{1}{2}f(n) \\
&\quad - \sum_{p=1}^{q}\frac{B_{2p}}{(2p)!}\left(f^{(2p-1)}(n) - f^{(2p-1)}(0)\right) \\
&\quad + \frac{1}{(2q)!}\int_0^n B_{2q}(x)\sum_{s=0}^{n-1} f^{(2q)}(x+s)dx
\end{aligned}$$

(Compare equation (5.168b) in [AW].)

***** In the last integral, should the limits be 0 and n, or 0 and 1 ?

Apply the Euler-Maclaurin integration formula above to the right-hand side of the equation

$$\frac{1}{z} = \int_0^\infty \frac{1}{(z+x)^2} dx$$

to get the series

$$\frac{1}{z} = \frac{1}{2} f(0) + \sum_{n=1}^{\infty} f(n) - \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} \left(\lim_{x \to \infty} f^{(2n-1)}(x) - f^{(2n-1)}(0) \right)$$

for $f(x) = (z+x)^{-2}$. Thus,

$$\frac{1}{z} = \frac{1}{2z^2} + \sum_{n=1}^{\infty} \frac{1}{(z+n)^2} - \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} \left(\frac{(2n)!}{z^{2n+1}} \right)$$

From equation (10.41) in [AW],

$$\sum_{n=1}^{\infty} \frac{1}{(z+n)^2} = F^{(1)}(z)$$

so we can write

$$F^{(1)}(z) = \frac{d}{dz} F(z) = \frac{1}{z} - \frac{1}{2z^2} + \sum_{n=1}^{\infty} \frac{B_{2n}}{z^{2n+1}}$$

(Compare equation (10.50) in [AW].) Integrating once we get

$$F(z) = C_1 + \log z + \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n z^{2n}}$$

It can be shown (see [AW]) that $C_1 = 0$. Since

$$F(z) = \frac{d}{dz} \log \Gamma(z+1)$$

we can integrate again to get

$$\log \Gamma(z+1) = C_2 + \left(z + \frac{1}{2} \right) \log z - z + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)(2n-1) z^{2n-1}}$$

for some constant C_2, where we have used the fact that

$$\frac{d}{dz} z (\log z - 1) = \log z.$$

We can solve for C_2 by substituting the expression above into the Legendre duplication formula

$$\Gamma(z+1)\Gamma(z+\frac{1}{2}) = 2^{-2z} \pi^{1/2} \Gamma(2z+1)$$

(this corrects a minor typo in equation (10.53)) from which one can derive that $C_2 = (1/2)\log(2\pi)$. Thus,

$$\log\Gamma(z+1) \;=\; \frac{\log 2\pi}{2} \;+\; \left(z \;+\; \frac{1}{2}\right)\log z \;-\; z \;+\; \sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)(2n-1)z^{2n-1}}$$

which is Stirling's series, an asymptotic series for the natural logarithm of the gamma function.

We can also derive a more commonly used expression for Stirling's series by exponentiating the series above. We get

$$\Gamma(z+1) \;=\; \sqrt{2\pi}z^{z+1/2}\exp(-z)\exp\left(\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)(2n-1)z^{2n-1}}\right)$$

We can simplify the last factor as follows. Recall the Taylor expansion

$$\log(1 \;+\; x) \;=\; x \;-\; \frac{x^2}{2} \;+\; \frac{x^3}{3} \;-\; \frac{x^4}{4} \;+\; \cdots$$

If we find an x such that

$$\sum_{n=1}^{\infty}(-)^{n+1}\frac{x^n}{n} \;=\; \sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)(2n-1)z^{2n-1}}$$

then we can write

$$\exp\left(\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)(2n-1)z^{2n-1}}\right) \;=\; 1+x$$

Although finding a closed-form expression is impossible, we can find a series in z for x. From the first terms, clearly

$$x \;=\; \frac{B_2}{2z} \;+\; \mathcal{O}(z^{-2})$$

and if we work out the expansion more systematically, we discover

$$\begin{aligned} x \;&=\; \frac{B_2}{2z} \;+\; \frac{B_2^2}{8z^2} \;+\; \mathcal{O}(z^{-3}) \\ &=\; \frac{1}{12z} \;+\; \frac{1}{288z^2} \;+\; \mathcal{O}(z^{-3}) \end{aligned}$$

Thus, we have that

$$\Gamma(z+1) \;=\; \sqrt{2\pi}z^{z+1/2}\exp(-z)\left(1 \;+\; \frac{1}{12z} \;+\; \frac{1}{288z^2} \;+\; \mathcal{O}(z^{-3})\right)$$

another form of Stirling's asymptotic series.

D.3 Watson's lemma

Given a power series that converges in some finite domain, when one integrates the power series term-by-term along a contour that extends outside of the domain, although the result is no longer convergent, nevertheless it is often still sensible as an asymptotic series.

Formally, this result is known as *Watson's lemma*:

> Suppose that $f(t)$ is a possibly multivalued function of t, which has the following expansion near $t = 0$:
>
> $$f(t) \; = \; \sum_{n=1}^{\infty} a_n t^{n/r-1}, \quad |t| \leq a, r > 0$$
>
> Furthermore, suppose that $f(t)$ is of, at most, exponential growth for large t, so that for suitable positive constants K, b,
>
> $$|f(t)| \; < \; K e^{b|t|}, \quad |t| \geq a$$
>
> Then for $|z|$ large, the function
>
> $$F(z) \; \equiv \; \int_0^{\infty} e^{-zt} f(t) dt$$
>
> has asymptotic series expansion
>
> $$\sum_{n=1}^{\infty} a_n \left(\int_0^{\infty} e^{-zt} t^{n/r-1} dt \right) \; = \; \sum_{n=1}^{\infty} a_n \Gamma(n/r) z^{-n/r} \quad \text{for } |\arg z| < \pi/2$$

Proof: Let $S_k(z)$ be the finite sum

$$S_k(z) \; \equiv \; \int_0^{\infty} e^{-zt} \left(\sum_{n=1}^{k} a_n t^{n/r-1} \right) dt$$

Since the sum is finite, we can exchange the sum and integral:

$$S_k(z) \; = \; \sum_{n=1}^{k} a_n \left(\int_0^{\infty} e^{-zt} t^{n/r-1} dt \right) \; = \; \sum_{n=1}^{k} a_n \Gamma(n/r) z^{-n/r}$$

To establish that the result is an asymptotic series expansion of $F(t)$, we need to show that

$$\lim_{z \rightarrow \infty} z^{n/r} \left(F(z) \; - \; S_n(z) \right) \; = \; 0$$

for $|\arg z| < \pi/2$. By virtue of the bound on $f(t)$, there must exist a constant C_k such that

$$\left| f(t) \; - \; \sum_{n=1}^{k} a_n t^{n/r-1} \right| \; \leq \; C_k e^{b|t|} |t|^{(k+1)/r-1}$$

Then,

$$
\begin{aligned}
|z|^{n/r}\,|F(z) \;-\; S_n(z)| &\le |z|^{n/r} C_n \int_0^\infty e^{-(|z|\cos\phi - b)t} t^{(n+1)/r-1} dt \\
&= \frac{|z|^{n/r} C_n}{(|z|\cos\phi - b)^{(n+1)/r}} \Gamma((n+1)/r)
\end{aligned}
$$

where $\phi = \arg z$. For $|\arg z| < \pi/2$, $\cos\phi > 0$, so for large enough $|z|$ the expression above is well-defined and goes to zero as $|z| \to \infty$. □

D.4 Method of steepest descent

Consider a contour integral of the form

$$
G(z) \;=\; \int_C g(t)\exp(zf(t))dt
$$

The *method of steepest descent* is a systematic procedure for generating an asymptotic series that approximates integrals of this form. Briefly, the method says that for large values of z, the dominant contribution to the integral $G(z)$ will come from values of t such that $f'(t) = 0$, known as *saddle points*, in the sense that such points will make a contribution to the integral that is proportional to the integrand evaluated at the saddle point.

Let us first see how this works in the special case of a contour located along the real line, so that t is real, involving real-valued functions $f(t)$, $g(t)$. (This will repeat an argument from section *** CITE ****.) Let t_0 be a point such that $f'(t_0) = 0$, and for simplicity let us assume $f''(t_0) < 0$. Then, locally, we can approximate the integral by a Gaussian. Expand

$$
\begin{aligned}
g(t) &= g(t_0) \;+\; (t-t_0)g'(t_0) \;+\; \frac{1}{2}(t-t_0)^2 g''(t_0) \;+\; \cdots \\
f(t) &= f(t_0) \;+\; (t-t_0)f'(t_0) \;+\; \frac{1}{2}(t-t_0)^2 f''(t_0) \;+\; \cdots \\
&= f(t_0) \;+\; \frac{1}{2}(t-t_0)^2 f''(t_0) \;+\; \cdots
\end{aligned}
$$

Now, define $s = \sqrt{z}(t - t_0)$, so that we can write the integral as

$$
\begin{aligned}
G(z) &= \int_{-\infty}^{\infty} dt\, (g(t_0) \;+\; (t-t_0)g'(t_0) \;+\; \cdots) \exp\left(z\left(f(t_0) \;+\; \frac{1}{2}(t-t_0)^2 f''(t_0) \;+\; \cdots\right)\right) \\
&= \exp(zf(t_0)) \int_{-\infty}^{\infty} \frac{ds}{\sqrt{z}} \left(g(t_0) \;+\; \frac{s}{\sqrt{z}} g'(t_0) \;+\; \mathcal{O}(z^{-1})\right) \exp\left(-\frac{1}{2}s^2 |f''(t_0)| \;+\; \mathcal{O}(z^{-1/2})\right) \\
&= \frac{\exp(zf(t_0))}{\sqrt{z}} \int_{-\infty}^{\infty} ds\, g(t_0) \exp\left(-\frac{1}{2}s^2|f''(t_0)|\right) \;+\; \mathcal{O}(z^{-1}) \\
&= \frac{\exp(zf(t_0))}{\sqrt{z}} g(t_0)\sqrt{2|f''(t_0)|\pi} \;+\; \mathcal{O}(z^{-1})
\end{aligned}
$$

Not only do we see that the dominant contribution comes from $t = t_0$ for large z, in the sense that the contribution to the integral is proportional to the integrand evaluated at t_0, but we also get an explicit expression for the contribution to the limit from the t_0 point, in the same limit.

The method of steepest descent generalizes to complex integrands and contour integrals. In such cases, the general idea is that in an integral over a complex exponential of the form $\exp(zf(t))$, for large z, the part of the integration contour that mostly just changes the phase will not significantly contribute to the integral, but rather will tend to cancel out. A little more systematically, if the integral involves integrating over all phases of the complex exponential, then the different contributions should sum to zero, on the grounds that

$$\int_0^{2\pi} \cos\theta d\theta \;=\; \int_0^{2\pi} \sin\theta d\theta \;=\; 0$$

At this same, level of approximation, the leading contribution to the integral should come from parts of the contour where the phase does not change significantly. See [AW][section 7.4] or [MF][section 4.6] for more information on the general complex case.

As described so far here, the method of steepest descent gives the leading-order term in an asymptotic series expansion of an integral, but can also be generalized to give higher-order terms in the same expansion.

One application will be to the gamma function. Recall the definite integral description of the gamma function:

$$\Gamma(z+1) \;=\; \int_0^\infty \exp(-t) t^z dt$$

Change integration variables to $t = \tau z$:

$$\Gamma(z+1) \;=\; z^{z+1} \int_0^\infty \exp(-z\tau z)\tau^z d\tau$$

and write

$$\exp(-\tau/z)\tau^z \;=\; \exp(-z\tau + z\log\tau)$$

giving an integral of the form

$$\int_C \exp(zf(t))dt$$

with $f(t) = -t + \log t$.

Another application of the method of steepest descent is to the Feynman path integral description of quantum mechanics and quantum field theory, where it is used to recover the classical limit.

Let us describe how to derive this leading contribution in the case of the gamma function, then we shall describe how to more systematically use these general ideas to create an asymptotic expansion for such contour integrals.

In the case of the gamma function, recall

$$\Gamma(z+1) \;=\; z^{z+1} \int_0^\infty \exp(zf(t))dt$$

for $f(t) = \log t - t$. Solving $\partial f/\partial t = 0$, we find that the only possible saddle point is at

$$\frac{1}{t} - 1 = 0$$

i.e. $t_0 = 1$. Let us expand $f(t)$ about this saddle point. Write $t = 1 + x$, then

$$\begin{aligned} f(t) &= \log(1+x) - (1+x) \\ &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) - (1+x) \\ &= -1 - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \end{aligned}$$

from which we see the leading order approximation to the integral should be proportional to $\exp(-z)$. We can now approximate the gamma function by

$$\begin{aligned} \Gamma(z+1) &= z^{z+1}\exp(zf(1))\int_{-1}^{\infty}\exp(zf''(1)x^2/2!)dx \\ &= z^{z+1}\exp(-z)\int_{-1}^{\infty}\exp(-zx^2/2)dx \\ &\cong z^{z+1}\exp(-z)\int_{-\infty}^{\infty}\exp(-zx^2/2)dx \\ &= z^{z+1}\exp(-z)\sqrt{\frac{2\pi}{z}} \\ &= \sqrt{2\pi z}z^z\exp(-z) \end{aligned}$$

which is the leading term in Stirling's expansion of the factorial function.

Reference [AW], in section 7.4, discusses how to treat a contour integral in which the contour does not lie along the real line. An important part of the treatment, which I will omit here but which is discussed in [AW], involves replacing the contour integral with an integral along an infinite line that is tangent to a particular direction at the saddle point.

Let us instead outline how to more systematically derive an asymptotic series, not just the leading term, using these methods. In terms of our original function $G(z)$, if our contour crosses a single saddle point t_0, and the contour line is along a "path of steepest descent" along which the imaginary part of $zf(t)$ is constant. Write

$$f(t) = f(t_0) - w^2$$

for some variable w, which is real Im $f(t) = \text{Im } f(t_0)$ everywhere along the contour. Then,

$$G(z) = \exp(zf(t_0))\int_C \exp(-zw^2)\left(\frac{dt}{dw}\right)dw$$

Assume that the contour C is such that the w integral can be taken to run over the real numbers from $-\infty$ to ∞. Next, we need to write dt/dw as a function of w, rather than t. In general, we can accomplish such an inversion at the power-series level, so write

$$\frac{dt}{dw} = \sum_{n=0}^{\infty} a_n w^n$$

for some constants a_n. (Note that from the definition of w above, one only expects even powers of w to appear in this power series.) Substituting in one has

$$\begin{aligned} G(z) &= \exp(zf(t_0)) \int_{-\infty}^{\infty} \exp(-zw^2) \sum_{n=0}^{\infty} a_n w^n dw \\ &= \exp(zf(t_0)) \sum_{n=0}^{\infty} a_n z^{-(n+1)/2} \Gamma\left(\frac{n+1}{2}\right) \\ &= \frac{\exp(zf(t_0))}{\sqrt{z}} \sum_{m=0}^{\infty} a_{2m} \Gamma\left(m+\frac{1}{2}\right)\left(\frac{1}{z}\right)^m \end{aligned}$$

where in the last line we have used the fact that only even powers of w appear in dt/dw.

Let us outline how to derive the Stirling series using this method. Recall there that

$$\Gamma(z+1) = z^{z+1} \int_0^{\infty} \exp(zf(t))dt$$

for $f(t) = \log t - t$, with only saddle point at $t_0 = 1$. Expand in a Taylor series about $f(t_0) = -1$ to get

$$f(t) = -1 - \frac{(t-1)^2}{2!} + 2\frac{(t-1)^3}{3!} - (3!)\frac{(t-1)^4}{4!} + \cdots$$

so in the notation above,

$$w^2 = \frac{(t-1)^2}{2} - \frac{(t-1)^3}{3} + \cdots$$

It can be shown (see [MF][section 4.6] for details, but note their a_0 at the bottom of p. 442 is $\sqrt{2}$ not $1/\sqrt{2}$) that

$$\frac{dt}{dw} = \sqrt{2}\left(1 + \frac{w^2}{6} + \frac{w^4}{216} + \cdots\right)$$

and plugging these values into the general expression for the asymptotic series we find

$$\begin{aligned} \Gamma(z+1) &= z^{z+1}\frac{\exp(-z)}{\sqrt{z}}\sqrt{2}\left[\Gamma(1/2) + \frac{1}{6}\Gamma(3/2)\frac{1}{z} + \frac{1}{216}\Gamma(5/2)\frac{1}{z^2} + \cdots\right] \\ &= \sqrt{2\pi} z^{z+1/2} \exp(-z)\left[1 + \frac{1}{12z} + \frac{1}{288z^2} + \cdots\right] \end{aligned}$$

and so we recover Stirling's series.

For more information, see [AW][section 7.4] or [MF][section 4.6].

D.5 Uniqueness (or lack thereof)

One very important property of asymptotic series is that they do *not* uniquely determine a function. It is easy to check, for example, that if $\mathrm{Re}(z) > 0$, then the same series can be simultaneously asymptotic to both $f(z)/\varphi(z)$ and $f(z)/\varphi(z) + \exp(-z)$.

This fact is very important in quantum field theory, and is a reflection of nonperturbative effects in the theory. Summing over Feynman diagrams yields a series in which the coupling constant of the theory plays the part of $1/z$. Now, a typical quantum field theory has ‘nonperturbative effects,’ which cannot be seen in a (perturbative) Feynman diagram expansion. Nonperturbative effects, which are not uniquely determined by the perturbative theory, are exponentially small in the coupling constant, *i.e.* multiplied by factors of $\exp(-1/g) = \exp(-z)$. Since the Feynman diagram expansion is only an asymptotic series, and the nonperturbative effects are exponentially small, adding nonperturbative effects does not change the asymptotic expansion, *i.e.* does not change the Feynman diagram expansion.

Properties of asymptotic series:

Asymptotic series can be added, multiplied, and integrated term-by-term. However, asymptotic series can only be differentiated term-by-term to obtain an asymptotic expansion for the derivative only if it is known that the derivative possesses an asymptotic expansion.

D.6 Summation of asymptotic series

How can we sum, in any sense, a divergent series?

One approach is as follows. Given a divergent series

$$F(z) \; = \; \sum_{n=0}^{\infty} A_n z^n$$

for some constants A_n, consider the related series

$$B(z) \; = \; \sum_{n=0}^{\infty} A_n \frac{z^n}{n!}$$

Depending upon how badly divergent the original series $F(z)$ was, one might hope that the new series $B(z)$ might actually converge in some region. Assuming that $B(z)$ converges and can be resummed, how might one recover $F(z)$? Well, use the formula

$$\int_0^\infty \exp(-t/z) t^n dt \; = \; z^{n+1} n!$$

to show that, *formally*,

$$zF(z) \; = \; \int_0^\infty \exp(-t/z) B(t) dt \tag{D.1}$$

To calculate $F(z)$ using the formal trick above, we need $B(t)$ for real positive values of t less than or of order z. So long as any singularities in $B(t)$ on the complex t plane are at distances greater than $|z|$ from the origin, this should be OK. (In quantum field theory, singularities in $B(t)$ are typically associated with nonperturbative effects – instantons – so again we see that nonperturbative effects limit the usefulness of resummation methods for the (asymptotic) Feynman series. See [Weinberg2] for more information.)

This particular resummation technique is known as *Borel summation.* If the integral on the right-hand-side exists at points outside the radius of convergence of the original series, then the Borel sum is defined to be

$$\frac{1}{z}\int_0^\infty \exp(-t/z)B(t)dt$$

(The original reference on Borel summation is, to our knowledge, Borel, *Leçons sur les Séries Divergentes* (1901) pp 97-115.)

As asymptotic series expansions do not uniquely determine functions, the reader should not be surprised to learn there are additional resummation techniques, which can yield different results.

For example, *Euler resummation* is to define the sum of the series $\sum A_n$ to be given by

$$\lim_{z\to 1^-}\sum_{n=0}^\infty A_n z^n$$

when this limit exists. For example, the Euler sum of the series

$$1 \,-\, 1 \,+\, 1 \,-\, 1 \,+\, 1 \,-\, \cdots$$

is given by

$$\lim_{z\to 1^-}\left(1 \,-\, z \,+\, z^2 \,-\, z^3 \,+\, \cdots\right) \;=\; \lim_{z\to 1^-}\frac{1}{1+z} \;=\; \frac{1}{2}$$

Further resummation methods are discussed in [Hardy, WW].

**** See Zinn-Justin pp 840-, section 37.3, 37.4 for more info on Borel.

D.7 Stokes' phenomenon

Stokes' phenomenon is the observation that the operations of analytic continuation and asymptotic series expansion do not commute with one another.

Let us work through a simple example, following [MF][section 5.3]. Consider the confluent hypergeometric function defined by

$$M(a,c;z) \;=\; \sum_{n=0}^\infty \frac{(a)_n}{(c)_n}\frac{z^n}{n!}$$

where $(a)_n$ is the Pochhammer symbol:

$$\begin{aligned}
(a)_n &= a(a+1)(a+2)\cdots(a+n-1) \\
&= \frac{(a+n-1)!}{(a-1)!} \;=\; \frac{\Gamma(a+n)}{\Gamma(a)} \\
(a)_0 &= 1
\end{aligned}$$

First, let us derive the leading term in an asymptotic expansion as $z \to +\infty$ along the real axis. The n term of the series above is given by

$$\frac{(a)_n}{(c)_n}\frac{z^n}{n!} = \frac{(a+n-1)!}{(a-1)!}\frac{(c-1)!}{(c+n-1)!}\frac{z^n}{n!}$$

It can be shown [AW][problem 8.3.8] that

$$\lim_{x\to\infty} x^{b-a}\frac{(x+a)!}{(x+b)!} = 1$$

so we see that for large n, the n term becomes

$$\begin{aligned} &\to \frac{(c-1)!}{(a-1)!}\frac{n^{(a-1)-(c-1)}}{n!}z^n \\ &\to \frac{(c-1)!}{(a-1)!}\frac{z^n}{(n-a+c)!} \end{aligned}$$

At large positive z, the terms with large n dominate, so that

$$M(a,c;z) \approx \sum_{n=1}^{\infty}\frac{(c-1)!}{(a-1)!}\frac{z^n}{(n-a+c)!}$$

Assume $a-c \in \mathbf{Z}$, and ignoring the first few terms (small compared to the rest), we have that

$$\begin{aligned} M(a,c;z) &\approx \frac{(c-1)!}{(a-1)!}\sum_{m=0}^{\infty}\frac{z^m}{m!}z^{a-c} \quad (m = n-a+c) \\ &= \frac{\Gamma(c)}{\Gamma(a)}z^{a-c}\exp(z) \end{aligned}$$

We have been sloppy, but a more careful analysis reveals this result is correct; the leading term in an asymptotic series expansion of $M(a,c;z)$ for $z \to +\infty$ along the real axis is given by the expression above.

Next, let us find the leading term in an asymptotic expansion of $M(a,c;z)$ for $z \to -\infty$ along the real axis. The fast way to do this is to use the identity

$$M(a,c;z) = \exp(z)M(c-a,c;-z)$$

Using this and our previous result, we see that as $z \to -\infty$ along the real axis,

$$M(a,c;z) \to \frac{\Gamma(c)}{\Gamma(c-a)}(-z)^{-a}$$

This is the leading term in an asymptotic series expansion of $M(a,c;z)$ for $z \to -\infty$.

But if we compare our results for the two different limits, we see that this is not what we would have gotten by analytically continuing either separately.

For example, if we started with the $z \to +\infty$ limit, and analytically continued, we would have found

$$\frac{\Gamma(c)}{\Gamma(a)}(-z)^{a-c}\exp(-z) \neq \frac{\Gamma(c)}{\Gamma(c-a)}(-z)^{-a}$$

Thus, analytic continuation does *not* commute with asymptotic series expansions. This is known as Stokes' phenomenon. This is very unlike convergent Taylor series, for example, where analytic continuation does commute with series expansion.

Let us also understand this more systematically by taking asymptotic series expansions at all $\arg(z)$, not just along the real axis. To do this, we shall use the integral representation [AW][problem 13.5.10a]

$$M(a,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 \exp(zt)t^{a-1}(1-t)^{c-a-1}dt$$

Deform the integration path to go from first, $t = 0$ to $t = -\infty\exp(-i\phi)$, then, from $t = -\infty\exp(-i\phi)$ to $t = 1$, where $z = |z|\exp(i\phi)$. Then we find that

$$M(a,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\left\{\int_0^{-\infty\exp(-i\phi)} \exp(zt)t^{a-1}(1-t)^{c-a-1}dt + \int_{-\infty\exp(-i\phi)}^1 \exp(zt)t^{a-1}(1-t)^{c-a-1}dt\right\}$$

When $0 < \phi < \pi$, make the following changes of variables. In the first integral, define w by

$$t = -\frac{w\exp(-i\phi)}{|z|} = -\frac{w}{z}$$

and in the second integral, define

$$t = 1 - \frac{u\exp(-i\phi)}{|z|} = 1 - \frac{u}{z}$$

Plugging these in, we find

$$\begin{aligned}
M(a,c;z) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\left\{\int_0^\infty \exp(-w)\left(\frac{-w}{z}\right)^{a-1}\left(1+\frac{w}{z}\right)^{c-a-1}\left(-\frac{dw}{z}\right)\right.\\
&\qquad \left. + \int_\infty^0 \exp(z(1-u/z))\left(1-\frac{u}{z}\right)^{a-1}\left(\frac{u}{z}\right)^{c-a-1}\left(-\frac{du}{z}\right)\right\}\\
&= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\left\{\frac{\exp(ia\pi)}{z^a}\int_0^\infty \exp(-w)w^{a-1}\left(1+\frac{w}{z}\right)^{c-a-1}dw\right.\\
&\qquad \left. + \frac{\exp(z)}{z^{c-a}}\int_0^\infty \exp(-u)\left(1-\frac{u}{z}\right)^{a-1}u^{c-a-1}du\right\}\\
&\to \frac{\Gamma(c)}{\Gamma(c-a)}\frac{\exp(ia\pi)}{z^a} + \frac{\Gamma(c)}{\Gamma(a)}\exp(z)z^{a-c}
\end{aligned}$$

The first term dominates when $\phi = \pi$, at which the second term is negligible. When $\phi = 0$, the opposite is true: the second term dominates, the other term is negligible. For $\phi = \pi/2$, the two terms are comparable.

The analysis can be repeated for $-\pi < \phi < 0$; but as it is very similar, for brevity we shall not repeat it here.

So, what we have found in general is that for general ϕ, the leading term is a combination of the two terms, but in the two limits, one dominates and the other is much less than the corrections, so that the leading term in the asymptotic series expansion is defined by only term, not both. Which term dominates, varies as ϕ changes. Thus, analytic continuation does not commute with asymptotic series expansion.

D.8 Exercises

1. Show that the function e^z has no asymptotic series expansion for z real, positive, and large.

2. Show that the function e^z has the asymptotic series expansion

$$0 \,+\, 0 \,+\, 0 \,+\, 0 \,+\, \cdots$$

   for z real, negative, and of large magnitude. The previous problem plus this one give an easy example of Stokes' phenomenon, namely that the same function can have different asymptotic series expansions as one goes in different directions on the complex plane.

3. (AW 8.3.8) Use the Stirling series for the gamma function to show that

$$\lim_{x \to \infty} x^{b-a} \frac{(x+a)!}{(x+b)!} \;=\; 1$$

4. Use Watson's lemma to derive an asymptotic series for

$$\int_0^\infty e^{-zt} \left(1 \,+\, t^2\right)^{1/2} dt$$

5. (AW 7.3.1) Using the method of steepest descent, evaluate the second Hankel function given by

$$H_\nu^{(2)}(s) \;=\; \frac{1}{\pi i} \int_{-\infty}^{0} \exp\left(\frac{s}{2}\left(z \,-\, \frac{1}{z}\right)\right) \frac{dz}{z^{\nu+1}}$$

   with contour **** FILL IN ****

6. (WW VIII.6) Show that the series

$$1 \,-\, 2! \,+\, 4! \,-\, \cdots$$

   is not Borel summable, whereas the series

$$1 \,+\, 0 \,-\, 2! \,+\, 0 \,+\, 4! \,+\, \cdots$$

   is Borel summable.

**** Also mention asymptotic series, steepest descent problems buried in the bulk of the text.

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