Elliptic functions

See [1][section 4.5] and [2] for more information.

1 Definition

An *elliptic function* is a single-valued doubly-periodic function of a single complex variable which is analytic except at poles and whose only singularities in the finite plane are poles.

Such functions are called elliptic because they define functions on the two-torus. Imagine building a two-torus (a doughnut) by taking a square and identifying opposing sides – that means a function on the complex plane which is periodic in two directions can be thought of as a function on a square, periodic at opposing sides, and hence is a function on the two-torus.

Given an angle φ, define

$$u = \int_0^{\varphi} \frac{d\theta}{\left(1 - m\sin^2\theta\right)^{1/2}}$$

where m is the *parameter*, a real number in the interval $0 \leq m \leq 1$. In terms of the elliptic integrals discussed in A-W section 5.8, $u = F(\sin\varphi|m)$ where F is the elliptic integral of the first kind. The angle φ corresponding to u is known as the *amplitude* of u, and is denoted am u. Then, define the *Jacobi elliptic functions*

$$\begin{aligned} \text{sn } u &= \sin\varphi \\ \text{cn } u &= \cos\varphi \\ \text{dn } u &= \left(1 - m\sin^2\varphi\right)^{1/2} \end{aligned}$$

In the case $m = 0$, note that

$$\begin{aligned} \text{sn } u &= \sin u \\ \text{cn } u &= \cos u \\ \text{dn } u &= 1 \end{aligned}$$

and in the special case $m = 1$,

$$\begin{aligned} \text{sn } u &= \tanh u \\ \text{cn } u &= \frac{1}{\cosh u} \\ \text{dn } u &= \frac{1}{\cosh u} \end{aligned}$$

Now, in order to be an elliptic function, these functions must possess a double periodicity. Let

$$K(m) \;=\; \int_0^{\pi/2} \frac{d\theta}{\left(1 \;-\; m\sin^2\theta\right)^{1/2}}$$

be the complete elliptic integral of the first kind, and define

$$K'(m) \;=\; \int_0^{\pi/2} \frac{d\theta}{\left(1 \;-\; m_1\sin^2\theta\right)^{1/2}}$$

where $m_1 = 1 - m$ is known as the *complementary parameter*. In this context, $K(m)$ is known as the real quarter-period, and $iK'(m)$ is known as the imaginary quarter-period. Then, it can be shown that

$$\begin{aligned}
\mathrm{sn}\; u \;&=\; \mathrm{sn}\;(u + 4K(m)) \;=\; \mathrm{sn}\;(u + 2iK'(m)) \;=\; \mathrm{sn}\;(u + 4K(m) + 4iK'(m)) \\
\mathrm{cn}\; u \;&=\; \mathrm{cn}\;(u + 4K(m)) \;=\; \mathrm{cn}\;(u + 4iK'(m)) \;=\; \mathrm{cn}\;(u + 2K(m) + 2iK'(m)) \\
\mathrm{dn}\; u \;&=\; \mathrm{dn}\;(u + 2K(m)) \;=\; \mathrm{dn}\;(u + 4iK'(m)) \;=\; \mathrm{dn}\;(u + 4K(m) + 4iK'(m))
\end{aligned}$$

Since these functions are periodic in two directions on the complex plane, they are elliptic functions.

Given these elliptic functions, one can define the additional Jacobi elliptic functions

$$\begin{array}{lll}
\mathrm{cd}\; u = \frac{\mathrm{cn}\; u}{\mathrm{dn}\; u} & \mathrm{dc}\; u = \frac{\mathrm{dn}\; u}{\mathrm{cn}\; u} & \mathrm{ns}\; u = \frac{1}{\mathrm{sn}\; u} \\
\mathrm{sd}\; u = \frac{\mathrm{sn}\; u}{\mathrm{dn}\; u} & \mathrm{nc}\; u = \frac{1}{\mathrm{cn}\; u} & \mathrm{ds}\; u = \frac{\mathrm{dn}\; u}{\mathrm{sn}\; u} \\
\mathrm{nd}\; u = \frac{1}{\mathrm{dn}\; u} & \mathrm{sc}\; u = \frac{\mathrm{sn}\; u}{\mathrm{cn}\; u} & \mathrm{cs}\; u = \frac{\mathrm{cn}\; u}{\mathrm{sn}\; u}
\end{array}$$

Put simply, if p, q, r are any three of the letters s, c, d, n, then

$$\mathrm{pq}\; u \;=\; \frac{\mathrm{pr}\; u}{\mathrm{qr}\; u}$$

with the convention that when any two letters are the same, *e.g.* pp u, then that is set to 1.

Define the *theta functions* as follows:

$$\begin{aligned}
\theta_1(z,q) \;&=\; 2q^{1/4}\sum_{n=0}^{\infty}(-)^n q^{n(n+1)}\sin[(2n+1)z] \\
&=\; \sum_{n=-\infty}^{\infty}(-)^{n-1/2}q^{(n+1/2)^2}\exp((2n+1)iz) \\
\theta_1(z+\pi,q) \;&=\; -\theta_1(z,q) \\
\theta_1(z+\pi\gamma,q) \;&=\; -N\theta_1(z,q)
\end{aligned}$$

$$
\begin{aligned}
\theta_2(z,q) &= 2q^{1/4}\sum_{n=0}^{\infty} q^{n(n+1)}\cos[(2n+1)z] \\
&= \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}\exp((2n+1)iz) \\
\theta_2(z+\pi,q) &= -\theta_2(z,q) \\
\theta_2(z+\pi\gamma,q) &= N\theta_2(z,q) \\
\theta_3(z,q) &= 1 \;+\; 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz) \\
&= \sum_{n=-\infty}^{\infty} q^{n^2}\exp(2niz) \\
\theta_3(z+\pi,q) &= \theta_3(z,q) \\
\theta_3(z+\pi\gamma,q) &= N\theta_3(z,q) \\
\theta_4(z,q) &= 1 \;+\; 2\sum_{n=1}^{\infty} (-)^n q^{n^2}\cos(2nz) \\
&= \sum_{n=-\infty}^{\infty} (-)^n q^{n^2}\exp(2niz) \\
\theta_4(z+\pi,q) &= \theta_4(z,q) \\
\theta_4(z+\pi\gamma,q) &= -N\theta_4(z,q) \\
\theta_2(z,q) &= \theta_1(z+(1/2)\pi,q) \\
\theta_3(z,q) &= \theta_4(z+(1/2)\pi,q)
\end{aligned}
$$

where $q = \exp(i\pi\gamma)$, Re $\gamma > 0$, $N = q^{-1}\exp(-2iz)$.

Some relations between squares of the theta functions:

$$
\begin{aligned}
\theta_1^2(z)\theta_4^2(0) &= \theta_3^2(z)\theta_2^2(0) \;-\; \theta_2^2(z)\theta_3^2(0) \\
\theta_2^2(z)\theta_4^2(0) &= \theta_4^2(z)\theta_2^2(0) \;-\; \theta_1^2(z)\theta_3^2(0) \\
\theta_3^2(z)\theta_4^2(0) &= \theta_4^2(z)\theta_3^2(0) \;-\; \theta_1^2(z)\theta_2^2(0) \\
\theta_4^2(z)\theta_4^2(0) &= \theta_3^2(z)\theta_2^2(0) \;-\; \theta_2^2(z)\theta_2^2(0) \\
\theta_3^4(0) &= \theta_2^4(0) \;+\; \theta_4^4(0)
\end{aligned}
$$

where we have omitted the q for brevity.

The theta functions are not precisely doubly-periodic: after periods, they shift by phases. However, such phases can be cancelled by taking ratios. For example, it can be shown that

$$
\begin{aligned}
\text{sn } u &= \frac{\theta_3(0,q)}{\theta_2(0,q)}\frac{\theta_1(z,q)}{\theta_4(z,q)} \\
\text{cn } u &= \frac{\theta_4(0,q)}{\theta_2(0,q)}\frac{\theta_2(z,q)}{\theta_4(z,q)}
\end{aligned}
$$

$$\mathrm{dn}\ u = \frac{\theta_4(0,q)}{\theta_3(0,q)}\frac{\theta_3(z,q)}{\theta_4(z,q)}$$

where

$$z = \frac{u}{(\theta_3(0,q))^2}$$

and $q = \exp(-\pi K'(m)/K(m))$.

In addition to the Jacobi elliptic functions, another class of elliptic functions known as the Weierstrass elliptic functions also exists, though we shall not discuss them here.

2 Exercises

1. Show that $\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1$.
2. Show that $\mathrm{sn}\ (-u) = -\mathrm{sn}\ u$.
3. Show that $\mathrm{sn}\ u = \mathrm{sn}\ (u + 4K(m))$.
4. Show that

$$\begin{aligned} \theta_1(z+\pi, q) &= -\theta_1(z,q) \\ \theta_1(z+\pi\gamma, q) &= -N\theta_1(z,q) \end{aligned}$$

5. Show that the ratio

$$\frac{\theta_1(z,q)}{\theta_4(z,q)}$$

   is invariant under $z \mapsto z + 2\pi$ and under $z \mapsto z + \pi\gamma$.

References

[1] P. Morse, H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, 1953.

[2] M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions*, AMS55.